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ON THE ADJOINT OF BOUNDED OPERATORS ON A SEMI-INNER PRODUCT SPACE

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Abstract. The notion of semi-inner product (SIP) spaces is a generalization of inner product (IP) spaces notion by reducing the positive definite property of the product to positive semi-definite. As in IP spaces, the existence of an adjoint of a linear operator on a SIP space is guaranteed when the operator is bounded. However, in contrast, a bounded linear operator on SIP space can have more than one adjoint linear operators. In this article we give an alternative proof of those results using the generalized Riesz Representation Theorem in SIP space. Further, the description of all adjoint operators of a bounded linear operator in SIP space is identified.

Key words and Phrases: semi-inner product space, adjoint, bounded operator, Riesz representation theorem

1. INTRODUCTION

Semi-inner product (SIP) space is a generalization of inner product (IP) space. The study of SIP spaces was pioneered by Krein in 1947 [9], and followed by Zaanen in 1950 [16]. In the definition of SIP space, the positive definiteness of IP was generalized into positive semi-definiteness. Hence,the stark difference between SIP space and inner product (IP) space is the present of non-zero, self-orthogonal elements. We call such elements as *neutral elements* of SIP space. SIP space also known as semi-Hilbertian space ([1],[4],[5]) or semi-unitary space [7]. SIP had been applied for the solution of general quadratic programming [14], also as a representation of quiver [7]. Latest studies on SIP showed the generalization of concepts in IP space to SIP space, such as properties of normal operators ([11],[5]), isometry and unitary properties [1], and closed operator [4]; also, geometrical aspects, such

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as metric on projections [2], Birkhoff-James orthogonality ([17],[12]) and numerical radius ([8],[6],[13]).

As in IP space, we can introduce the notion of adjoint operator for a linear operator on SIP space. In the case of finite dimensional IP space, every linear operator has a unique adjoint operator. In contrast, there exists a linear operator on finite dimensional SIP space that does not have an adjoint operator. In addition, adjoint of a linear operator in SIP, if it exists, is not unique. The adjoint is unique if the SIP is IP.

In this paper we study the class of linear operators in finite dimensional SIP space that have adjoint. We discuss several necessary and sufficient conditions for the existence of adjoint operator in finite dimensional SIP space. One of the necessary and sufficient conditions is that the linear operator being bounded. For this purpose, we derive Generalized Riesz Representation Theorem in SIP spaces. Arias et.al [1] gave one distinguished adjoint operator in SIP which lead to the discoveries of some properties of normal operator ([11], [5], [15]). Here, we give the description of all adjoint operators of a bounded linear operator on SIP space. In addition it is shown that the class of bounded linear operators on SIP spaces forms a vector space.

2. BASIC PROPERTIES OF SEMI-INNER PRODUCT SPACES

In this section, we recall the definition of a SIP space [10] and show some of its basic properties.

Definition 2.1. *A Semi-Inner Product space (SIP) is a vector space U over the field $\mathbb{C}$ equipped with a SIP, a mapping $[\cdot,\cdot] : U \times U \to \mathbb{C}$, which satisfies*

(SIP1) symmetrical conjugate: $[x, y] = \overline{[y, x]}$, $\forall x, y \in U$
(SIP2) linearity on first variable:

$$\begin{aligned} [x + y, z] &= [x, z] + [y, z] \\ [\alpha x, z] &= \alpha[x, z]; \forall x, y, z \in U \text{ and } \alpha \in \mathbb{C} \end{aligned}$$

(SIP3) positive semi-definite: $[x, x] \geq 0$, $\forall x \in U$.

Let U denote a IP space with IP defined by $\langle\cdot,\cdot\rangle$ and A be a self-adjoint positive semi-definite linear operator on U. A mapping defined by

$$[x, y]_A = \langle Ax, y\rangle$$

for all $x, y \in U$, will be a SIP [1]. On the other hand, let $[\cdot,\cdot]$ be a SIP on finite dimensional vector space U, then there exists $\langle\cdot,\cdot\rangle$ an IP on U and a self-adjoint positive semi-definite linear operator on U, say A, such that $[x, y] = \langle Ax, y\rangle$, for all $x, y \in U$ [7].

