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Existence of Fixed Points in Neutrosophic Bipolar Fuzzy Metric Spaces with Application

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Abstract. In this paper, we aim to introduce and explain the concept of neutrosophic fuzzy bipolar metric spaces and provide some fixed point results in this new setting. Additionally, we give a detailed presentation of the proof of fixed point theorems using covariant maps, showing how our findings extend and generalize those established in the existing body of literature. The derived results are supported with a suitable example and an application.

Key words and Phrases: Neutrosophic fuzzy bipolar metric space, fixed point, covariant mapping.

1. INTRODUCTION

Zadeh [1] was the pioneer behind the introduction of fuzzy sets. Fuzzy theory's rapid evolution has simplified its application, enabling us to explore uncertainty levels in nature through a rigorous and mathematical approach. The incorporation of both probabilistic metric space and fuzzy concepts led to the introduction of fuzzy metric space. This pioneering work, attributed to Kaleva and Seikkala [2] characterizes FMS as a representation of distance between two points through non-negative fuzzy numbers.

FMS has since found practical applications in diverse fields, including fixed point theory, image and signal processing, medical imaging, and decision-making. Schweizer and Sklar [3] introduced the concept of continuous t-norms, which laid the foundation for further developments in the field. Kramosil and Michalek [4]

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extended the study by comparing probabilistic and statistical generalizations of metric spaces, incorporating the concept of fuzziness. Garbiec [5] interpreted the fuzzy concept of Banach contraction, and this idea has since been explored in various contexts.

With the emergence of intuitionistic fuzzy sets (IFS), these versatile sets have permeated the various domains of fuzzy set theory. Park [6] extends this reach by defining the intuitionistic Fuzzy Metric Space (IFMS), a broadened idia of FMS. Park [6] incorporates George and Veeramani's [7] insights by applying t-norm and t-conorm operations to FMS, effectively defining IFMS and exploring its foundational characteristics. Many researchers work on FMS and IFMS, see references [8, 9, 10]. In 2024 Gupta, Saini and Gondhi [11] combine MIF metric with soft and proved FP theorem in this newly defined space, also see [12, 13].

Mutlu and Gurdal [14] introduced bipolar metric spaces and proved fixed point theorems, initiating a trend in utilizing different types of contractions in bipolar metric spaces, as evidenced by various researchers, see in references [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. The utilization of intuitionistic fuzzy bipolar metric spaces for solving integral equations is another noteworthy development, emphasizing membership functions within fuzzy metric spaces.

Kirişci and Simşek [27] introduced the concept of neutrosophic metric spaces (NMSs), which extended the framework of intuitionistic fuzzy metric spaces by incorporating the principles of neutrosophy. They explored various topological properties of these spaces and established numerous fixed point results for contraction mappings within NMSs. Several other researchers have also pursued related advancements, enriching the theory with additional generalizations and applications, see [28, 29, 30, 31].

The paper's structure is as follows: The first section provides an introduction, the second section covers basic definitions in the preliminaries, the third section presents the neutrosophic fuzzy bipolar metric space and some fixed-point results, accompanied by illustrative examples. In the fourth section, we demonstrate applications, including the proof of existence and uniqueness of a solution for a first-order ordinary differential equation.

2. PRELIMINARIES

Definition 2.1. [15] *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous triangle norm (CTN) if:*

(1) $\varpi * \aleph = \aleph * \varpi$*, for all* $\varpi, \aleph \in [0,1]$*;*
(2) $*$ *is continuous;*
(3) $\varpi * 1 = \varpi$*, for all* $\varpi \in [0,1]$*;*
(4) $(\varpi * \aleph) * \omega = \varpi * (\aleph * \omega)$*, for all* $\varpi, \aleph, \omega \in [0,1]$*;*
(5) If $\varpi \leq \aleph$ *and* $\zeta \leq \omega$*, with* $\varpi, \aleph, \zeta, \omega \in [0,1]$*, then* $\varpi * \zeta \leq \omega * \aleph$*.*

Definition 2.2. [15] *A continuous triangle co-norm (CTCN) is defined as a binary operation* $\circ : [0,1] \times [0,1] \to [0,1]$ *satisfying the following properties:*

(1) $\varpi \circ \aleph = \aleph \circ \varpi$, *for all* $\varpi, \aleph \in [0,1]$;
(2) *The operation* $\circ$ *is continuous;*
(3) $\varpi \circ 0 = 0$, *for all* $\varpi \in [0,1]$;
(4) $(\varpi \circ \aleph) \circ \omega = \varpi \circ (\aleph \circ \omega)$, *for all* $\varpi, \aleph, \omega \in [0,1]$;
(5) *If* $\varpi \leq \aleph$ *and* $\zeta \leq \omega$, *with* $\varpi, \aleph, \zeta, \omega \in [0,1]$, *then* $\varpi \circ \zeta \leq \aleph \circ \omega$.

Definition 2.3. [4] *Let* $*$ *be a CTN,* $\circ$ *be a CTCN, and* $\mathbb{H}, \mathbb{O}$ *be fuzzy sets on* $\Omega \times \Omega \times (0, +\infty)$, *where* $\Omega \neq \phi$. *If* $(\Omega, \mathbb{H}, \mathbb{O}, *, \circ)$ *fulfills the below conditions for all* $\aleph, \varpi \in \Omega$ *and* $\psi, \varphi > 0$:

(1) $\mathbb{H}(\aleph, \varpi, \varphi) + \mathbb{O}(\aleph, \varpi, \varphi) \leq 1$;
(2) $\mathbb{H}(\aleph, \varpi, \varphi) > 0$;
(3) $\mathbb{H}(\aleph, \varpi, \varphi) = 1$ *if and only if* $\aleph = \varpi$;
(4) $\mathbb{H}(\aleph, \varpi, \varphi) = \mathbb{H}(\varpi, \aleph, \varphi)$;
(5) $\mathbb{H}(\aleph, \aleph_0, (\varphi + \psi)) \geq \mathbb{H}(\aleph, \varpi, \varphi) * \mathbb{H}(\varpi, \aleph_0, \psi)$;
(6) $\mathbb{H}(\aleph, \varpi, \cdot)$ *is an increasing function of* $\mathbf{R}^+$ *and* $\lim_{\varphi \to +\infty} \mathbb{H}(\aleph, \varpi, \varphi) = 1$;
(7) $\mathbb{O}(\aleph, \varpi, \varphi) > 0$;
(8) $\mathbb{O}(\aleph, \varpi, \varphi) = 0$ *if and only if* $\aleph = \varpi$;
(9) $\mathbb{O}(\aleph, \varpi, \varphi) = \mathbb{O}(\varpi, \aleph, \varphi)$;
(10) $\mathbb{O}(\aleph, \aleph_0, (\varphi + \psi)) \leq \mathbb{O}(\aleph, \varpi, \varphi) \circ \mathbb{O}(\varpi, \aleph_0, \psi)$;
(11) $\mathbb{O}(\aleph, \varpi, \cdot)$ *is a decreasing function of* $\mathbf{R}^+$ *and* $\lim_{\varphi \to +\infty} \mathbb{O}(\aleph, \varpi, \varphi) = 0$.

Then, $(\Omega, \mathbb{H}, \mathbb{O}, *, \circ)$ is an intuitionistic fuzzy metric space.

Definition 2.4. [4] *Let* $*$ *be a CTN,* $\circ$ *be a CTCN,* $b \geq 1$ *and* $\mathbb{H}, \mathbb{O}$ *be fuzzy sets on* $\Omega \times \Omega \times (0, +\infty)$ *where* $\Omega \neq \phi$. *If* $(\Omega, \mathbb{H}, \mathbb{O}, *, \circ, b)$ *fulfills the below conditions for all* $\aleph, \varpi \in \Omega$ *and* $\psi, \varphi > 0$:

(1) $\mathbb{H}(\aleph, \varpi, \varphi) + \mathbb{O}(\aleph, \varpi, \varphi) \leq 1$;
(2) $\mathbb{H}(\aleph, \varpi, \varphi) > 0$;
(3) $\mathbb{H}(\aleph, \varpi, \varphi) = 1$ *if and only if* $\aleph = \varpi$;
(4) $\mathbb{H}(\aleph, \varpi, \varphi) = \mathbb{H}(\varpi, \aleph, \varphi)$;
(5) $\mathbb{H}(\aleph, \aleph_0, b(\varphi + \psi)) \geq \mathbb{H}(\aleph, \varpi, \varphi) * \mathbb{H}(\varpi, \aleph_0, \psi)$;
(6) $\mathbb{H}(\aleph, \varpi, \cdot)$ *is an increasing function of* $\mathbf{R}^+$ *and* $\lim_{\varphi \to +\infty} \mathbb{H}(\aleph, \varpi, \varphi) = 1$;
(7) $\mathbb{O}(\aleph, \varpi, \varphi) > 0$;
(8) $\mathbb{O}(\aleph, \varpi, \varphi) = 0$ *if and only if* $\aleph = \varpi$;
(9) $\mathbb{O}(\aleph, \varpi, \varphi) = \mathbb{O}(\varpi, \aleph, \varphi)$;
(10) $\mathbb{O}(\aleph, \aleph_0, b(\varphi + \psi)) \leq \mathbb{O}(\aleph, \varpi, \varphi) \circ \mathbb{O}(\varpi, \aleph_0, \psi)$;
(11) $\mathbb{O}(\aleph, \varpi, \cdot)$ *is a decreasing function of* $\mathbf{R}^+$ *and* $\lim_{\varphi \to +\infty} \mathbb{O}(\aleph, \varpi, \varphi) = 0$.

Then, $(\Omega, \mathbb{H}, \mathbb{O}, *, \circ, b)$ is an intuitionistic fuzzy b-metric space.

Definition 2.5. [19] *Let* Ω *and* Ξ *be non-void sets and* $\mu : \Omega \times \Xi \to [0, +\infty)$ *be a function, such that*

(1) $\mu(\aleph, \varpi) = 0$ *if and only if* $\aleph = \varpi$*, for all* $(\aleph, \varpi) \in \Omega \times \Xi$*;*
(2) $\mu(\aleph, \varpi) = \mu(\varpi, \aleph)$*, for all* $(\aleph, \varpi) \in \Omega \cap \Xi$*;*
(3) $\mu(\aleph, \varpi) \leq \mu(\aleph, \varpi_0) + \mu(\aleph_1, \varpi_0) + \mu(\aleph_1, \varpi)$*, for all* $\aleph, \aleph_1 \in \Omega$ *and* $\varpi_0, \varpi \in \Xi$*.*

The pair (Ω, Ξ, μ) is called a bipolar metric space.

Definition 2.6. [19] *Let* Ω *and* Ξ *be two non-void sets. We say that the quadruple* $(\Omega, \Xi, \mathbb{H}, *)$ *is a fuzzy bipolar metric space if* $*$ *is a continuous t-norm and* $\mathbb{H}$ *is a fuzzy set on* $\Omega \times \Xi \times (0, \infty)$*, fulfilling the following conditions for all* $\rho, \omega, r > 0$*:*

(1) $\mathbb{H}(\aleph, \varpi, \varphi) > 0$ *for all* $(\aleph, \varpi) \in \Omega \times \Xi$*;*
(2) $\mathbb{H}(\aleph, \varpi, \varphi) = 1$ *if and only if* $\aleph = \varpi$ *for* $\aleph \in \Omega$ *and* $\varpi \in \Xi$*;*
(3) $\mathbb{H}(\aleph, \varpi, \varphi) = \mathbb{H}(\varpi, \aleph, \varphi)$ *for all* $\aleph, \varpi \in \Omega \cap \Xi$*;*
(4) $\mathbb{H}(\aleph_1, \varphi_2, \rho+\omega+r) \geq \mathbb{H}(\aleph_1, \varphi_1, \rho) * \mathbb{H}(\aleph_2, \varphi_1, \omega) * \mathbb{H}(\aleph_2, \varphi_2, r)$ *for all* $\aleph_1, \aleph_2 \in \Omega$ *and* $\varphi_1, \varphi_2 \in \Xi$*;*
(5) $\mathbb{H}(\aleph, \varpi, \cdot) : [0, \infty) \to [0, 1]$ *is left continuous;*
(6) $\mathbb{H}(\aleph, \varpi, \cdot)$ *is non-decreasing for all* $\aleph \in \Omega$ *and* $\varpi \in \Xi$*.*

Definition 2.7. [14] *Let* $\Omega \neq \phi$*,* $\Xi \neq \phi$ *be two sets and* $*$ *be a CTN,* $\circ$ *be a CTCN, and* $\mathbb{H}$*,* $\mathbb{O}$ *be neutrosophic sets on* $\Omega \times \Xi \times (0, +\infty)$*, said to be an intuitionistic fuzzy bipolar metric on* $\Omega \times \Xi$*, if for all* $\aleph, \hat{a} \in \Omega$*,* $\varpi, \hat{s} \in \Xi$ *and* $\varphi, \zeta, \hat{w} > 0$*, the following conditions are satisfied:*

(1) $\mathbb{H}(\aleph, \varpi, \varphi) + \mathbb{O}(\aleph, \varpi, \varphi) \leq 1$*;*
(2) $\mathbb{H}(\aleph, \varpi, \varphi) > 0$*;*
(3) $\mathbb{H}(\aleph, \varpi, \varphi) = 1$ *for all* $\varphi > 0$*, if and only if* $\aleph = \varpi$*;*
(4) $\mathbb{H}(\aleph, \varpi, \varphi) = \mathbb{H}(\varpi, \aleph, \varphi)$*;*
(5) $\mathbb{H}(\aleph, \hat{s}, \varphi + \hat{w} + \zeta) \geq \mathbb{H}(\aleph, \varpi, \varphi) * \mathbb{H}(\hat{a}, \varpi, \hat{w}) * \mathbb{H}(\hat{a}, \hat{s}, \zeta)$*;*
(6) $\mathbb{H}(\aleph, \varpi, \cdot) : (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\varphi \to +\infty} \mathbb{H}(\aleph, \varpi, \varphi) = 1$*;*
(7) $\mathbb{H}(\aleph, \varpi, \cdot)$ *is an increasing function;*
(8) $\mathbb{O}(\aleph, \varpi, \varphi) < 1$*;*
(9) $\mathbb{O}(\aleph, \varpi, \varphi) = 0$ *for all* $\varphi > 0$*, if and only if* $\aleph = \varpi$*;*
(10) $\mathbb{O}(\aleph, \varpi, \varphi) = \mathbb{O}(\varpi, \aleph, \varphi)$*;*
(11) $\mathbb{O}(\aleph, \hat{s}, \varphi + \hat{w} + \zeta) \leq \mathbb{O}(\aleph, \varpi, \varphi) \circ \mathbb{O}(\hat{a}, \varpi, \hat{w}) \circ \mathbb{O}(\hat{a}, \hat{s}, \zeta)$*;*
(12) $\mathbb{O}(\aleph, \varpi, \cdot) : (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\varphi \to +\infty} \mathbb{O}(\aleph, \varpi, \varphi) = 0$*;*
(13) $\mathbb{O}(\aleph, \varpi, \cdot)$ *is a decreasing function;*
(14) *If* $\varphi \leq 0$*, then* $\mathbb{H}(\aleph, \varpi, \varphi) = 0$ *and* $\mathbb{O}(\aleph, \varpi, \varphi) = 1$*.*

Then, $(\Omega, \Xi, \mathbb{M}, *, \circ)$ is called an intuitionistic fuzzy bipolar metric space.

