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A NOTE ON SOME ENDPOINT ESTIMATES OF COMMUTATORS OF FRACTIONAL INTEGRAL OPERATORS

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Abstract. It is known that fractional integral operators are not bounded from Lebesgue integrable functions to Lebesgue space for some particular related exponent. Based on some recent results by Schikorra, Spector, and Van Schaftingen, we investigate commutators of fractional integral operators on Lebesgue integrable functions. We establish a weak type estimates for these commutators generated by essentially bounded functions. Under the same assumption, we also prove that the norm of these commutators are dominated by the norm of the Riesz transform.

Key words and Phrases: fractional integral operators, commutators, Riesz transform, Lebesgue spaces.

1. INTRODUCTION

Fractional integral operators are mathematical operators that extend the concept of integration to non-integer orders. This fractional integral operators also often referred to as Riesz potential was introduced by Marcel Riesz in the 1920s. The Riesz potential is a specific type of integral operators and defined as the convolution of a function with the fundamental solution of the Laplace operator raised to a fractional power. Let $n \geq 1$ and $0 < \alpha < n$, then the fractional integral operator of order α, I_α, has the form

$$I_\alpha f(x) \equiv \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \quad (x \in \mathbb{R}^n)$$

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with $\gamma(\alpha)=\pi^{\frac{n}{2}}2^{\alpha}\Gamma\left(\frac{\alpha}{2}\right)/\Gamma\left(\frac{n-2}{2}\right)$ and f is a function such that this integral is well-defined. The constant $\gamma(\alpha)$ is called a normalization constant and it is to guarantee that I_α satisfy the semigroup properties, that is $I_{\alpha_1+\alpha_2}=I_{\alpha_1}I_{\alpha_2},\ 0<\alpha_1+\alpha_2<n$.

On the other hand commutators are operators that measure the failure of two operators to commute with each other. Commutators of A and B, denoted by $[A,B]$ is defined as $[A,B]=AB-BA$. The commutators generated by a function b and the fractional integral operator I_α is defined as

$$[b,I_\alpha]f=b\cdot I_\alpha f-I_\alpha(bf)=\int_{\mathbb{R}^n}\frac{(b(x)-b(y))}{|x-y|^{n-\alpha}}f(y)\,dy.$$

The study of boundedness of fractional integral operators is closely related to the regularity of solution of Laplace equation [8]. It is known that I_α is not bounded from $L^{np/(n-\alpha p)}$ to L^p if $1<p<n/\alpha$ [8]. Since then many authors have studying the similar estimates for commutators of fractional integral operators in different setting [1, 2, 3, 4, 5, 6, 7, 9]. In this paper, we will investigate an estimation of this commutators on the endpoint $p=1$ based of the result on [10]. These results also had been generalized for a vector-valued function working on Lorentz spaces [11]. We establish both the weak type and norm estimates for the commutators of fractional integral operators under the assumption that it is generated by essentially bounded functions.

2. PRELIMINARY RESULTS

The main function space that we will considered here is the classical Lebesgue spaces. Two of the fundamental results on the boundedness of fractional integral operators are as follow.

Lemma 2.1 (Hedberg Lemma [8])**.** *Let $f\in L^p(\mathbb{R}^n)$ with $1\le p<n/\alpha$. Define $q>0$ by $1/q=1/p-\alpha/n$, then there exists $C>0$ such that*

$$|I_\alpha f(x)|\le C\|f\|_{L^p}^{1-p/q}Mf(x)^{p/q}$$

Theorem 2.2 (Hardy-Littlewood-Sobolev Inequality [8])**.** *Let $0<\alpha<n,\ 1\le p<n/\alpha$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$.*

*(1) **(Strong Type)** If $1<p<n/\alpha$ then there exists $C>0$ such that*

$$\|I_\alpha f\|_{L^q(\mathbb{R}^n)}\le C\|f\|_{L^p(\mathbb{R}^n)},\quad \forall f\in L^p(\mathbb{R}^n)$$

*(2) **(Weak Type)** There exist $C>0$ such that for all $\lambda>0$,*

$$|\{x\in\mathbb{R}^n:|I_\alpha f(x)|>\lambda\}|\le\left(\frac{C}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}\right)^{\frac{n}{n-\alpha}},\ \forall f\in L^1(\mathbb{R}^n)$$

We examine one generalization of Lebesgue space termed weak Lebesgue space to understand why the second inequality of previous theorem is referred to as a weak type.

Definition 2.3. *Let $0 < p < \infty$. Then a weak Lebesgue spaces $\mathsf{WL}^p(\mathbb{R}^n)$ is a space that contains all measurable function f with the following norm*

$$\|f\|_{\mathsf{WL}^p(\mu)} \equiv \sup_{\lambda>0} \lambda \cdot |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{\frac{1}{p}} < \infty$$

In other words, the weak type vesion of Hardy-Littlewood-Sobolev inequality says that I_α is bounded from L^1 to $\mathsf{WL}^{\frac{n}{n-\alpha}}$.