It is easy to see that Cauchy-Schwarz inequality also holds on SIP space, that is for SIP space U, we will have

$$|[x, y]|^2 \leq [x, x][y, y], \ \forall x, y \in U. \tag{1}$$

Every SIP will induce a semi-norm on SIP space, a mapping $\|x\| = [x,x]^{1/2}$, for all $x \in U$. It is easy to show that a semi-norm $\|x\|$ in vector space U will satisfy the following condition.

(SN1) (Positive semi-definite) $\|x\| \geq 0$, $\forall x \in U$;
(SN2) (Homogeneous) $\|cx\| = |c|\|x\|$, $\forall x \in U$ and scalar c;
(SN3) (Triangle inequality) $\|x+y\| \leq \|x\| + \|y\|$, $\forall x, y \in U$.

An element $x \in U$ which satisfies $[x,x] = 0$ is called a *neutral element* of U. Bovdi [7] named the set of all neutral elements of U as the isotropic part.

Definition 2.2. *Let U be a finite dimensional SIP space.* Isotropic part *of U is the subset of U which contains all neutral elements of U, that is*

$$U_0 = \{x \in U \mid [x,x] = 0.\} \tag{2}$$

The isotropic part U_0 is a subspace of U [7]. It is easy to see that if A is a positive semi-definite linear operator on U as an IP space that induces the SIP space U, then $U_0 = N(A)$. Also, for every $x \in U_0$, we have $[x,y] = \langle Ax, y\rangle = \langle 0, y\rangle = 0, \forall y \in U$.

Definition 2.3. *Let U be a SIP space with a SIP $[x,y]_A = \langle Ax, y\rangle$, for all $x, y \in U$ induced from the IP by a semi-definite positive operator A. An element $x \in U$ is said to be A-orthogonal to $y \in U$ relative to SIP $[\cdot,\cdot]_A$ if*

$$[x,y]_A = \langle Ax, y\rangle = 0$$

and we denote $x \perp_A y$. [17]

A set in SIP space is an *orthogonal set* if all its elements are A-orthogonal to each other. A *unit* elements in SIP space is an element which has semi-norm of 1. An *orthonormal set* in SIP space is orthogonal set whose elements are unit elements. Hence, a set $\mathcal{O} = \{x_i \in U | i = 1, 2, \ldots, n\}$ is an orthonormal set if every $x_i, x_j \in \mathcal{O}$ satisfy $[x_i, x_j] = \delta_{i,j}$, where $\delta_{i,j}$ is Kronecker-delta function defined by

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

Let U be a n dimensional SIP space, then there exists a basis of U, where m of its elements are orthonormal and the rest are neutral elements of U [7]. Such basis is called the *m-orthonormal basis* of U.

The Example 1 show a SIP in $\mathbb{C}^2$, its isotropic part, and 1-orthonormal basis.

Example 1

Let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{C}^2,$$

then

$$[x,y] = \overline{y_1} x_1 = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x, y \right\rangle \tag{3}$$

is a SIP in $\mathbb{C}^2$, where $\langle x, y\rangle$ denotes the standard IP in $\mathbb{C}^2$. The isotropic part of $\mathbb{C}^2$ is the set

$$U_0 = \left\{ \begin{bmatrix} 0 \\ s \end{bmatrix} \middle| s \in \mathbb{C} \right\}.$$

We can see that the 1-orthonormal basis of $\mathbb{C}^2$ is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Next is the definition of adjoint operator in SIP space modified from the definition mentioned by Krein [9].

Definition 2.4. *Let U be a SIP space with SIP $[\cdot,\cdot]$ and $\mathcal{T} : U \to U$ is a linear operator. A linear operator $\mathcal{S} : U \to U$ is called adjoint operator of $\mathcal{T}$ if*

$$[\mathcal{T}(x), y] = [x, \mathcal{S}(y)],$$

for all $x, y \in U$.