3. FIXED POINT THEOREMS ON NEUTROSOPHIC FUZZY BIPOLAR METRIC SPACE

Definition 3.1. *Let* $\Omega \neq \phi$*,* $\Xi \neq \phi$ *be two sets and* $*$ *be a CTN,* $\circ$ *be a CTCN, and* $\mathbb{M} = \{< a, \mathbb{H}(a), \mathbb{O}(a), \mathbb{S}(a) >: a \in \Omega \cap \Xi\}$ *be neutrosophic set on* $\Omega \times \Xi \times (0, +\infty)$*, said to be an neutrosophic fuzzy bipolar metric on* $\Omega \times \Xi$*, if for all* $\aleph, \hat{a} \in \Omega$*,* $\varpi, \hat{s} \in \Xi$ *and* $\varphi, \zeta, \hat{w} > 0$*, the following conditions are satisfied:*

(1) $\mathbb{H}(\aleph,\varpi,\varphi)+\mathbb{O}(\aleph,\varpi,\varphi)+\mathbb{S}(\aleph,\varpi,\varphi)\leq 3$;
(2) $\mathbb{H}(\aleph,\varpi,\varphi)>0$;
(3) $\mathbb{H}(\aleph,\varpi,\varphi)=1$ *for all* $\varphi>0$, *if and only if* $\aleph=\varpi$;
(4) $\mathbb{H}(\aleph,\varpi,\varphi)=\mathbb{H}(\varpi,\aleph,\varphi)$;
(5) $\mathbb{H}(\aleph,\hat{s},\varphi+\hat{w}+\zeta)\geq\mathbb{H}(\aleph,\varpi,\varphi)*\mathbb{H}(\hat{a},\varpi,\hat{w})*\mathbb{H}(\hat{a},\hat{s},\zeta)$;
(6) $\mathbb{H}(\aleph,\varpi,\cdot):(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\varphi\to+\infty}\mathbb{H}(\aleph,\varpi,\varphi)=1$;
(7) $\mathbb{H}(\aleph,\varpi,\cdot)$ *is an increasing function;*
(8) $\mathbb{O}(\aleph,\varpi,\varphi)<1$;
(9) $\mathbb{O}(\aleph,\varpi,\varphi)=0$ *for all* $\varphi>0$, *if and only if* $\aleph=\varpi$;
(10) $\mathbb{O}(\aleph,\varpi,\varphi)=\mathbb{O}(\varpi,\aleph,\varphi)$;
(11) $\mathbb{O}(\aleph,\hat{s},\varphi+\hat{w}+\zeta)\leq\mathbb{O}(\aleph,\varpi,\varphi)\circ\mathbb{O}(\hat{a},\varpi,\hat{w})\circ\mathbb{O}(\hat{a},\hat{s},\zeta)$;
(12) $\mathbb{O}(\aleph,\varpi,\cdot):(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\varphi\to+\infty}\mathbb{O}(\aleph,\varpi,\varphi)=0$;
(13) $\mathbb{O}(\aleph,\varpi,\cdot)$ *is a decreasing function;*
(14) $\mathbb{S}(\aleph,\varpi,\varphi)<1$;
(15) $\mathbb{S}(\aleph,\varpi,\varphi)=0$ *for all* $\varphi>0$, *if and only if* $\aleph=\varpi$;
(16) $\mathbb{S}(\aleph,\varpi,\varphi)=\mathbb{S}(\varpi,\aleph,\varphi)$;
(17) $\mathbb{S}(\aleph,\hat{s},\varphi+\hat{w}+\zeta)\leq\mathbb{S}(\aleph,\varpi,\varphi)\circ\mathbb{S}(\hat{a},\varpi,\hat{w})\circ\mathbb{S}(\hat{a},\hat{s},\zeta)$;
(18) $\mathbb{S}(\aleph,\varpi,\cdot):(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\varphi\to+\infty}\mathbb{S}(\aleph,\varpi,\varphi)=0$;
(19) $\mathbb{S}(\aleph,\varpi,\cdot)$ *is a decreasing function;*
(20) *If* $\varphi\leq 0$, *then* $\mathbb{H}(\aleph,\varpi,\varphi)=0$, $\mathbb{O}(\aleph,\varpi,\varphi)=1$, *and* $\mathbb{S}(\aleph,\varpi,\varphi)=1$.

Then, $(\Omega,\Xi,\mathbb{M},*,\circ)$ is called an neutrosophic fuzzy bipolar metric space.

Example 3.2. *Let* $\Omega=\{5,7,9,11\}$ *and* $\Xi=\{4,6,8,10\}$. *Define* $\mathbb{M}=\{<a,\mathbb{H}(a),\mathbb{O}(a),\mathbb{S}(a)>: a\in\Omega\cap\Xi\}$ *in short* $\mathbb{M}=(\mathbb{H},\mathbb{O},\mathbb{S})$ *are neutrosophic set from* $\Omega\times\Xi\times(0,+\infty)\to[0,1]$ *as*

$$\mathbb{H}(\aleph,\varpi,\varphi)=\begin{cases}1, & if\ \aleph=\varpi\\ \dfrac{\varphi}{\varphi+\max\{\aleph,\varpi\}}, & otherwise;\end{cases}$$

$$\mathbb{O}(\aleph,\varpi,\varphi)=\begin{cases}0, & if\ \aleph=\varpi\\ \dfrac{\max\{\aleph,\varpi\}}{\max\{\aleph,\varpi\}+\varphi}, & otherwise;\end{cases}$$

$$\mathbb{S}(\aleph,\varpi,\varphi)=\begin{cases}0, & if\ \aleph=\varpi\\ \dfrac{\max\{\aleph,\varpi\}}{\varphi}, & otherwise.\end{cases}$$

Then, $(\Omega,\Xi,\mathbb{M},*,\circ)$*is an neutrosophic fuzzy bipolar metric space with CTN,* $\nu*\mu=\nu\mu$ *and CTCN,* $\nu\circ\mu=\max\{\nu,\mu\}$.

Proof. Here, we prove properties 3.1(5), 3.1(11) and 3.1(17) of Definition 3.1, others are obvious. Let $\aleph=9$, $\varpi=6$, $\hat{a}=7$ and $\hat{s}=8$. Then,

$$\mathbb{H}(9,8,\varphi+\hat{w}+\zeta)=\frac{\varphi+\hat{w}+\zeta}{\varphi+\hat{w}+\zeta+\max\{9,8\}}=\frac{\varphi+\hat{w}+\zeta}{\varphi+\hat{w}+\zeta+9}.$$

On the other hand,

$$\mathbb{H}(9,6,\varphi)=\frac{\varphi}{\varphi+\max\{9,6\}}=\frac{\varphi}{\varphi+9},$$
$$\mathbb{H}(6,7,\hat{w})=\frac{\hat{w}}{\hat{w}+\max\{6,7\}}=\frac{\hat{w}}{\hat{w}+7}$$

and

$$\mathbb{H}(7,8,\zeta)=\frac{\zeta}{\zeta+\max\{7,8\}}=\frac{\zeta}{\zeta+8}.$$

Therefore,

$$\frac{\varphi+\hat{w}+\zeta}{\varphi+\hat{w}+\zeta+9}\geq\frac{\varphi}{\varphi+9}.\frac{\hat{w}}{\hat{w}+7}.\frac{\zeta}{\zeta+8}.$$

Hence,
$\mathbb{H}(\aleph,\hat{s},\varphi+\hat{w}+\zeta)\geq\mathbb{H}(\aleph,\varpi,\varphi)*\mathbb{H}(\hat{a},\varpi,\hat{w})*\mathbb{H}(\hat{a},\hat{s},\zeta)$, for all $\varphi,\hat{w},\zeta>0$.
Now,

$$\mathbb{O}(9,8,\varphi+\hat{w}+\zeta)=\frac{\max\{9,8\}}{\varphi+\hat{w}+\zeta+\max\{9,8\}}=\frac{9}{\varphi+\hat{w}+\zeta+9}.$$

We take,

$$\mathbb{O}(9,6,\varphi)=\frac{\max\{9,6\}}{\varphi+\max\{9,6\}}=\frac{9}{\varphi+9},$$
$$\mathbb{O}(6,7,\hat{w})=\frac{\max\{6,7\}}{\hat{w}+\max\{6,7\}}=\frac{7}{\hat{w}+7}$$

and

$$\mathbb{O}(7,8,\zeta)=\frac{\max\{7,8\}}{\zeta+\max\{7,8\}}=\frac{8}{\zeta+8}.$$

That is,

$$\frac{9}{\varphi+\hat{w}+\zeta+9}\leq\max\left\{\frac{9}{\varphi+9},\frac{7}{\hat{w}+7},\frac{8}{\zeta+8}\right\}.$$

Therefore, $\mathbb{O}(\aleph,\hat{s},\varphi+\hat{w}+\zeta)\leq\mathbb{O}(\aleph,\varpi,\varphi)\circ\mathbb{O}(\hat{a},\varpi,\hat{w})\circ\mathbb{O}(\hat{a},\hat{s},\zeta)$, for all $\varphi,\hat{w},\zeta>0$.

Now,

$$\mathbb{S}(9,8,\varphi+\hat{w}+\zeta)=\frac{\max\{9,8\}}{\varphi+\hat{w}+\zeta}=\frac{9}{\varphi+\hat{w}+\zeta}.$$

On the other hand,

$$\mathbb{S}(9,6,\varphi)=\frac{\max\{9,6\}}{\varphi}=\frac{9}{\varphi},$$
$$\mathbb{S}(6,7,\hat{w})=\frac{\max\{6,7\}}{\hat{w}}=\frac{7}{\hat{w}}$$

and

$$\mathbb{S}(7,8,\zeta)=\frac{\max\{7,8\}}{\zeta}=\frac{8}{\zeta}.$$

That is,

$$\frac{9}{\varphi+\hat{w}+\zeta}\leq\max\left\{\frac{9}{\varphi},\frac{7}{\hat{w}},\frac{8}{\zeta}\right\}.$$

Therefore, $\mathbb{S}(\aleph, \hat{s}, \varphi + \hat{w} + \zeta) \leq \mathbb{S}(\aleph, \varpi, \varphi) \circ \mathbb{S}(\hat{a}, \varpi, \hat{w}) \circ \mathbb{S}(\hat{a}, \hat{s}, \zeta)$, for all $\varphi, \hat{w}, \zeta > 0$.

Hence, $(\Omega, \Xi, \mathbb{M}, *, \circ)$ is an neutrosophic fuzzy bipolar metric space. □

Definition 3.3. *Let $\mathcal{P} : \Omega_1 \cup \Xi_1 \to \Omega_2 \cup \Xi_2$ be a mapping, where (Ω_1, Ξ_1) and (Ω_2, Ξ_2) are pairs of sets. If $\mathcal{P}(\Omega_1) \subseteq \Omega_2$ and $\mathcal{P}(\Xi_1) \subseteq \Xi_2$, then $\mathcal{P}$ is called a covariant map, and this is written as $\mathcal{P} : (\Omega_1, \Xi_1, \mathbb{M}_1, *, \circ) \rightrightarrows (\Omega_2, \Xi_2, \mathbb{M}_2, *, \circ)$, where $\mathbb{M}_1 = (\mathbb{H}_1, \mathbb{O}_1, \mathbb{S}_1)$ and $\mathbb{M}_2 = (\mathbb{H}_2, \mathbb{O}_2, \mathbb{S}_2)$.*

Example 3.4. *If $\Omega_1 \cup \Xi_1 = X = \{0, 1\}$, then $\mathcal{P}(X) = \{\{\}, \{0\}, \{1\}, X\}$. Suppose $\mathbb{H}(0) = \{\}$ and $\mathbb{H}(1) = X$. Then $\mathcal{P}(\mathbb{H})$ is the function which sends any subset U of X to its image $\mathbb{H}(U)$, which in this case means $\{\} \to \mathbb{H}(\{\}) = \{\}$, where $\to$ denotes the mapping under $\mathcal{P}(\mathbb{H})$, so this could also be written as $(\mathcal{P}(\mathbb{H}))(\{\}) = \{\}$. For the other values, $\{0\} \to \mathbb{H}(\{0\}) = \{\mathbb{H}(0)\} = \{\{\}\}$, $\{1\} \to \mathbb{H}(\{1\}) = \{\mathbb{H}(1)\} = \{X\}$,$\{0, 1\} \to \mathbb{H}(\{0, 1\}) = \{\mathbb{H}(0), \mathbb{H}(1)\} = \{\{\}, X\}$.*

Definition 3.5. *Let $(\Omega, \Xi, \mathbb{M}, *, \circ)$ be an neutrosophic fuzzy bipolar metric space.*

(1) Within the set $\Omega \cup \Xi$, a point $\aleph$ is categorized as a left point if it belongs to Ω, and right point if it belongs to Ξ, and a central point if it belongs to both.

(2) A sequence denoted as $\{\aleph_\alpha\}$ and belonging to the set Ω is termed a left sequence, while a sequence denoted as $\{\alpha_n\}$ and belonging to the set Ξ is referred to as a right sequence.

(3) A sequence denoted as $\{\aleph_\alpha\} \subset \Omega \cup \Xi$ is considered to converge to a point $\aleph$ if and only if $\{\aleph_\alpha\}$ is a left sequence, $\aleph$ is a right point and

$$\lim_{\alpha \to +\infty} \mathbb{H}(\aleph_\alpha, \aleph, \varphi) = 1,$$
$$\lim_{\alpha \to +\infty} \mathbb{O}(\aleph_\alpha, \aleph, \varphi) = 0,$$
$$\lim_{\alpha \to +\infty} \mathbb{S}(\aleph_\alpha, \aleph, \varphi) = 0,$$

for all $\varphi > 0$.