In particular we see that this theorem concerns L^p strong-type estimates for Riesz potential with the exception of $p = 1$. Consider the following counterexample. If $p = 1,\ q = n/(n-\alpha)$, then take the function

$$f(x) = \begin{cases} 1 & |x| \leq 1, \\ 0 & |x| > 1. \end{cases}$$

It can be shown that $f \in L^1$, but $\|I_\alpha f\|_{\frac{n}{n-\alpha}} = \infty$. For $p = 1$, we then ask if there is a substitute for this inequality. If we restrict the function to a space called a Hardy space $\mathcal{H}^p(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) \ : \ Rf \in L^p(\mathbb{R}^n;\mathbb{R}^n)\}$, then we have the following result by Stein and Weiss [12]. The space $L^p(\mathbb{R}^n;\mathbb{R}^n)$ itself is defined as the space that contains all vector-valued function F such that $|F| \in L^p(\mathbb{R}^n)$. This result will involve whats called a Riesz transform of a function.

Definition 2.4. *The Riesz transformation of f, denoted by Rf, is a vector-valued operator where its component is defined by*

$$R_j f(x) = \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy, \quad j = 1, \ldots, n.$$

Theorem 2.5 ([12]). *Let $0 < \alpha < n$ and $1 \leq p < n/\alpha$. Then there exists a constant $C = C(\alpha, p, n) > 0$ such that*

$$\|I_\alpha f\|_{L^{\frac{np}{n-\alpha p}}(\mathbb{R}^n)} \leq C \left(\|f\|_{L^p(\mathbb{R}^n)} + \|Rf\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \right)$$

for all $f \in \mathcal{H}^p(\mathbb{R}^n)$. Here $\mathcal{H}^p(\mathbb{R}^n)$ denotes the Hardy space.

Following the previous result, Schikorra *et al.* then gives another replacement that is stronger than before by removing the norm of f on the right side of inequality.

Theorem 2.6 ([10]). *Let $n \geq 2$ and $0 < \alpha < n$. Then there exists a constant $C = C(\alpha, n) > 0$ such that*

$$\|I_\alpha f\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \leq C \|Rf\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}$$

for $f \in C_c^\infty(\mathbb{R}^n)$ that satisfy $Rf \in L^1(\mathbb{R}^n;\mathbb{R}^n)$.

The space $C_c^\infty(\mathbb{R}^n)$ means the function that are infinitely differentiable and have compact support. Now we move to the result concerning commutators of fractional integral operators that is generated by a function in a function spaces called BMO.

Definition 2.7. *Function $b \in L^1_{loc}(\mathbb{R}^n)$ is the element of a space called $\mathsf{BMO}(\mathbb{R}^n)$ (Bounded Mean Oscillation) if it satisfy*

$$\|b\|_{\mathsf{BMO}} = \sup_Q m_Q(|f - m_Q(f)|) < \infty$$

with $m_Q(f)$ denoted an average of f over cube Q:

$$m_Q(f) = \frac{1}{|Q|}\int_Q f(y)\,dy$$

The boundedness of commutators of fractional integral operators is given in the following theorem. This can be seen as a L^p-strong type estimate.

Theorem 2.8 ([3, 4])**.** *Let $0 < \alpha < n$, $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. If $b \in \mathsf{BMO}(\mathbb{R}^n)$, then there exists constant $C = C(\alpha, N) > 0$ such that*

$$\|[b, I_\alpha]f\|_{L^q(\mathbb{R}^n)} \leq C\|b\|_{\mathsf{BMO}(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)} \tag{1}$$

for $f \in L^p(\mathbb{R}^n)$.

Later we will try to expand the previous estimate to the domain $p = 1$. Looking at Theorem 2.6, our first guess will be that the norm of f on the right hand side need to be replaced by the norm of Rf. But if we do that, then the estimate will still not hold. Consider the following counterexample. Let $n = 1$, choose $b = \log|2 + x|$ and $f(x) = x$, for $x \in [-1, 1]$ and 0 elsewhere. Then it can be shown that $b \in \mathsf{BMO}$ and $Rf \in L^1$, but $[b, I_\alpha]f \notin L^{n/(n-\alpha)}$.

The same problem also occurs when we look at the weak-type estimate. The commutators $[b, I_\alpha]$ is not bounded from L^1 to $\mathsf{WL}^{n/(n-\alpha)}$ if $b \in \mathsf{BMO}(\mathbb{R}^n)$. In particular, take into account the following counterexample. For $n = 1$ and $0 < \alpha < 1$, choose $b = \log|1 + x| \in \mathsf{BMO}(\mathbb{R})$ and $f(x) = \delta(x)$. Then for all $\lambda > 0$,

$$\begin{aligned}
&\lambda\left|\{x \in \mathbb{R} : |[b, I_\alpha]f(x)| > \lambda\}\right|^{1-\alpha} \\
&= \lambda\left|\left\{x \in \mathbb{R} : (\log|1 + x|)/|x|^{1-\alpha}| > \lambda\right\}\right|^{1-\alpha} \\
&\geq \lambda\left|\left\{x > e : (\log x/x)^{1-\alpha} > \lambda\right\}\right|^{1-\alpha} \to \infty,
\end{aligned}$$

when $\lambda^{1/(1-\alpha)} \to 0$. Instead, the operators $[b, I_\alpha]$ satisfy whats called a weak $L\log L$-estimate.