(4) A sequence $\{(\aleph_\alpha, \alpha_\alpha)\} \subset \Omega \times \Xi$ is called a bisequence. If both sequences $\{\aleph_\alpha\}$ and $\{\alpha_\alpha\}$ converge, then the bisequence $\{(\aleph_\alpha, \alpha_\alpha)\}$ is called convergent within the space $\Omega \times \Xi$.

(5) If $\{\aleph_\alpha\}$ and $\{\alpha_\alpha\}$ both converge to a common point $\alpha \in \Omega \cap \Xi$, then the bisequence $\{(\aleph_\alpha, \alpha_\alpha)\}$ is labeled as biconvergent. A sequence $\{(\aleph_\alpha, \alpha_\alpha)\}$ is a Cauchy bisequence if

$$\lim_{\alpha, \kappa \to +\infty} \mathbb{H}(\aleph_\alpha, \alpha_\kappa, \varphi) = 1,$$
$$\lim_{\alpha, \kappa \to +\infty} \mathbb{O}(\aleph_\alpha, \alpha_\kappa, \varphi) = 0,$$
$$\lim_{\alpha, \kappa \to +\infty} \mathbb{S}(\aleph_\alpha, \alpha_\kappa, \varphi) = 0,$$

for all $\varphi > 0$.

(6) An neutrosophic fuzzy bipolar metric space is said to be complete if every Cauchy bisequence is convergent.

Example 3.6. *Let $\Omega = (1, \infty)$ and $\Xi = [-1, 1]$. Define $\mathbb{H} : \Omega \times \Xi \to \mathbf{R}^+$ as $\mathbb{H}(\aleph, \varpi) = |\aleph^2 - \varpi^2|$. Then $(\Omega, \Xi, \mathbb{M}, *, \circ)$ is an neutrosophic fuzzy bipolar metric space.*
Note that the left sequence $(1 + \frac{1}{\alpha})$ converges to right points 1 and -1.

Lemma 3.7. *Let $\{\aleph_\alpha\}$ be a Cauchy sequence in the neutrosophic fuzzy bipolar metric space $(\Omega, \Xi, \mathbb{M}, *, \circ)$ such that $\aleph_\alpha \neq \aleph_\kappa$, whenever $\kappa, \alpha \in \mathbf{N}$ with $\alpha \neq \kappa$. Then the sequence $\{\aleph_\alpha\}$ can converge to, at most, one limit point.*

Proof. Assume that, $\aleph_\alpha \to \aleph \in \Xi$ and $\aleph_\alpha \to \varpi \in \Omega \cap \Xi$, for $\aleph \neq \varpi$.
Then,

$$\lim_{\alpha \to +\infty} \mathbb{H}(\aleph_\alpha, \aleph, \varphi) = 1, \lim_{\alpha \to +\infty} \mathbb{O}(\aleph_\alpha, \aleph, \varphi) = 0, \lim_{\alpha \to +\infty} \mathbb{S}(\aleph_\alpha, \aleph, \varphi) = 0$$

and

$$\lim_{\alpha \to +\infty} \mathbb{H}(\aleph_\alpha, \varpi, \varphi) = 1, \lim_{\alpha \to +\infty} \mathbb{O}(\aleph_\alpha, \varpi, \varphi) = 0, \lim_{\alpha \to +\infty} \mathbb{S}(\aleph_\alpha, \varpi, \varphi) = 0,$$

for all $\varphi > 0$.

Suppose,

$$\begin{aligned}
\mathbb{H}(\aleph, \varpi, \varphi) &\geq \mathbb{H}\left(\aleph, \aleph_\alpha, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_\alpha, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, \varpi, \frac{\varphi}{3}\right) \\
&\to 1 * 1 * 1, \ \textit{as } \alpha \to +\infty, \\
\mathbb{O}(\aleph, \varpi, \varphi) &\leq \mathbb{O}\left(\aleph, \aleph_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_\alpha, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, \varpi, \frac{\varphi}{3}\right) \\
&\to 0 \circ 0 \circ 0, \ \textit{as } \alpha \to +\infty, \\
\mathbb{S}(\aleph, \varpi, \varphi) &\leq \mathbb{S}\left(\aleph, \aleph_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_\alpha, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, \varpi, \frac{\varphi}{3}\right) \\
&\to 0 \circ 0 \circ 0, \ \textit{as } \alpha \to +\infty.
\end{aligned}$$

That is,

$$\mathbb{H}(\aleph, \varpi, \varphi) \geq 1 * 1 * 1 = 1, \mathbb{O}(\aleph, \varpi, \varphi) \leq 0 \circ 0 \circ 0 = 0, \mathbb{S}(\aleph, \varpi, \varphi) \leq 0 \circ 0 \circ 0 = 0.$$

Hence $\aleph = \varpi$. ☐

Lemma 3.8. *Let $(\Omega, \Xi, \mathbb{M}, *, \circ)$ be an neutrosophic fuzzy bipolar metric space. If for some $0 < \theta < 1$ and for any $\aleph, \varpi \in \Omega$, $\varphi > 0$,*

$$\mathbb{H}(\aleph, \varpi, \varphi) \geq \mathbb{H}(\aleph, \varpi, \frac{\varphi}{\theta}), \mathbb{O}(\aleph, \varpi, \varphi) \leq \mathbb{O}(\aleph, \varpi, \frac{\varphi}{\theta}), \mathbb{S}(\aleph, \varpi, \varphi) \leq \mathbb{S}(\aleph, \varpi, \frac{\varphi}{\theta}), \quad (1)$$

then $\aleph = \varpi$.

Proof. Equation (1) implies that,

$$\mathbb{H}(\aleph,\varpi,\varphi)\geq\mathbb{H}(\aleph,\varpi,\frac{\varphi}{\theta^{\alpha}}),\quad \mathbb{O}(\aleph,\varpi,\varphi)\leq\mathbb{O}(\aleph,\varpi,\frac{\varphi}{\theta^{\alpha}})\ \textit{and}$$

$$\mathbb{S}(\aleph,\varpi,\varphi)\leq\mathbb{S}(\aleph,\varpi,\frac{\varphi}{\theta^{\alpha}}),\alpha\in\mathbf{N},\varphi>0.$$

Now,

$$\mathbb{H}(\aleph,\varpi,\varphi)\geq\lim_{\alpha\to+\infty}\mathbb{H}(\aleph,\varpi,\frac{\varphi}{\theta^{\alpha}})=1,$$
$$\mathbb{O}(\aleph,\varpi,\varphi)\leq\lim_{\alpha\to+\infty}\mathbb{O}(\aleph,\varpi,\frac{\varphi}{\theta^{\alpha}})=0,$$
$$\mathbb{S}(\aleph,\varpi,\varphi)\leq\lim_{\alpha\to+\infty}\mathbb{S}(\aleph,\varpi,\frac{\varphi}{\theta^{\alpha}})=0.$$

Also, by the Definition 3.1, we get $\aleph=\varpi$. □

Theorem 3.9. *Suppose* $(\Omega,\Xi,\mathbb{M},*,\circ)$ *is a complete neutrosophic fuzzy bipolar metric space with* $0<\theta<1$. *Let* $\mathcal{P}:\Omega\cup\Xi\to\Omega\cup\Xi$ *be a mapping satisfying:*

(1) $\mathcal{P}(\Omega)\subseteq\Omega$ *and* $\mathcal{P}(\Xi)\subseteq\Xi$;
(2) $\mathbb{H}(\mathcal{P}\aleph,\mathcal{P}\varpi,\theta\varphi)\geq\mathbb{H}(\aleph,\varpi,\varphi),\mathbb{O}(\mathcal{P}\aleph,\mathcal{P}\varpi,\theta\varphi)\leq\mathbb{O}(\aleph,\varpi,\varphi)$, $\mathbb{S}(\mathcal{P}\aleph,\mathcal{P}\varpi,\theta\varphi)\leq\mathbb{S}(\aleph,\varpi,\varphi)$, *for all* $\aleph\in\Omega$, $\varpi\in\Xi$, *and* $\varphi>0$.

Then $\mathcal{P}$ *has a unique fixed point.*

Proof. Let $\aleph_0\in\Omega$ and $\varpi_0\in\Xi$ and assume that $\mathcal{P}(\aleph_\alpha)=\aleph_{\alpha+1}$ and $\mathcal{P}(\varpi_\alpha)=\varpi_{\alpha+1}$ for all $\alpha\in\mathbf{N}\cup\{0\}$.
We get, $(\aleph_\alpha,\varpi_\alpha)$ as a bisequence on the neutrosophic fuzzy bipolar metric space $(\Omega,\Xi,\mathbb{M},*,\circ)$. Now, we have

$$\mathbb{H}(\aleph_1,\varpi_1,\varphi)=\mathbb{H}(\mathcal{P}\aleph_0,\mathcal{P}\varpi_0,\varphi)\geq\mathbb{H}\left(\aleph_0,\varpi_0,\frac{\varphi}{\theta}\right),$$
$$\mathbb{O}(\aleph_1,\varpi_1,\varphi)=\mathbb{O}(\mathcal{P}\aleph_0,\mathcal{P}\varpi_0,\varphi)\leq\mathbb{O}\left(\aleph_0,\varpi_0,\frac{\varphi}{\theta}\right)$$

and

$$\mathbb{S}(\aleph_1,\varpi_1,\varphi)=\mathbb{S}(\mathcal{P}\aleph_0,\mathcal{P}\varpi_0,\varphi)\leq\mathbb{S}\left(\aleph_0,\varpi_0,\frac{\varphi}{\theta}\right),$$

for all $\varphi>0$ and $\alpha\in\mathbf{N}$. By simple induction, we get

$$\mathbb{H}(\aleph_\alpha,\varpi_\alpha,\varphi)=\mathbb{H}(\mathcal{P}\aleph_{\alpha-1},\mathcal{P}\varpi_{\alpha-1},\varphi)\geq\mathbb{H}\left(\aleph_{\alpha-1},\varpi_{\alpha-1},\frac{\varphi}{\theta}\right)$$
$$\geq\mathbb{H}\left(\aleph_{\alpha-2},\varpi_{\alpha-2},\frac{\varphi}{\theta^2}\right)\geq\mathbb{H}\left(\aleph_{\alpha-3},\varpi_{\alpha-3},\frac{\varphi}{\theta^3}\right)\geq\ldots\geq\mathbb{H}\left(\aleph_0,\varpi_0,\frac{\varphi}{\theta^\alpha}\right),$$
$$\mathbb{O}(\aleph_\alpha,\varpi_\alpha,\varphi)=\mathbb{O}(\mathcal{P}\aleph_{\alpha-1},\mathcal{P}\varpi_{\alpha-1},\varphi)\leq\mathbb{O}\left(\aleph_{\alpha-1},\varpi_{\alpha-1},\frac{\varphi}{\theta}\right)$$
$$\leq\mathbb{O}\left(\aleph_{\alpha-2},\varpi_{\alpha-2},\frac{\varphi}{\theta^2}\right)\leq\mathbb{O}\left(\aleph_{\alpha-3},\varpi_{\alpha-3},\frac{\varphi}{\theta^3}\right)\leq\ldots\leq\mathbb{O}\left(\aleph_0,\varpi_0,\frac{\varphi}{\theta^\alpha}\right)$$

and

$$\mathbb{S}(\aleph_\alpha,\varpi_\alpha,\varphi)=\mathbb{S}(\mathcal{P}\aleph_{\alpha-1},\mathcal{P}\varpi_{\alpha-1},\varphi)\leq\mathbb{S}\left(\aleph_{\alpha-1},\varpi_{\alpha-1},\frac{\varphi}{\theta}\right)$$

$$\leq \mathbb{S}\left(\aleph_{\alpha-2}, \varpi_{\alpha-2}, \frac{\varphi}{\theta^2}\right) \leq \mathbb{S}\left(\aleph_{\alpha-3}, \varpi_{\alpha-3}, \frac{\varphi}{\theta^3}\right) \leq \ldots \leq \mathbb{S}\left(\aleph_0, \varpi_0, \frac{\varphi}{\theta^\alpha}\right).$$

We obtain,

$$\mathbb{H}(\aleph_\alpha, \varpi_\alpha, \varphi) \geq \mathbb{H}(\aleph_0, \varpi_0, \frac{\varphi}{\theta^\alpha}), \quad \mathbb{O}(\aleph_\alpha, \varpi_\alpha, \varphi) \leq \mathbb{O}(\aleph_0, \varpi_0, \frac{\varphi}{\theta^\alpha}),$$

$$\mathbb{S}(\aleph_\alpha, \varpi_\alpha, \varphi) \leq \mathbb{S}(\aleph_0, \varpi_0, \frac{\varphi}{\theta^\alpha})$$

and

$$\mathbb{H}(\aleph_{\alpha+1}, \varpi_\alpha, \varphi) \geq \mathbb{H}(\aleph_1, \varpi_0, \frac{\varphi}{\theta^\alpha}), \quad \mathbb{O}(\aleph_{\alpha+1}, \varpi_\alpha, \varphi) \leq \mathbb{O}(\aleph_1, \varpi_0, \frac{\varphi}{\theta^\alpha}),$$

$$\mathbb{S}(\aleph_{\alpha+1}, \varpi_\alpha, \varphi) \leq \mathbb{S}(\aleph_1, \varpi_0, \frac{\varphi}{\theta^\alpha}).$$

Letting, $\alpha < \kappa$, for $\alpha, \kappa \in \mathbf{N}$. Then,

$$\begin{aligned}\mathbb{H}(\aleph_\alpha, \varpi_\kappa, \varphi) &\geq \mathbb{H}\left(\aleph_\alpha, \varpi_\alpha, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, \varpi_\alpha, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, \varpi_\kappa, \frac{\varphi}{3}\right) * \ldots \\ &\geq \mathbb{H}\left(\aleph_\alpha, \varpi_\alpha, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, \varpi_\alpha, \frac{\varphi}{3}\right) * \ldots * \\ &\quad \mathbb{H}\left(\aleph_{\kappa-1}, \varpi_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}\right) * \mathbb{H}\left(\aleph_\kappa, \varpi_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}\right) * \mathbb{H}\left(\aleph_\kappa, \varpi_\kappa, \frac{\varphi}{3^{\kappa-1}}\right), \\ \mathbb{O}(\aleph_\alpha, \varpi_\kappa, \varphi) &\leq \mathbb{O}\left(\aleph_\alpha, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, \varpi_\kappa, \frac{\varphi}{3}\right) \circ \ldots \\ &\leq \mathbb{O}\left(\aleph_\alpha, \varpi_\alpha, \frac{\varphi}{3}\right) * \mathbb{O}\left(\aleph_{\alpha+1}, \varpi_\alpha, \frac{\varphi}{3}\right) \ldots \circ \\ &\quad \mathbb{O}\left(\aleph_{\kappa-1}, \varpi_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}\right) \circ \mathbb{O}\left(\aleph_\kappa, \varpi_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}\right) \circ \mathbb{O}\left(\aleph_\kappa, \varpi_\kappa, \frac{\varphi}{3^{\kappa-1}}\right)\end{aligned}$$

and

$$\begin{aligned}\mathbb{S}(\aleph_\alpha, \varpi_\kappa, \varphi) &\leq \mathbb{S}\left(\aleph_\alpha, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, \varpi_\kappa, \frac{\varphi}{3}\right) \circ \ldots \\ &\leq \mathbb{S}\left(\aleph_\alpha, \varpi_\alpha, \frac{\varphi}{3}\right) * \mathbb{S}\left(\aleph_{\alpha+1}, \varpi_\alpha, \frac{\varphi}{3}\right) \ldots \circ \\ &\quad \mathbb{S}\left(\aleph_{\kappa-1}, \varpi_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}\right) \circ \mathbb{S}\left(\aleph_\kappa, \varpi_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}\right) \circ \mathbb{S}\left(\aleph_\kappa, \varpi_\kappa, \frac{\varphi}{3^{\kappa-1}}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{H}(\aleph_\alpha, \varpi_\kappa, \varphi) \geq{} & \mathbb{H}\left(\aleph_\alpha, \varpi_\alpha, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, \varpi_\alpha, \frac{\varphi}{3}\right) * \ldots * \mathbb{H}\left(\aleph_{\kappa-1}, \varpi_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}\right) * \\ & \mathbb{H}\left(\aleph_\kappa, \varpi_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}\right) * \mathbb{H}\left(\aleph_\kappa, \varpi_\kappa, \frac{\varphi}{3^{\kappa-1}}\right) \geq \mathbb{H}\left(\aleph_0, \varpi_0, \frac{\varphi}{3\theta^\alpha}\right) * \\ & \mathbb{H}\left(\aleph_1, \varpi_0, \frac{\varphi}{3\theta^\alpha}\right) * \cdots * \mathbb{H}\left(\aleph_0, \varpi_0, \frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}\right) * \mathbb{H}\left(\aleph_1, \varpi_0, \frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}\right) * \\ & \mathbb{H}\left(\aleph_0, \varpi_0, \frac{\varphi}{3^{\kappa-1}\theta^\kappa}\right), \\ \mathbb{O}(\aleph_\alpha, \varpi_\kappa, \varphi) \leq{} & \mathbb{O}\left(\aleph_\alpha, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \ldots \circ \mathbb{O}(\aleph_{\kappa-1}, \varpi_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}) \circ \\ & \mathbb{O}\left(\aleph_\kappa, \varpi_{\kappa-1}, \frac{\varphi}{3^{\kappa-1}}\right) * \mathbb{O}\left(\aleph_\kappa, \varpi_\kappa, \frac{\varphi}{3^{\kappa-1}}\right) \leq \mathbb{O}\left(\aleph_0, \varpi_0, \frac{\varphi}{3\theta^\alpha}\right) \circ\end{aligned}$$

$$\mathbb{O}\left(\aleph_1,\varpi_0,\frac{\varphi}{3\theta^\alpha}\right)\circ\cdots\circ\mathbb{O}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}\right)\circ\mathbb{O}\left(\aleph_1,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}\right)\circ \mathbb{O}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa}}\right)$$

and

$$\begin{aligned}\mathbb{S}(\aleph_\alpha,\varpi_\kappa,\varphi)&\leq\mathbb{S}\left(\aleph_\alpha,\varpi_\alpha,\frac{\varphi}{3}\right)\circ\mathbb{S}\left(\aleph_{\alpha+1},\varpi_\alpha,\frac{\varphi}{3}\right)\circ\ldots\circ\mathbb{S}\left(\aleph_{\kappa-1},\varpi_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}\right)\circ\\ &\mathbb{S}\left(\aleph_\kappa,\varpi_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}\right)*\mathbb{S}\left(\aleph_\kappa,\varpi_\kappa,\frac{\varphi}{3^{\kappa-1}}\right)\\ &\leq\mathbb{S}\left(\aleph_0,\varpi_0,\frac{\varphi}{3\theta^\alpha}\right)\circ\mathbb{S}\left(\aleph_1,\varpi_0,\frac{\varphi}{3\theta^\alpha}\right)\circ\cdots\circ\mathbb{S}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}\right)\circ\\ &\mathbb{S}\left(\aleph_1,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}\right)\circ\mathbb{S}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa}}\right),\end{aligned}$$

implies that,

$$\begin{aligned}\mathbb{H}(\aleph_\alpha,\varpi_\kappa,\varphi)&\geq\mathbb{H}\left(\aleph_0,\varpi_0,\frac{\varphi}{3\theta^\alpha}\right)*\mathbb{H}\left(\aleph_1,\varpi_0,\frac{\varphi}{3\theta^\alpha}\right)*\cdots*\mathbb{H}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}\right)*\\ &\mathbb{H}\left(\aleph_1,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}\right)*\mathbb{H}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa}}\right),\\ \mathbb{O}(\aleph_\alpha,\varpi_\kappa,\varphi)&\leq\mathbb{O}\left(\aleph_0,\varpi_0,\frac{\varphi}{3\theta^\alpha}\right)\circ\mathbb{O}\left(\aleph_1,\varpi_0,\frac{\varphi}{3\theta^\alpha}\right)\circ\cdots\circ\mathbb{O}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}\right)\circ\\ &\mathbb{O}\left(\aleph_1,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}\right)\circ\mathbb{O}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa}}\right)\end{aligned}$$

and

$$\begin{aligned}\mathbb{S}(\aleph_\alpha,\varpi_\kappa,\varphi)&\leq\mathbb{S}\left(\aleph_0,\varpi_0,\frac{\varphi}{3\theta^\alpha}\right)\circ\mathbb{S}\left(\aleph_1,\varpi_0,\frac{\varphi}{3\theta^\alpha}\right)\circ\cdots\circ\mathbb{S}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}\right)\circ\\ &\mathbb{S}\left(\aleph_1,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa-1}}\right)\circ\mathbb{S}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}\theta^{\kappa}}\right).\end{aligned}$$

As $\alpha,\kappa\to+\infty$, we deduce

$$\begin{aligned}\lim_{\alpha,\kappa\to+\infty}\mathbb{H}(\aleph_\alpha,\varpi_\kappa,\varphi)&=1*1*\cdots*1=1,\\ \lim_{\alpha,\kappa\to+\infty}\mathbb{O}(\aleph_\alpha,\varpi_\kappa,\varphi)&=0\circ0\circ\cdots\circ0=0,\\ \textit{and}\ \lim_{\alpha,\kappa\to+\infty}\mathbb{S}(\aleph_\alpha,\varpi_\kappa,\varphi)&=0\circ0\circ\cdots\circ0=0.\end{aligned}$$

Which implies that the bisequence $(\aleph_\alpha,\varpi_\alpha)$ is a Cauchy bisequence. Since $(\Omega,\Xi,\mathbb{M},*,\circ)$ is a complete neutrosophic fuzzy bipolar metric space, we have that $\{\aleph_\alpha\}\to\aleph$ and $\{\varpi_\alpha\}\to\aleph$, where $\aleph\in\Omega\cap\Xi$. Using Lemmas 3.7 and 3.8, we get

$$\begin{aligned}\mathbb{H}(\aleph,\mathcal{P}\aleph,\varphi)&\geq\mathbb{H}\left(\aleph,\aleph_{\alpha+1},\frac{\varphi}{3}\right)*\mathbb{H}\left(\aleph_{\alpha+1},\aleph_{\alpha+1},\frac{\varphi}{3}\right)*\mathbb{H}\left(\aleph_{\alpha+1},\mathcal{P}\aleph,\frac{\varphi}{3}\right)\\ &=\mathbb{H}\left(\aleph,\aleph_{\alpha+1},\frac{\varphi}{3}\right)*\mathbb{H}\left(\mathcal{P}\aleph_\alpha,\mathcal{P}\aleph_\alpha,\frac{\varphi}{3}\right)*\mathbb{H}\left(\mathcal{P}\aleph_\alpha,\mathcal{P}\aleph,\frac{\varphi}{3}\right)\\ &\to1*1*1=1\quad\text{as}\quad\alpha\to+\infty,\\ \mathbb{O}(\aleph,\mathcal{P}\aleph,\varphi)&\leq\mathbb{O}\left(\aleph,\aleph_{\alpha+1},\frac{\varphi}{3}\right)\circ\mathbb{O}\left(\aleph_{\alpha+1},\aleph_{\alpha+1},\frac{\varphi}{3}\right)\circ\mathbb{O}\left(\aleph_{\alpha+1},\mathcal{P}\aleph,\frac{\varphi}{3}\right)\end{aligned}$$

$$= \mathbb{O}\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\mathcal{P}\aleph_{\alpha+1}, \mathcal{P}\aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\mathcal{P}\aleph_{\alpha+1}, \mathcal{P}\aleph, \frac{\varphi}{3}\right)$$
$$\to 0 \circ 0 \circ 0 = 0 \quad \text{as} \quad \alpha \to +\infty \quad \text{and}$$
$$\mathbb{S}(\aleph, \mathcal{P}\aleph, \varphi) \le \mathbb{S}\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, \mathcal{P}\aleph, \frac{\varphi}{3}\right)$$
$$= \mathbb{S}\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\mathcal{P}\aleph_{\alpha+1}, \mathcal{P}\aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\mathcal{P}\aleph_{\alpha+1}, \mathcal{P}\aleph, \frac{\varphi}{3}\right)$$
$$\to 0 \circ 0 \circ 0 = 0 \quad \text{as} \quad \alpha \to +\infty.$$

Hence, $\mathcal{P}\aleph = \aleph$.

Now, we examine the uniqueness. Let $\mathcal{P}\omega = \omega$, for some $\omega \in \Omega \cap \Xi$, then

$$1 \ge \mathbb{H}(\omega, \aleph, \varphi) = \mathbb{H}(\mathcal{P}\omega, \mathcal{P}\aleph, \varphi) \ge \mathbb{H}\left(\omega, \aleph, \frac{\varphi}{\theta}\right) = \mathbb{H}\left(\mathcal{P}\omega, \mathcal{P}\aleph, \frac{\varphi}{\theta}\right)$$
$$\ge \mathbb{H}\left(\omega, \aleph, \frac{\varphi}{\theta^2}\right) \ge \cdots \ge \mathbb{H}\left(\omega, \aleph, \frac{\varphi}{\theta^\alpha}\right) \to 1 \quad \text{as} \quad \alpha \to +\infty \textit{ and}$$
$$0 \le \mathbb{O}(\omega, \aleph, \varphi) = \mathbb{O}(\mathcal{P}\omega, \mathcal{P}\aleph, \varphi) \le \mathbb{O}\left(\omega, \aleph, \frac{\varphi}{\theta}\right) = \mathbb{O}\left(\mathcal{P}\omega, \mathcal{P}\aleph, \frac{\varphi}{\theta}\right)$$
$$\le \mathbb{O}\left(\omega, \aleph, \frac{\varphi}{\theta^2}\right) \le \cdots \le \mathbb{O}\left(\omega, \aleph, \frac{\varphi}{\theta^\alpha}\right) \to 0 \quad \text{as} \quad \alpha \to +\infty,$$
$$0 \le \mathbb{S}(\omega, \aleph, \varphi) = \mathbb{S}(\mathcal{P}\omega, \mathcal{P}\aleph, \varphi) \le \mathbb{S}\left(\omega, \aleph, \frac{\varphi}{\theta}\right) = \mathbb{S}\left(\mathcal{P}\omega, \mathcal{P}\aleph, \frac{\varphi}{\theta}\right)$$
$$\le \mathbb{S}\left(\omega, \aleph, \frac{\varphi}{\theta^2}\right) \le \cdots \le \mathbb{S}\left(\omega, \aleph, \frac{\varphi}{\theta^\alpha}\right) \to 0 \quad \text{as} \quad \alpha \to +\infty,$$

by using Lemmas 3.7 and 3.8, $\aleph = \omega$. □

Theorem 3.10. *Let $(\Omega, \Xi, \mathbb{M}_b, *, \circ)$ be a complete neutrosophic fuzzy bipolar metric space such that*

$$\lim_{\varphi \to \infty} \mathbb{H}(\aleph, \varpi, \varphi) = 1, \lim_{\varphi \to \infty} \mathbb{O}(\aleph, \varpi, \varphi) = 0 \textit{ and } \lim_{\varphi \to \infty} \mathbb{S}(\aleph, \varpi, \varphi) = 0,$$

for all $\aleph \in \Omega, \varpi \in \Xi$.

Let $\mathcal{P}, \mathcal{Q} : \Xi \cup \Omega \to \Xi \cup \Omega$ be two mappings satisfying

(1) $\mathcal{P}(\Xi) \subseteq \Xi$, $\mathcal{Q}(\Xi) \subseteq \Xi$, and $\mathcal{P}(\Omega) \subseteq \Omega$, $\mathcal{Q}(\Omega) \subseteq \Omega$.
(2) $\mathbb{H}(\mathcal{P}(\aleph), \mathcal{Q}(\varpi), \theta\varphi) \ge \mathbb{H}(\aleph, \varpi, \varphi)$, $\mathbb{O}(\mathcal{P}(\aleph), \mathcal{Q}(\varpi), \theta\varphi) \le \mathbb{O}(\aleph, \varpi, \varphi)$ and $\mathbb{S}(\mathcal{P}(\aleph), \mathcal{Q}(\varpi), \theta\varphi) \le \mathbb{S}(\aleph, \varpi, \varphi)$ for all $\aleph \in \Xi$, $\varpi \in \Omega$ and $\varphi > 0$, where $0 < \theta < 1$

Then, $\mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point.