Theorem 2.9 ([5])**.** *If $b \in \mathsf{BMO}(\mathbb{R}^n)$, $0 < \alpha < n$, $\Phi(t) = t(1 + \log^+ t)$, then there exist $C > 0$ such that for all $\lambda > 0$,*

$$\begin{aligned}
|\{x \in \mathbb{R} : |[b, I_\alpha]f(x)| > \lambda\}|^{1/q} &\leq C\Phi(\Phi(\|b\|_{\mathsf{BMO}}))\|\Phi(|f(\cdot)/\lambda)\|_1 \\
&\quad\times\left(1 + \frac{\alpha}{n}\log^+\|\Phi(|f(\cdot)/\lambda)\|_1\right).
\end{aligned}$$

3. MAIN RESULTS

Our result will try to compensate the failure of Theorem 2.8 for $p = 1$ by changing the domain spaces of function b. We give more restriction to b by assuming b be a function that is essentialy bounded or in other words $b \in L^\infty(\mathbb{R}^n)$.

Theorem 3.1. *Let $n \geq 2,\ 0 < \alpha < n$ and $b \in L^{\infty}(\mathbb{R}^n)$. Then there exist a constant $C = C(\alpha, N) > 0$ such that*

$$\|[b, I_\alpha]f\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \leq C\|b\|_{L^\infty(\mathbb{R}^n)}\|R|f|\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)},$$

for $f \in C_c^\infty(\mathbb{R}^n)$ that satisfy $R|f| \in L^1(\mathbb{R}^n;\mathbb{R}^n)$

Proof. Using the definition of commutators, triangle inequality for norm, and Theorem 2.6, we get

$$\begin{aligned}
\|[b, I_\alpha]f\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} &= \|bI_\alpha f - I_\alpha(bf)\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \\
&\leq \|bI_\alpha f\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} + \|I_\alpha(bf)\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \\
&\leq C\|b\|_{L^\infty(\mathbb{R}^n)}\left(\|I_\alpha f\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} + \|I_\alpha|f|\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)}\right) \\
&\leq C\|b\|_{L^\infty(\mathbb{R}^n)}\|R|f|\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)},
\end{aligned}$$

as desired. □

For the weak-type estimate, we also do a similar thing by assuming b is essentialy bounded. The result is as follows.

Theorem 3.2. *Let $b \in L^\infty(\mathbb{R}^n),\ 0 < \alpha < n$. Then for all $\lambda > 0$*

$$|\{x \in \mathbb{R} : |[b, I_\alpha]f(x)| > \lambda\}| \leq \left(\frac{C}{\lambda}\|b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^1(\mathbb{R}^n)}\right)^{\frac{n}{n-\alpha}}.$$

Proof. Utilizing the definition of commutators, triangle inequality, and the weak-type Hardy-Littlewood-Sobolev inequality, we obtain

$$\begin{aligned}
|\{x \in \mathbb{R} : |[b, I_\alpha]f(x)| > \lambda\}| &= |\{x \in \mathbb{R} : |bI_\alpha f(x) - I_\alpha(bf)(x)| > \lambda\}| \\
&= |\{x \in \mathbb{R} : |bI_\alpha f(x)| + |I_\alpha(bf)(x)| > \lambda\}| \\
&= \left|\left\{x \in \mathbb{R} : 2|I_\alpha(|f|)(x)| > \frac{\lambda}{\|b\|_{L^\infty(\mathbb{R}^n)}}\right\}\right| \\
&\leq C_1\left(\frac{C\|b\|_{L^\infty(\mathbb{R}^n)}\|f\|^{\alpha/n}_{L^1(\mathbb{R}^n)}}{\lambda}\right)^{\frac{n}{n-\alpha}}\|f\|_{L^1(\mathbb{R}^n)} \\
&\leq \left(\frac{C}{\lambda}\|b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^1(\mathbb{R}^n)}\right)^{\frac{n}{n-\alpha}},
\end{aligned}$$

as desired. □

4. CONCLUSION

We have proved a strong and weak-type estimates of commutators of fractional integral operators in Lebesgue space that holds for $p = 1$. For this endpoint estimate to work, we assume that the commutators is generated by an essentially bounded function. The strong-type estimate implies that the norm of commutators of a function will be dominated by the norm of its Riesz transform. The weak-type estimate says that the commutators is bounded from L^1 to $L^{\frac{n}{n-\alpha}}$.

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