Proof. Fix $\aleph_0 \in \Xi$ and $\varpi_0 \in \Omega$ and assume that $\mathcal{P}(\aleph_{2\alpha}) = \aleph_{2\alpha+1}$, $\mathcal{Q}(\aleph_{2\alpha+1}) = \aleph_{2\alpha+2}$, $\mathcal{P}(\varpi_{2\alpha}) = \varpi_{2\alpha+1}$ and $\mathcal{Q}(\varpi_{2\alpha+1}) = \varpi_{2\alpha+2}$ for all $\alpha \in \mathbf{N} \cup \{0\}$. Then, we get $(\aleph_\alpha, \varpi_\alpha)$ as a bisequence on the neutrosophic fuzzy bipolar metric space $(\Xi, \Omega, \mathbb{M}, *, \circ)$. Now, we have

$$\mathbb{H}(\aleph_1, \varpi_1, \varphi) = \mathbb{H}(\mathcal{P}(\aleph_0), \mathcal{Q}(\varpi_0), \varphi) \ge \mathbb{H}\left(\aleph_0, \varpi_0, \frac{\varphi}{\theta}\right),$$
$$\mathbb{O}(\aleph_1, \varpi_1, \varphi) = \mathbb{O}(\mathcal{P}(\aleph_0), \mathcal{Q}(\varpi_0), \varphi) \le \mathbb{O}\left(\aleph_0, \varpi_0, \frac{\varphi}{\theta}\right),$$

$$\mathbb{S}(\aleph_1, \varpi_1, \varphi) = \mathbb{S}(\mathcal{P}(\aleph_0), \mathcal{Q}(\varpi_0), \varphi) \leq \mathbb{S}\left(\aleph_0, \varpi_0, \frac{\varphi}{\theta}\right),$$

for all $\varphi > 0$ and $\alpha \in \mathbf{N}$. By induction, we obtain

$$\begin{aligned}
\mathbb{H}(\aleph_{2\alpha+1}, \varpi_{2\alpha+1}, \varphi) &= \mathbb{H}(\mathcal{P}(\aleph_{2\alpha}), \mathcal{Q}(\varpi_{2\alpha}), \varphi) \geq \mathbb{H}\left(\aleph_0, \varpi_0, \frac{\varphi}{\theta^{2\alpha+1}}\right),\\
\mathbb{H}(\aleph_{2\alpha+1}, \varpi_{2\alpha+2}, \varphi) &= \mathbb{H}(\mathcal{P}(\aleph_{2\alpha}), \mathcal{Q}(\varpi_{2\alpha+1}), \varphi) \geq \mathbb{H}\left(\aleph_0, \varpi_1, \frac{\varphi}{\theta^{2\alpha+1}}\right),\\
\mathbb{O}(\aleph_{2\alpha+1}, \varpi_{2\alpha+1}, \varphi) &= \mathbb{O}(\mathcal{P}(\aleph_{2\alpha}), \mathcal{Q}(\varpi_{2\alpha}), \varphi) \leq \mathbb{O}\left(\aleph_0, \varpi_0, \frac{\varphi}{\theta^{2\alpha+1}}\right),\\
\mathbb{O}(\aleph_{2\alpha+1}, \varpi_{2\alpha+2}, \varphi) &= \mathbb{O}(\mathcal{P}(\aleph_{2\alpha}), \mathcal{Q}(\varpi_{2\alpha+1}), \varphi) \leq \mathbb{O}\left(\aleph_0, \varpi_1, \frac{\varphi}{\theta^{2\alpha+1}}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{S}(\aleph_{2\alpha+1}, \varpi_{2\alpha+1}, \varphi) &= \mathbb{S}(\mathcal{P}(\aleph_{2\alpha}), \mathcal{Q}\left(\varpi_{2\alpha}), \varphi\right) \leq \mathbb{S}(\aleph_0, \varpi_0, \frac{\varphi}{\theta^{2\alpha+1}}\right),\\
\mathbb{S}(\aleph_{2\alpha+1}, \varpi_{2\alpha+2}, \varphi) &= \mathbb{S}(\mathcal{P}(\aleph_{2\alpha}), \mathcal{Q}(\varpi_{2\alpha+1}), \varphi) \leq \mathbb{S}\left(\aleph_0, \varpi_1, \frac{\varphi}{\theta^{2\alpha+1}}\right),
\end{aligned}$$

for all $\varphi > 0$ and $\alpha \in \mathbf{N}$.

Letting $\alpha < \beta$, for $\alpha, \beta \in \mathbf{N}$. Then, from the definition of the neutrosophic fuzzy bipolar metric space, we get

$$\begin{aligned}
\mathbb{H}(\aleph_\alpha, \varpi_\beta, \varphi) &\geq \mathbb{H}\left(\aleph_\alpha, \varpi_\alpha, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, \varpi_\alpha, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, \varpi_\beta, \frac{\varphi}{3}\right) \ldots\\
&\geq \mathbb{H}\left(\aleph_\alpha, \varpi_\alpha, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, \varpi_\alpha, \frac{\varphi}{3}\right) * \ldots\\
&\quad \mathbb{H}\left(\aleph_{\beta-1}, \varpi_{\beta-1}, \frac{\varphi}{3^{\beta-1}}\right) * \mathbb{H}\left(\aleph_\beta, \varpi_{\beta-1}, \frac{\varphi}{3^{\beta-1}}\right) * \mathbb{H}\left(\aleph_{\beta-1}, \varpi_\beta, \frac{\varphi}{3^\beta}\right),\\
\mathbb{O}(\aleph_\alpha, \varpi_\beta, \varphi) &\leq \mathbb{O}\left(\aleph_\alpha, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, \varpi_\beta, \frac{\varphi}{3}\right) \ldots\\
&\leq \mathbb{O}\left(\aleph_\alpha, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \ldots\\
&\quad \mathbb{O}\left(\aleph_{\beta-1}, \varpi_{\beta-1}, \frac{\varphi}{3^{\beta-1}}\right) \circ \mathbb{O}\left(\aleph_\beta, \varpi_{\beta-1}, \frac{\varphi}{3^{\beta-1}}\right) \circ \mathbb{O}\left(\aleph_{\beta-1}, \varpi_\beta, \frac{\varphi}{3^\beta}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{S}(\aleph_\alpha, \varpi_\beta, \varphi) &\leq \mathbb{S}\left(\aleph_\alpha, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, \varpi_\beta, \frac{\varphi}{3}\right) \ldots\\
&\leq \mathbb{S}\left(\aleph_\alpha, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \ldots\\
&\quad \mathbb{S}\left(\aleph_{\beta-1}, \varpi_{\beta-1}, \frac{\varphi}{3^{\beta-1}}\right) \circ \mathbb{S}\left(\aleph_\beta, \varpi_{\beta-1}, \frac{\varphi}{3^{\beta-1}}\right) \circ \mathbb{S}\left(\aleph_{\beta-1}, \varpi_\beta, \frac{\varphi}{3^\beta}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{H}(\aleph_\alpha, \varpi_\beta, \varphi) &\geq \mathbb{H}\left(\aleph_0, \varpi_0, \frac{\varphi}{3\theta^\alpha}\right) * \mathbb{H}\left(\aleph_0, \varpi_1, \frac{\varphi}{3\theta^{\alpha+1}}\right) \ldots \mathbb{H}\left(\aleph_0, \varpi_0, \frac{\varphi}{3^{\beta-1}\theta^{\beta-1}}\right),\\
\mathbb{O}(\aleph_\alpha, \varpi_\beta, \varphi) &\leq \mathbb{O}\left(\aleph_0, \varpi_0, \frac{\varphi}{3\theta^\alpha}\right) * \mathbb{O}\left(\aleph_0, \varpi_1, \frac{\varphi}{3\theta^{\alpha+1}}\right) \ldots \mathbb{O}\left(\aleph_0, \varpi_0, \frac{\varphi}{3^{\beta-1}\theta^{\beta-1}}\right)
\end{aligned}$$

and

$$\mathbb{S}(\aleph_\alpha, \varpi_\beta, \varphi) \leq \mathbb{S}\left(\aleph_0, \varpi_0, \frac{\varphi}{3\theta^\alpha}\right) * \mathbb{S}\left(\aleph_0, \varpi_1, \frac{\varphi}{3\theta^{\alpha+1}}\right) \ldots \mathbb{S}\left(\aleph_0, \varpi_0, \frac{\varphi}{3^{\beta-1}\theta^{\beta-1}}\right).$$

As $\alpha, \beta \to \infty$, we get

$$\mathbb{H}(\aleph_\alpha, \varpi_\beta, \varphi) \geq 1, \mathbb{O}(\aleph_\alpha, \varpi_\beta, \varphi) \leq 0, \mathbb{S}(\aleph_\alpha, \varpi_\beta, \varphi) \leq 0 \quad \text{for all} \quad \varphi > 0.$$

Thus, the bisequence $(\aleph_\alpha, \varpi_\alpha)$ is a Cauchy bisequence. Since $(\Xi, \Omega, \mathbb{M}, *, \circ)$ is a complete neutrosophic fuzzy bipolar metric space. By Lemma 3.7, the bisequence $(\aleph_\alpha, \varpi_\alpha)$ is a biconvergent sequence. Therefore, $\aleph_\alpha \to u$ and $\varpi_\alpha \to u$, where $u \in \Xi \cap \Omega$. By Lemma 3.8, both sequences $\aleph_\alpha$ and ϖ_α have a unique limit. From the triangular property of neutrosophic bipolar metric spaces, we have

$$\begin{aligned}
\mathbb{H}(\mathcal{P}u, u, \varphi) &\geq \mathbb{H}\left(\mathcal{P}(u), \varpi_{\alpha+1}, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, \varpi_{\alpha+1}, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, u, \frac{\varphi}{3}\right) \\
&= \mathbb{H}\left(\mathcal{P}u, \mathcal{Q}(\varpi_\alpha), \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, \varpi_{\alpha+1}, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, u, \frac{\varphi}{3}\right) \\
&\geq \mathbb{H}\left(u, \varpi_\alpha, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, \varpi_{\alpha+1}, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, u, \frac{\varphi}{3}\right), \\
\mathbb{O}(\mathcal{P}u, u, \varphi) &\leq \mathbb{O}\left(\mathcal{P}(u), \varpi_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, \varpi_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, u, \frac{\varphi}{3}\right) \\
&= \mathbb{O}\left(\mathcal{P}u, \mathcal{Q}(\varpi_\alpha), \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, \varpi_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, u, \frac{\varphi}{3}\right) \\
&\leq \mathbb{O}\left(u, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, \varpi_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, u, \frac{\varphi}{3}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{S}(\mathcal{P}u, u, \varphi) &\leq \mathbb{S}\left(\mathcal{P}(u), \varpi_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, \varpi_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, u, \frac{\varphi}{3}\right) \\
&= \mathbb{S}\left(\mathcal{P}u, \mathcal{Q}(\varpi_\alpha), \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, \varpi_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, u, \frac{\varphi}{3}\right) \\
&\leq \mathbb{S}\left(u, \varpi_\alpha, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, \varpi_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, u, \frac{\varphi}{3}\right),
\end{aligned}$$

for all $\alpha \in \mathbf{N}$ and $\varphi > 0$ and as $\alpha \to \infty$, we get

$\mathbb{H}(\mathcal{P}u, u, \varphi) \to 1 * 1 * 1 = 1, \mathbb{O}(\mathcal{P}u, u, \varphi) \to 0 \circ 0 \circ 0 = 0, \mathbb{S}(\mathcal{P}u, u, \varphi) \to 0 \circ 0 \circ 0 = 0.$

From Definition 3.1 condition (2), $\mathcal{P}(u) = u$. Again,

$$\begin{aligned}
\mathbb{H}(u, \mathcal{Q}(u), \varphi) = \mathbb{H}(\mathcal{P}(u), \mathcal{Q}(u), \varphi) \geq \mathbb{H}\left(u, u, \frac{\varphi}{\theta}\right) = 1, \\
\mathbb{O}(u, \mathcal{Q}(u), \varphi) = \mathbb{O}(\mathcal{P}(u), \mathcal{Q}(u), \varphi) \leq \mathbb{O}\left(u, u, \frac{\varphi}{\theta}\right) = 0, \\
\mathbb{S}(u, \mathcal{Q}(u), \varphi) = \mathbb{S}(\mathcal{P}(u), \mathcal{Q}(u), \varphi) \leq \mathbb{S}\left(u, u, \frac{\varphi}{\theta}\right) = 0.
\end{aligned}$$

Therefore, $\mathcal{Q}(u) = u$. Hence, u is a common fixed point of $\mathcal{P}$ and $\mathcal{Q}$.
Let $v \in \Xi \cap \Omega$ be another fixed point of $\mathcal{P}$ and $\mathcal{Q}$. Then,

$$\begin{aligned}
\mathbb{H}(u, v, \varphi) = \mathbb{H}(\mathcal{P}(u), \mathcal{Q}(v), \varphi) \geq \mathbb{H}\left(u, v, \frac{\varphi}{\theta}\right), \\
\mathbb{O}(u, v, \varphi) = \mathbb{O}\Big(\mathcal{P}(u), \mathcal{Q}(v), \varphi) \leq \mathbb{O}(u, v, \frac{\varphi}{\theta}\Big),
\end{aligned}$$

$$\mathbb{S}(u,v,\varphi)=\mathbb{S}\left(\mathcal{P}(u),\mathcal{Q}(v),\varphi\right)\leq \mathbb{S}(u,v,\frac{\varphi}{\theta}\right),$$

for $0<\theta<1$ and for all $\varphi>0$. By Lemma 3.8, we have $u=v$. □

Example 3.11. *Let $\Xi=[0,2]$ and $\Omega=\{0\}\cup\mathbf{N}-\{1,2\}$. Define,*

$$\mathbb{H}=\frac{\varphi}{(\varphi+|\aleph-\varpi|)},\quad \mathbb{O}=\frac{|\aleph-\varpi|}{(\varphi+|\aleph-\varpi|)},\quad \mathbb{S}=\frac{|\aleph-\varpi|}{\varphi}$$

for all $\varphi>0,\aleph\in\Xi$ and $\varpi\in\Omega$. Clearly, $(\Xi,\Omega,\mathbb{M},*,\circ)$ is a complete neutrosophic fuzzy bipolar metric space, where $*$ is a continuous t-norm defined as $p*q=pq$.

Let $\mathcal{P},\mathcal{Q}:\Xi\cup\Omega\to\Xi\cup\Omega$ be mappings defined by

$$\mathcal{P}(\aleph)=\begin{cases}2-\aleph, & if\ \aleph\in[0,2],\\ 2, & if\ \aleph\in\mathbf{N}-\{1,2\};\end{cases}$$

$$\mathcal{Q}(\aleph)=\begin{cases}\aleph, & if\ \aleph\in[0,2],\\ 2, & if\ \aleph\in\mathbf{N}-\{1,2\},\end{cases}$$

for all $\aleph\in\Xi\cup\Omega$. Now, suppose that $\theta=\frac{1}{2}$, then for all $\varphi>0$, we discuss the following cases:

Case 1. *If $\aleph\in[0,2]$ and $\varpi\in\mathbf{N}-\{1,2\}$, then*

$$\begin{aligned}\mathbb{H}(\mathcal{P}(\aleph),\mathcal{Q}(\varpi),\theta\varphi)&=\mathbb{H}(2-\aleph,2,\theta\varphi)=\frac{\theta\varphi}{\theta\varphi+|2-\aleph-2|}\\ &\geq\frac{\varphi}{\varphi+|\aleph-\varpi|}=\mathbb{H}(\aleph,\varpi,\varphi),\\ \mathbb{O}(\mathcal{P}(\aleph),\mathcal{Q}(\varpi),\theta\varphi)&=\mathbb{O}(2-\aleph,2,\theta\varphi)=\frac{|2-\aleph-2|}{\theta\varphi+|2-\aleph-2|}\\ &\leq\frac{|\aleph-\varpi|}{\varphi+|\aleph-\varpi|}=\mathbb{O}(\aleph,\varpi,\varphi),\\ \mathbb{S}(\mathcal{P}(\aleph),\mathcal{Q}(\varpi),\theta\varphi)&=\mathbb{S}(2-\aleph,2,\theta\varphi)=\frac{|2-\aleph-2|}{\theta\varphi}\\ &\leq\frac{|\aleph-\varpi|}{\varphi}=\mathbb{S}(\aleph,\varpi,\varphi).\end{aligned}$$

Case 2. *If $\aleph\in\mathbf{N}-\{1,2\}$ and $\varpi\in[0,2]$, then*

$$\begin{aligned}\mathbb{H}(\mathcal{P}(\aleph),\mathcal{Q}(\varpi),\theta\varphi)&=\mathbb{H}(2,\varpi,\theta\varphi)=\frac{\theta\varphi}{\theta\varphi+|2-\varpi|}\\ &\geq\frac{\varphi}{\varphi+|\aleph-\varpi|}=\mathbb{H}(\aleph,\varpi,\varphi),\\ \mathbb{O}(\mathcal{P}(\aleph),\mathcal{Q}(\varpi),\theta\varphi)&=\mathbb{O}(2,\varpi,\theta\varphi)=\frac{|2-\varpi|}{\theta\varphi+|2-\varpi|}\\ &\leq\frac{|\aleph-\varpi|}{\varphi+|\aleph-\varpi|}=\mathbb{O}(\aleph,\varpi,\varphi),\end{aligned}$$

$$\mathbb{S}(\mathcal{P}(\aleph), \mathcal{Q}(\varpi), \theta\varphi) = \mathbb{S}(2, \varpi, \theta\varphi) = \frac{|2-\varpi|}{\theta\varphi}$$
$$\leq \frac{|\aleph-\varpi|}{\varphi} = \mathbb{S}(\aleph, \varpi, \varphi).$$

Therefore, all the conditions are fulfilled by $\mathcal{P}$ and $\mathcal{Q}$. By Theorem 3.10, $\mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point, i.e., $\aleph = 1$.

Definition 3.12. *Let $(\Omega, \Xi, \mathbb{M}, *, \circ)$ be an neutrosophic fuzzy bipolar metric space. A map $\mathcal{P} : \Omega \cup \Xi \to \Omega \cup \Xi$ is an NFB (neutrosophic fuzzy bipolar)-contraction if we can find $0 < \theta < 1$ satisfying*

$$\frac{1}{\mathbb{H}(\mathcal{P}\aleph, \mathcal{P}\varpi, \varphi)} - 1 \leq \theta\left[\frac{1}{\mathbb{H}(\aleph, \varpi, \varphi)} - 1\right], \quad \mathbb{O}(\mathcal{P}\aleph, \mathcal{P}\varpi, \varphi) \leq \theta\mathbb{O}(\aleph, \varpi, \varphi)$$
$$\mathbb{S}(\mathcal{P}\aleph, \mathcal{P}\varpi, \varphi) \leq \theta\mathbb{S}(\aleph, \varpi, \varphi), \text{ for all } \aleph \in \Omega, \varpi \in \Xi \text{ and } \varphi > 0.$$

Theorem 3.13. *Let $(\Omega, \Xi, \mathbb{M}, *, \circ)$be a complete neutrosophic fuzzy bipolar metric space. Let $\mathcal{P} : \Omega \cup \Xi \to \Omega \cup \Xi$ be a mapping satisfying:*

(1) $\mathcal{P}(\Omega) \subseteq \Omega$ and $\mathcal{P}(\Xi) \subseteq \Xi$;
(2) $\mathcal{P}$ is IFB-contraction, for all $\aleph \in \Omega$, $\varpi \in \Xi$, and $\varphi > 0$.

Then, $\mathcal{P}$ has a unique fixed point.

Proof. Let $\aleph_0 \in \Omega$ and $\varpi_0 \in \Xi$ and assume that $\mathcal{P}(\aleph_\alpha) = \aleph_{\alpha+1}$ and $\mathcal{P}(\varpi_\alpha) = \varpi_{\alpha+1}$ for all $\alpha \in \mathbf{N} \cup \{0\}$.
Then, we get $(\aleph_\alpha, \varpi_\alpha)$ as a bisequence on the neutrosophic fuzzy bipolar metric space $(\Omega, \Xi, \mathbb{M}, *, \circ)$. By using contraction for all $\varphi > 0$, we deduce

$$\frac{1}{\mathbb{H}(\aleph_\alpha, \varpi_\alpha, \varphi)} - 1 = \frac{1}{\mathbb{H}(\mathcal{P}\aleph_{\alpha-1}, \mathcal{P}\varpi_{\alpha-1}, \varphi)} - 1$$
$$\leq \theta\left[\frac{1}{\mathbb{H}(\aleph_{\alpha-1}, \varpi_{\alpha-1}, \varphi)} - 1\right]$$
$$= \frac{\theta}{\mathbb{H}(\aleph_{\alpha-1}, \varpi_{\alpha-1}, \varphi)} - \theta,$$

implies that,

$$\frac{1}{\mathbb{H}(\aleph_\alpha, \varpi_\alpha, \varphi)} \leq \frac{\theta}{\mathbb{H}(\aleph_{\alpha-1}, \varpi_{\alpha-1}, \varphi)} + (1-\theta)$$
$$\leq \frac{\theta^2}{\mathbb{H}(\aleph_{\alpha-2}, \varpi_{\alpha-2}, \varphi)} + \theta(1-\theta) + (1-\theta).$$

Continuing this way, we deduce

$$\frac{1}{\mathbb{H}(\aleph_\alpha, \varpi_\alpha, \varphi)} \leq \frac{\theta^\alpha}{\mathbb{H}(\aleph_0, \varpi_0, \varphi)} + \theta^{\alpha-1}(1-\theta) + \theta^{\alpha-2}(1-\theta) + \cdots + \theta(1-\theta) + (1-\theta)$$

$$\leq \frac{\theta^{\alpha}}{\mathbb{H}(\aleph_0,\varpi_0,\varphi)} + (\theta^{\alpha-1}+\theta^{\alpha-2}+\cdots+\theta+1)(1-\theta).$$
$$\leq \frac{\theta^{\alpha}}{\mathbb{H}(\aleph_0,\varpi_0,\varphi)} + (1-\theta^{\alpha}).$$

We obtain

$$\frac{1}{\frac{\theta^{\alpha}}{\mathbb{H}(\aleph_0,\varpi_0,\varphi)} + (1-\theta^{\alpha})} \leq \mathbb{H}(\aleph_{\alpha},\varpi_{\alpha},\varphi),$$

$$\begin{aligned}
\mathbb{O}(\aleph_{\alpha},\varpi_{\alpha},\varphi) &= \mathbb{O}(\mathcal{P}\aleph_{\alpha-1},\mathcal{P}\varpi_{\alpha-1},\varphi) \leq \theta\mathbb{O}(\aleph_{\alpha-1},\varpi_{\alpha-1},\varphi) = \mathbb{O}(\mathcal{P}\aleph_{\alpha-2},\mathcal{P}\varpi_{\alpha-2},\varphi)\\
&\leq \theta^2\mathbb{O}(\aleph_{\alpha-2},\varpi_{\alpha-2},\varphi) \leq \cdots \leq \theta^{\alpha}\mathbb{O}(\aleph_0,\varpi_0,\varphi)\\
\mathbb{O}(\aleph_{\alpha+1},\varpi_{\alpha},\varphi) &= \mathbb{O}(\mathcal{P}\aleph_{\alpha},\mathcal{P}\varpi_{\alpha-1},\varphi) \leq \theta\mathbb{O}(\aleph_{\alpha},\varpi_{\alpha-1},\varphi) = \mathbb{O}(\mathcal{P}\aleph_{\alpha-1},\mathcal{P}\varpi_{\alpha-2},\varphi)\\
&\leq \theta^2\mathbb{O}(\aleph_{\alpha-1},\varpi_{\alpha-2},\varphi) \leq \cdots \leq \theta^{\alpha}\mathbb{O}(\aleph_1,\varpi_0,\varphi),\\
\mathbb{S}(\aleph_{\alpha},\varpi_{\alpha},\varphi) &= \mathbb{S}(\mathcal{P}\aleph_{\alpha-1},\mathcal{P}\varpi_{\alpha-1},\varphi) \leq \theta\mathbb{S}(\aleph_{\alpha-1},\varpi_{\alpha-1},\varphi) = \mathbb{S}(\mathcal{P}\aleph_{\alpha-2},\mathcal{P}\varpi_{\alpha-2},\varphi)\\
&\leq \theta^2\mathbb{S}(\aleph_{\alpha-2},\varpi_{\alpha-2},\varphi) \leq \cdots \leq \theta^{\alpha}\mathbb{S}(\aleph_0,\varpi_0,\varphi)\\
\mathbb{S}(\aleph_{\alpha+1},\varpi_{\alpha},\varphi) &= \mathbb{S}(\mathcal{P}\aleph_{\alpha},\mathcal{P}\varpi_{\alpha-1},\varphi) \leq \theta\mathbb{S}(\aleph_{\alpha},\varpi_{\alpha-1},\varphi) = \mathbb{S}(\mathcal{P}\aleph_{\alpha-1},\mathcal{P}\varpi_{\alpha-2},\varphi)\\
&\leq \theta^2\mathbb{S}(\aleph_{\alpha-1},\varpi_{\alpha-2},\varphi) \leq \cdots \leq \theta^{\alpha}\mathbb{S}(\aleph_1,\varpi_0,\varphi).
\end{aligned}$$

Let $\alpha < \kappa$, for $\alpha,\kappa \in \mathbf{N}$. Then,

$$\begin{aligned}
\mathbb{H}(\aleph_{\alpha},\varpi_{\kappa},\varphi) \geq &\mathbb{H}\left(\aleph_{\alpha},\varpi_{\alpha},\frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1},\varpi_{\alpha},\frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1},\varpi_{\kappa},\frac{\varphi}{3}\right) \cdots \geq \mathbb{H}\left(\aleph_{\alpha},\varpi_{\alpha},\frac{\varphi}{3}\right) *\\
&\mathbb{H}\left(\aleph_{\alpha+1},\varpi_{\alpha},\frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\kappa-1},\varpi_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}\right) * \mathbb{H}\left(\aleph_{\kappa},\varpi_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}\right) *\\
&\mathbb{H}\left(\aleph_{\kappa},\varpi_{\kappa},\frac{\varphi}{3^{\kappa-1}}\right),\\
\mathbb{O}(\aleph_{\alpha},\varpi_{\kappa},\varphi) \leq &\mathbb{O}\left(\aleph_{\alpha},\varpi_{\alpha},\frac{\varphi}{3}\right) * \mathbb{O}\left(\aleph_{\alpha+1},\varpi_{\alpha},\frac{\varphi}{3}\right) * \mathbb{O}\left(\aleph_{\alpha+1},\varpi_{\kappa},\frac{\varphi}{3}\right) \cdots \leq \mathbb{O}\left(\aleph_{\alpha},\varpi_{\alpha},\frac{\varphi}{3}\right) *\\
&\mathbb{O}\left(\aleph_{\alpha+1},\varpi_{\alpha},\frac{\varphi}{3}\right) \cdots * \mathbb{O}\left(\aleph_{\kappa-1},\varpi_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}\right) * \mathbb{O}\left(\aleph_{\kappa},\varpi_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}\right) *\\
&\mathbb{O}\left(\aleph_{\kappa},\varpi_{\kappa},\frac{\varphi}{3^{\kappa-1}}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{S}(\aleph_{\alpha},\varpi_{\kappa},\varphi) \leq &\mathbb{S}\left(\aleph_{\alpha},\varpi_{\alpha},\frac{\varphi}{3}\right) * \mathbb{S}\left(\aleph_{\alpha+1},\varpi_{\alpha},\frac{\varphi}{3}\right) * \mathbb{S}\left(\aleph_{\alpha+1},\varpi_{\kappa},\frac{\varphi}{3}\right) \cdots \leq \mathbb{S}\left(\aleph_{\alpha},\varpi_{\alpha},\frac{\varphi}{3}\right) *\\
&\mathbb{S}\left(\aleph_{\alpha+1},\varpi_{\alpha},\frac{\varphi}{3}\right) \cdots * \mathbb{S}\left(\aleph_{\kappa-1},\varpi_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}\right) * \mathbb{S}\left(\aleph_{\kappa},\varpi_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}\right) * \mathbb{S}\left(\aleph_{\kappa},\varpi_{\kappa},\frac{\varphi}{3^{\kappa-1}}\right).
\end{aligned}$$

Therefore,

$$\mathbb{H}(\aleph_{\alpha},\varpi_{\kappa},\varphi) \geq \mathbb{H}\left(\aleph_{\alpha},\varpi_{\alpha},\frac{\varphi}{3}\right)*\mathbb{H}\left(\aleph_{\alpha+1},\varpi_{\alpha},\frac{\varphi}{3}\right)*\cdots*\mathbb{H}\left(\aleph_{\kappa-1},\varpi_{\kappa-1},\frac{\varphi}{3^{k-1}}\right)*\mathbb{H}\left(\aleph_{\kappa},\varpi_{\kappa-1},\frac{\varphi}{3^{k-1}}\right) *$$

$$\mathbb{H}\left(\aleph_{\kappa},\varpi_{\kappa},\frac{\varphi}{3^{k-1}}\right)\geq\frac{1}{\frac{\theta^{\alpha}}{\mathbb{H}\left(\aleph_0,\varpi_0,\frac{\varphi}{3}\right)}+(1-\theta^{\alpha})}*\frac{1}{\frac{\theta^{\alpha}}{\mathbb{H}\left(\aleph_1,\varpi_0,\frac{\varphi}{3}\right)}+(1-\theta^{\alpha})}*$$
$$\frac{1}{\frac{\theta^{\alpha}}{\mathbb{H}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{k-1}}\right)}+(1-\theta^{k-1})}*\frac{1}{\frac{\theta^{\alpha}}{\mathbb{H}\left(\aleph_1,\varpi_0,\frac{\varphi}{3^{k-1}}\right)}+(1-\theta^{k-1})}*$$
$$\frac{1}{\frac{\theta^{\alpha}}{\mathbb{H}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{k-1}}\right)}+(1-\theta^{k})},$$

$$\mathbb{O}(\aleph_{\alpha},\varpi_{\kappa},\varphi)\leq\mathbb{O}\left(\aleph_{\alpha},\varpi_{\alpha},\frac{\varphi}{3}\right)\circ\mathbb{O}\left(\aleph_{\alpha+1},\varpi_{\alpha},\frac{\varphi}{3}\right)\circ\ldots\mathbb{O}\left(\aleph_{\kappa-1},\varpi_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}\right)\circ$$
$$\mathbb{O}\left(\aleph_{\kappa},\varpi_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}\right)\circ\mathbb{O}\left(\aleph_{\kappa},\varpi_{\kappa},\frac{\varphi}{3^{\kappa-1}}\right)\circ\leq\theta^{\alpha}\mathbb{O}\left(\aleph_0,\varpi_0,\frac{\varphi}{3}\right)\circ$$
$$\theta^{\alpha}\mathbb{O}\left(\aleph_1,\varpi_0,\frac{\varphi}{3^{\kappa-1}}\right)\circ\ldots\theta^{\kappa-1}\mathbb{O}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}}\right)\circ\theta^{\kappa-1}\mathbb{O}\left(\aleph_1,\varpi_0,\frac{\varphi}{3^{\kappa-1}}\right)\circ$$
$$\theta^{\kappa}\mathbb{O}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}}\right)$$

and

$$\mathbb{S}(\aleph_{\alpha},\varpi_{\kappa},\varphi)\leq\mathbb{S}\left(\aleph_{\alpha},\varpi_{\alpha},\frac{\varphi}{3}\right)\circ\mathbb{S}\left(\aleph_{\alpha+1},\varpi_{\alpha},\frac{\varphi}{3}\right)\circ\ldots\mathbb{S}\left(\aleph_{\kappa-1},\varpi_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}\right)\circ$$
$$\mathbb{S}\left(\aleph_{\kappa},\varpi_{\kappa-1},\frac{\varphi}{3^{\kappa-1}}\right)\circ\mathbb{S}\left(\aleph_{\kappa},\varpi_{\kappa},\frac{\varphi}{3^{\kappa-1}}\right)\circ\leq\theta^{\alpha}\mathbb{S}\left(\aleph_0,\varpi_0,\frac{\varphi}{3}\right)\circ\theta^{\alpha}\mathbb{S}\left(\aleph_1,\varpi_0,\frac{\varphi}{3^{\kappa-1}}\right)\circ\ldots$$
$$\theta^{\kappa-1}\mathbb{S}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}}\right)\circ\theta^{\kappa-1}\mathbb{S}\left(\aleph_1,\varpi_0,\frac{\varphi}{3^{\kappa-1}}\right)\circ\theta^{\kappa}\mathbb{S}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}}\right),$$

implies that,

$$\mathbb{H}(\aleph_k,\varpi_k,\varphi)\geq\frac{1}{\frac{\theta^{\alpha}}{\mathbb{H}\left(\aleph_0,\varpi_0,\frac{\varphi}{3}\right)}+(1-\theta^{\alpha})}*\frac{1}{\frac{\theta^{\alpha}}{\mathbb{H}\left(\aleph_1,\varpi_0,\frac{\varphi}{3}\right)}+(1-\theta^{\alpha})}*$$
$$\frac{1}{\frac{\theta^{\kappa-1}}{\mathbb{H}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{k-1}}\right)}+(1-\theta^{k-1})}*\frac{1}{\frac{\theta^{\kappa-1}}{\mathbb{H}\left(\aleph_1,\varpi_0,\frac{\varphi}{3^{k-1}}\right)}+(1-\theta^{k-1})}*$$
$$\frac{1}{\frac{\theta^{\kappa}}{\mathbb{H}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{k-1}}\right)}+(1-\theta^{k})},$$

$$\mathbb{O}(\aleph_k,\varpi_k,\varphi)=\theta^{\alpha}\mathbb{O}\left(\aleph_0,\varpi_0,\frac{\varphi}{3}\right)\circ\theta^{\alpha}\mathbb{O}\left(\aleph_1,\varpi_0,\frac{\varphi}{3^{\kappa-1}}\right)\circ\ldots$$
$$\theta^{\kappa-1}\mathbb{O}\left(\aleph_0,\varpi_0,\frac{\varphi}{3^{\kappa-1}}\right)\circ\theta^{\kappa-1}\mathbb{O}\left(\aleph_1,\varpi_0,\frac{\varphi}{3^{\kappa-1}}\right)\circ$$

$$\theta^{\kappa}\mathbb{O}\left(\aleph_0, \varpi_0, \frac{\varphi}{3^{\kappa-1}}\right)$$

and

$$\begin{aligned}\mathbb{S}(\aleph_k, \varpi_k, \varphi) =&\theta^{\alpha}\mathbb{S}\left(\aleph_0, \varpi_0, \frac{\varphi}{3}\right) \circ \theta^{\alpha}\mathbb{S}\left(\aleph_1, \varpi_0, \frac{\varphi}{3^{\kappa-1}}\right) \circ \dots \\ &\circ \theta^{\kappa-1}\mathbb{S}\left(\aleph_0, \varpi_0, \frac{\varphi}{3^{\kappa-1}}\right) \circ \theta^{\kappa-1}\mathbb{S}\left(\aleph_1, \varpi_0, \frac{\varphi}{3^{\kappa-1}}\right) \circ \theta^{\kappa}\mathbb{S}\left(\aleph_0, \varpi_0, \frac{\varphi}{3^{\kappa-1}}\right).\end{aligned}$$

As $\alpha, \kappa \to +\infty$, we deduce

$$\begin{aligned}\lim_{\alpha,\kappa\to+\infty} \mathbb{H}(\aleph_\alpha, \varpi_\kappa, \varphi) &= 1 * 1 * \dots * 1 = 1, \\ \lim_{\alpha,\kappa\to+\infty} \mathbb{O}(\aleph_\alpha, \varpi_\kappa, \varphi) &= 0 \circ 0 \circ \dots \circ 0 = 0 \textit{ and} \\ \lim_{\alpha,\kappa\to+\infty} \mathbb{S}(\aleph_\alpha, \varpi_\kappa, \varphi) &= 0 \circ 0 \circ \dots \circ 0 = 0.\end{aligned}$$

Which implies that the bisequence $(\aleph_\alpha, \varpi_\alpha)$ is a Cauchy bisequence. Since $(\Omega, \Xi, \mathbb{M}, *, \circ)$ is a complete neutrosophic fuzzy bipolar metric space, then $\{\aleph_\alpha\} \to \aleph$ and $\{\varpi_\alpha\} \to \aleph$, where $\aleph \in \Omega \cap \Xi$.

$$\begin{aligned}\mathbb{H}(\aleph, \mathcal{P}\aleph, \varphi) &\geq \mathbb{H}\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) * \mathbb{H}\left(\aleph_{\alpha+1}, \mathcal{P}\aleph, \frac{\varphi}{3}\right) \\ &= \mathbb{H}\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) * \mathbb{H}\left(\mathcal{P}\aleph_\alpha, \mathcal{P}\aleph_\alpha, \frac{\varphi}{3}\right) * \mathbb{H}\left(\mathcal{P}\aleph_\alpha, \mathcal{P}\aleph, \frac{\varphi}{3}\right) \\ &\geq \mathbb{H}\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) * \frac{1}{\dfrac{\theta^{\alpha+1}}{\mathbb{H}\left(\aleph_0, \varpi_0, \frac{\varphi}{3}\right)} + (1 - \theta^{\alpha+1})} * \mathbb{H}\left(\mathcal{P}\aleph_\alpha, \mathcal{P}\aleph, \frac{\varphi}{3}\right)\end{aligned}$$

$$\to 1 * 1 * 1 = 1 \textit{ as } \alpha \to \infty,$$

$$\begin{aligned}\mathbb{O}(\aleph, \mathcal{P}\aleph, \varphi) &\leq \mathbb{O}\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\aleph_{\alpha+1}, \mathcal{P}\aleph, \frac{\varphi}{3}\right) \\ &= \mathbb{O}\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\mathcal{P}\aleph_{\alpha+1}, \mathcal{P}\aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\mathcal{P}\aleph_{\alpha+1}, \mathcal{P}\aleph, \frac{\varphi}{3}\right) \\ &\leq \mathbb{O}\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \theta^{\alpha+1}\mathbb{O}\left(\aleph_0, \varpi_0, \frac{\varphi}{3}\right) \circ \mathbb{O}\left(\mathcal{P}\aleph_{\alpha+1}, \mathcal{P}\aleph, \frac{\varphi}{3}\right)\end{aligned}$$

and

$$\begin{aligned}\mathbb{S}(\aleph, \mathcal{P}\aleph, \varphi) &\leq \mathbb{S}\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\aleph_{\alpha+1}, \mathcal{P}\aleph, \frac{\varphi}{3}\right) \\ &= \mathbb{S}\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\mathcal{P}\aleph_{\alpha+1}, \mathcal{P}\aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\mathcal{P}\aleph_{\alpha+1}, \mathcal{P}\aleph, \frac{\varphi}{3}\right) \\ &\leq \mathbb{S}\left(\aleph, \aleph_{\alpha+1}, \frac{\varphi}{3}\right) \circ \theta^{\alpha+1}\mathbb{S}\left(\aleph_0, \varpi_0, \frac{\varphi}{3}\right) \circ \mathbb{S}\left(\mathcal{P}\aleph_{\alpha+1}, \mathcal{P}\aleph, \frac{\varphi}{3}\right).\end{aligned}$$

Hence, $\mathcal{P}\aleph = \aleph$. Let $\mathcal{P}\varpi = \varpi$ for some $\varpi \in \Omega$, then

$$\frac{1}{\mathbb{H}(\aleph,\omega,\varphi)}-1 \leq \frac{1}{\mathbb{H}(\mathcal{P}\aleph,\mathcal{P}\omega,\varphi)}-1 \leq \theta\Big[\frac{1}{\mathbb{H}(\aleph,\omega,\varphi)}-1\Big] < \frac{1}{\mathbb{H}(\aleph,\omega,\varphi)}-1,$$

$$\mathbb{O}(\aleph,\omega,\varphi)=\mathbb{O}(\mathcal{P}\aleph,\mathcal{P}\omega,\varphi)\leq\theta\mathbb{O}(\aleph,\omega,\varphi)<\mathbb{O}(\aleph,\omega,\varphi)$$

and

$$\mathbb{S}(\aleph,\omega,\varphi)=\mathbb{S}(\mathcal{P}\aleph,\mathcal{P}\omega,\varphi)\leq\theta\mathbb{S}(\aleph,\omega,\varphi)<\mathbb{S}(\aleph,\omega,\varphi),$$

this is a contraction as well. Therefore, $\mathbb{H}(\aleph,\varpi,\varphi)=1$, $\mathbb{O}(\aleph,\varpi,\varphi)=0$, $\mathbb{S}(\aleph,\varpi,\varphi)=0$. Hence, $\aleph=\varpi$. □

Example 3.14. *Let* $\Omega=[0,1]$ *and* $\Xi=\{0\}\cup\mathbf{N}-\{1\}$. *Define* $\mathbb{M}:\Omega\times\Xi\times(0,+\infty)\to[0,1]$ *as*

$$\mathbb{H}(\aleph,\varpi,\varphi)=\frac{\varphi}{\varphi+|\aleph-\varpi|},\ \mathbb{O}(\aleph,\varpi,\varphi)=\frac{|\aleph-\varpi|}{\varphi+|\aleph-\varpi|},\ \mathbb{S}(\aleph,\varpi,\varphi)=\frac{|\aleph-\varpi|}{\varphi}.$$

Then, $(\Omega,\Xi,\mathbb{M},*,\circ)$ *is a complete neutrosophic fuzzy bipolar metric space with CTN* $\nu*\aleph=\nu\aleph$ *and CTCN* $\nu\circ\aleph=\max\{\nu,\aleph\}$.

Proof. Define,

$$\mathcal{P}:(\Omega,\Xi,\mathbb{M},*,\circ)\rightrightarrows(\Omega,\Xi,\mathbb{M},*,\circ)$$

by

$$\mathcal{P}(\aleph)=\begin{cases}\dfrac{1-3^{-\aleph}}{5},\ if\ \aleph\in[0,1]\\ 0,\ if\ \aleph\in\mathbf{N}-\{1\},\end{cases}$$

for all $\aleph\in\Omega\cup\Xi$ and take $\theta\in\left[\frac{1}{2},1\right)$, then

$$\begin{aligned}\mathbb{H}(\mathcal{P}\aleph,\mathcal{P}\varpi,\theta\varphi)=\mathbb{H}\left(\frac{1-3^{-\aleph}}{5},\frac{1-3^{-\varpi}}{5},\theta\varphi\right)&=\frac{\theta\varphi}{\theta\varphi+\left|\frac{1-3^{-\aleph}}{5}-\frac{1-3^{-\varpi}}{5}\right|}\\ &=\frac{\theta\varphi}{\theta\varphi+\left|\frac{3^{-\aleph}-3^{-\varpi}}{5}\right|}\geq\frac{\theta\varphi}{\theta\varphi+\frac{|\aleph-\varpi|}{5}}\\ &=\frac{5\theta\varphi}{5\theta\varphi+|\aleph-\varpi|}\geq\frac{\varphi}{\varphi+|\aleph-\varpi|}=\mathbb{H}(\aleph,\varpi,\varphi),\end{aligned}$$

$$\mathbb{O}(\mathcal{P}\aleph,\mathcal{P}\varpi,\theta\varphi)=\mathbb{O}\left(\frac{1-3^{-\aleph}}{5},\frac{1-3^{-\varpi}}{5},\theta\varphi\right)=\frac{\left|\frac{1-3^{-\aleph}}{5}-\frac{1+3^{-\varpi}}{5}\right|}{\theta\varphi+\left|\frac{1-3^{-\aleph}}{5}-\frac{1+3^{-\varpi}}{5}\right|}$$

$$= \frac{\dfrac{|3^{-\aleph} - 3^{-\varpi}|}{5}}{\theta\varphi + \dfrac{|3^{-\aleph} - 3^{-\varpi}|}{5}} \leq \frac{|3^{-\aleph} - 3^{-\varpi}|}{5\theta\varphi + |3^{-\aleph} - 3^{-\varpi}|}$$

$$\leq \frac{|\aleph - \varpi|}{5\theta\varphi + |\aleph - \varpi|} \leq \frac{|\aleph - \varpi|}{\theta\varphi + |\aleph - \varpi|} = \mathbb{O}(\aleph, \varpi, \varphi)$$

and

$$\mathbb{S}(\mathcal{P}\aleph, \mathcal{P}\varpi, \theta\varphi) = \mathbb{S}\left(\frac{1 - 3^{-\aleph}}{5}, \frac{1 - 3^{-\varpi}}{5}, \theta\varphi\right) = \frac{\left|\dfrac{1 - 3^{-\aleph}}{5} - \dfrac{1 + 3^{-\varpi}}{5}\right|}{\theta\varphi}$$

$$= \frac{\dfrac{|3^{-\aleph} - 3^{-\varpi}|}{5}}{\theta\varphi} \leq \frac{|3^{-\aleph} - 3^{-\varpi}|}{5\theta\varphi} \leq \frac{|\aleph - \varpi|}{5\theta\varphi}$$

$$\leq \frac{|\aleph - \varpi|}{\theta\varphi} = \mathbb{S}(\aleph, \varpi, \varphi).$$

Therefore, the conditions of Theorem 3.13 are fulfilled, and 0 is the only fixed point for $\mathcal{P}$. □

4. APPLICATION

Let $\Omega = \mathcal{C}([c, a], [0, +\infty))$ be the set of all continuous functions defined on $[c, a]$ with values in the interval $[0, +\infty)$, and $\Xi = \mathcal{C}([c, a], (-\infty, 0])$ be the set of all continuous functions defined on $[c, a]$ with values in the interval $(-\infty, 0]$.

Theorem 4.1. *Suppose the integral equation:*

$$\aleph(\chi) = \varpi(\chi) + \delta \int_a^c \mathcal{R}(\chi, \rho)\aleph(\chi)\, d\rho \quad \text{for } \chi, \rho \in [c, a], \tag{2}$$

where $\delta > 0$, $\varpi(\rho)$ is a fuzzy function of $\rho : \rho \in [c, a]$ and $\mathcal{R} : \mathcal{C}([c, a] \times \mathbf{R}) \to \mathbf{R}^+$.

Define, $\mathbb{H}$, $\mathbb{O}$ and $\mathbb{S}$ by

$$\mathbb{H}(\aleph(\chi), \varpi(\chi), \varphi) = \begin{cases} \dfrac{\varphi}{\varphi + \max_{\chi \in [c,a]} |\aleph(\chi) - \varpi(\chi)|} & if\ \varphi > 0 \\ 0\ if\ \varphi = 0; \end{cases}$$

$$\mathbb{O}(\aleph(\chi), \varpi(\chi), \varphi) = \begin{cases} \dfrac{\max_{\chi \in [c,a]} |\aleph(\chi) - \varpi(\chi)|}{\varphi + \max_{\chi \in [c,a]} |\aleph(\chi) - \varpi(\chi)|} & if\ \varphi > 0 \\ 0\ if\ \varphi = 0 \end{cases}$$

and

$$\mathbb{S}(\aleph(\chi), \varpi(\chi), \varphi) = \begin{cases} \dfrac{\max_{\chi \in [c,a]} |\aleph(\chi) - \varpi(\chi)|}{\varphi} & if\ \varphi > 0, \\ 0\ if\ \varphi = 0 \end{cases}$$

with CTN and CTCN defined by $\nu * \aleph = \nu \cdot \aleph$ *and* $\nu \circ \aleph = \max(\nu, \aleph)$. *Then* $(\Xi, \Omega, \mathbb{M}, *, \circ)$ *is a complete neutrosophic fuzzy bipolar metric space.*

Suppose that $|\mathcal{R}(\chi,\rho)\aleph(\chi) - \mathcal{R}(\chi,\rho)\varpi(\chi)| \leq |\aleph(\chi) - \varpi(\chi)|$ *for* $\aleph \in \Omega$, $\varpi \in \Xi$, $\theta \in (0,1)$ *and all* $\chi, \rho \in [c,a]$. *Also, let* $\mathcal{R}(\chi,\rho)(\delta \int_c^a d\rho) \leq \theta < 1$.

Then, the integral Equation (2) *has a unique solution.*

Proof. Define $\mathcal{P} : (\Xi, \Omega, \mathbb{H}, \mathbb{O}, *, \circ) \to (\Xi, \Omega, \mathbb{H}, \mathbb{O}, *, \circ)$ by

$$\mathcal{P}\aleph(\chi) = \varpi(\chi) + \delta \int_c^a \mathcal{R}(\chi,\rho)\aleph(\chi)d\rho, \ for \ all \ \chi, \rho \in [c,a].$$

Now, for all $\aleph, \varpi \in \Omega \cup \Xi$, we deduce

$$\begin{aligned}
\mathbb{H}(\mathcal{P}\aleph(\chi), \mathcal{P}\varpi(\chi), \theta\varphi) &= \frac{\theta\varphi}{\theta\varphi + \max_{\chi\in[c,a]} |\mathcal{P}\aleph(\chi) - \mathcal{P}\varpi(\chi)|} \\
&= \frac{\theta\varphi}{\theta\varphi + \max_{\chi\in[c,a]} \left[|\varpi(\chi) + \delta \int_c^a \mathcal{R}(\chi,\rho)\aleph(\chi)d\rho - \varpi(\chi) - \delta \int_c^a \mathcal{R}(\chi,\rho)\varpi(\chi)d\rho|\right]} \\
&= \frac{\theta\varphi}{\theta\varphi + \max_{\chi\in[c,a]} \left[|\delta \int_c^a \mathcal{R}(\chi,\rho)\aleph(\chi)d\rho - \delta \int_c^a \mathcal{R}(\chi,\rho)\varpi(\chi)d\rho|\right]} \\
&= \frac{\theta\varphi}{\theta\varphi + \max_{\chi\in[c,a]} \left[|\mathcal{R}(\chi,\rho)\aleph(\chi) - \mathcal{R}(\chi,\rho)\varpi(\chi)| \quad (\delta \int_c^a d\rho)\right]} \\
&\geq \frac{\varphi}{\varphi + \max_{\chi\in[c,a]} \left[|\aleph(\chi) - \varpi(\chi)|\right]} \\
&\geq \mathbb{H}(\aleph(\chi), \varpi(\chi), \varphi),
\end{aligned}$$

$$\begin{aligned}
\mathbb{O}(\mathcal{P}\aleph(\chi), \mathcal{P}\varpi(\chi), \theta\varphi) &= \frac{\max_{\chi\in[c,a]} |\mathcal{P}\aleph(\chi) - \mathcal{P}\varpi(\chi)|}{\theta\varphi + \max_{\chi\in[c,a]} |\mathcal{P}\aleph(\chi) - \mathcal{P}\varpi(\chi)|} \\
&= \frac{\max_{\chi\in[c,a]} \left[|\varpi(\chi) + \delta \int_c^a \mathcal{R}(\chi,\rho)\aleph(\chi)d\rho - \varpi(\chi) - \delta \int_c^a \mathcal{R}(\chi,\rho)\varpi(\chi)d\rho|\right]}{\theta\varphi + \max_{\chi\in[c,a]} \left[|\varpi(\chi) + \delta \int_c^a \mathcal{R}(\chi,\rho)\aleph(\chi)d\rho - \varpi(\chi) - \delta \int_c^a \mathcal{R}(\chi,\rho)\varpi(\chi)d\rho|\right]} \\
&= \frac{\max_{\chi\in[c,a]} \left[|\delta \int_c^a \mathcal{R}(\chi,\rho)\aleph(\chi)d\rho - \delta \int_c^a \mathcal{R}(\chi,\rho)\varpi(\chi)d\rho|\right]}{\theta\varphi + \max_{\chi\in[c,a]} \left[|\delta \int_c^a \mathcal{R}(\chi,\rho)\aleph(\chi)d\rho - \delta \int_c^a \mathcal{R}(\chi,\rho)\varpi(\chi)d\rho|\right]} \\
&= \frac{\max_{\chi\in[c,a]} \left[|\mathcal{R}(\chi,\rho)\aleph(\chi) - \mathcal{R}(\chi,\rho)\varpi(\chi)| \quad (\delta \int_c^a d\rho)\right]}{\theta\varphi + \max_{\chi\in[c,a]} \left[|\mathcal{R}(\chi,\rho)\aleph(\chi) - \mathcal{R}(\chi,\rho)\varpi(\chi)| \quad (\delta \int_c^a d\rho)\right]} \\
&\geq \frac{\max_{\chi\in[c,a]} \left[|\aleph(\chi) - \varpi(\chi)|\right]}{\varphi + \max_{\chi\in[c,a]} \left[|\aleph(\chi) - \varpi(\chi)|\right]} \geq \mathbb{O}(\aleph(\chi), \varpi(\chi), \varphi)
\end{aligned}$$

and

$$\mathbb{S}(\mathcal{P}\aleph(\chi), \mathcal{P}\varpi(\chi), \theta\varphi) = \frac{\max_{\chi\in[c,a]} |\mathcal{P}\aleph(\chi) - \mathcal{P}\varpi(\chi)|}{\theta\varphi}$$

$$= \frac{\max_{\chi\in[c,a]}\left[|\varpi(\chi)+\delta\int_c^a \mathcal{R}(\chi,\rho)\aleph(\chi)d\rho \quad -\varpi(\chi)-\delta\int_c^a \mathcal{R}(\chi,\rho)\varpi(\chi)d\rho|\right]}{\theta\varphi}$$
$$= \frac{\max_{\chi\in[c,a]}\left[|\delta\int_c^a \mathcal{R}(\chi,\rho)\aleph(\chi)d\rho \quad -\delta\int_c^a \mathcal{R}(\chi,\rho)\varpi(\chi)d\rho|\right]}{\theta\varphi}$$
$$= \frac{\max_{\chi\in[c,a]}\left[|\mathcal{R}(\chi,\rho)\aleph(\chi)-\mathcal{R}(\chi,\rho)\varpi(\chi)| \quad (\delta\int_c^a d\rho)\right]}{\theta\varphi}$$
$$\geq \frac{\max_{\chi\in[c,a]}\left[|\aleph(\chi)-\varpi(\chi)|\right]}{\varphi} \geq \mathbb{S}(\aleph(\chi),\varpi(\chi),\varphi).$$

Therefore, the conditions of Theorem 3.9 are fulfilled and operator $\mathcal{P}$ has a unique fixed point. □

Example 4.2. *Assume the following integral equation:*

$$\aleph(\chi) = |\cos\chi| + \frac{1}{7}\int_0^1 \rho\aleph(\rho)d\rho, \quad \textit{for all } \rho\in[0,1].$$

Then, it has a unique solution in $\Omega\cup\Xi$.

Proof. Let $\mathcal{P} : (\Omega,\Xi,\mathbb{M},*,\circ) \rightrightarrows (\Omega,\Xi,\mathbb{M},*,\circ)$ be defined by

$$\mathcal{P}\aleph(\chi) = |\sin\chi| + \frac{1}{7}\int_0^1 \rho\aleph(\rho)d\rho,$$

and set $\mathcal{R}(\chi,\rho)\aleph(\chi) = \frac{1}{7}\rho\aleph(\rho)$ and $\mathcal{R}(\chi,\rho)\varpi(\chi) = \frac{1}{7}\rho\varpi(\rho)$, where $\aleph,\varpi\in\Omega\cup\Xi$ and for all $\chi,\rho\in[0,1]$. Then, we have

$$|\mathcal{R}(\chi,\rho)\aleph(\chi)-\mathcal{R}(\chi,\rho)\varpi(\chi)| = \left|\frac{1}{7}\rho\aleph(\rho)-\frac{1}{7}\rho\varpi(\rho)\right| = \frac{1}{7}\rho|\aleph(\rho)-\varpi(\rho)| \leq |\aleph(\rho)-\varpi(\rho)|,$$

and

$$\frac{1}{7}\int_0^1 \rho d\rho = \frac{1}{7}\left(\frac{(1)^2}{2}-\frac{(0)^2}{2}\right) = \frac{1}{7} = \theta < 1, \quad \text{where } \delta = \frac{1}{7}.$$

Therefore, the conditions of Theorem 3.9 are fulfilled. Hence, $\mathcal{P}$ has a unique solution in $\Omega\cup\Xi$. □

CONCLUSION

This study introduces and explores a novel mathematical structure called the neutrosophic fuzzy bipolar metric space, which extends and generalizes existing fuzzy and bipolar metric space frameworks and formulate new definitions and properties of neutrosophic fuzzy bipolar metric spaces. Establish several fixed point theorems using covariant maps and neutrosophic fuzzy bipolar-contractions. Provide proofs and examples to validate the theoretical findings. Present an application to demonstrate the existence and uniqueness of solutions to certain differential equations within the developed framework. The results unify and generalize many existing fixed point results in fuzzy and neutrosophic metric spaces, making a significant contribution to the fixed point theory in uncertain environments.

FUTURE SCOPE

The present work opens several avenues for future research. One promising direction is the extension to multivalued or set-valued mappings, where fixed point results can be investigated under the framework of neutrosophic fuzzy bipolar metric spaces. Another significant area involves the application of this theory to real-world problems in fields such as engineering, decision-making, and artificial intelligence, especially where uncertainty and bipolar behavior are inherent. Additionally, the current structure can be generalized to other types of metric spaces, including neutrosophic fuzzy bipolar b-metric spaces, cone metric spaces, and probabilistic metric spaces, to broaden the applicability of the theory. From a computational perspective, there is scope for designing algorithmic approaches to numerically compute fixed points within neutrosophic bipolar contexts. Finally, future work could focus on the exploration of dynamic systems, particularly analyzing the stability and convergence behavior of iterative processes within such enriched metric frameworks.

Abbrevations

FMS-Fuzzy metric space
FP-Fixed point
IFS-Intuitionistic fuzzy sets
IFMS-Intuitionistic fuzzy metric space
NS-Neutrosohic set
NMS-Neutrosophic metric space
MIFMS-modified intuitionistic fuzzy metric space

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