In the next example, we will show that in a finite dimensional SIP space not all linear operator has adjoint.

Example 2 Let $\mathbb{C}^2$ be a SIP space with SIP defined in Example 1. The operator $\mathcal{S} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is the adjoint of linear operator $\mathcal{T} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ since for every $x, y \in \mathbb{C}^2$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we have

$$[\mathcal{T}(x), y] = \langle A\mathcal{T}(x), y\rangle = \overline{y_1}x_1 = \langle Ax, \mathcal{S}(y)\rangle = [x, \mathcal{S}(y)].$$

However, the operator $\mathcal{R} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ does not have adjoint, since for $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$[\mathcal{R}(x), y] = \langle A\mathcal{R}(x), y\rangle = 1,$$

while every linear operator $\mathcal{Q} : U \to U$ will give

$$[x, \mathcal{Q}(y)] = \langle Ax, \mathcal{Q}(y)\rangle = 0.$$

Definition 2.5. *Let U be a SIP space and $\|\cdot\|$ be the seminorm induced by SIP on U. A linear operator $\mathcal{T} : U \to U$ is called* bounded *if there is a positive real number c that satisfy*

$$\|\mathcal{T}(x)\| \leq c\|x\|, \text{ for all } x \in U. \tag{4}$$

Bovdi et.al. [7] give equivalence conditions of bounded properties in SIP space in the following lemma.

Lemma 2.6. *Let U be a finite dimensional SIP space and U_0 is its isotropic part. Let $\mathcal{T}$ be a linear operator on U. The following statements are equivalent.*

(i) The operator $\mathcal{T}$ is bounded
(ii) The subspace U_0 is $\mathcal{T}$-invariant

(iii) *The matrix of $\mathcal{T}$ in each m-orthonormal basis has the lower block triangular form*

$$T = \begin{bmatrix} S_1 & 0 \\ S_2 & S_3 \end{bmatrix},$$

where S_1 size is $m \times m$

Note that in finite dimensional SIP space, an operator might not be bounded as shown on Example 3.

Example 3

Let $\mathbb{C}^2$ be the SIP space as defined on Example 1, then the operator $\mathcal{T} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is bounded. On the other hand, the operator $\mathcal{S} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is unbounded, since $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ will give

$$\|\mathcal{S}(x)\| = 1 > 0 = c\|x\|,$$

for every $c > 0$.

3. MAIN RESULTS

The main results of this article are concerning adjoint operators on SIP spaces, the existence and description. A necessary and sufficient condition for a linear operator on finite dimensional SIP space has an adjoint is it is bounded. In this section, we show an alternative proof of this result using Generalized Riesz Representation Theorem on SIP spaces. For that, first we will discuss about the class of bounded operators on SIP spaces and a generalization of Riesz Representation Theorem on SIP spaces.

3.1. **Bounded Operators in SIP.**

The next theorem shows that the set bounded linear operator on SIP space forms a vector space.

Theorem 3.1. *Let U be finite dimensional SIP space over field $\mathbb{C}$ and define*

$$\mathcal{B}(U) = \{\mathcal{T} : U \to U | \mathcal{T} \text{ is bounded linear operator on } U\}. \tag{5}$$

Then, $\mathcal{B}(U)$ equipped with addition and action on operators forms a vector space.

Proof. First, it is obvious that the operator 0 is bounded, so $\mathcal{B}(U)$ is not empty. Next, we will show that $\mathcal{B}(U)$ is closed under addition and scalar multiplication. Since $\mathcal{B}(U)$ is the set of all bounded operators, then for $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{B}(U)$, there are $c_1, c_2 > 0$ such that

$$\|\mathcal{T}_1(x)\| \leq c_1\|x\| \text{ and } \|\mathcal{T}_2(x)\| \leq c_2\|x\|$$

for all $x \in U$. We can see that

$$\begin{aligned}\|(\mathcal{T}_1 + \mathcal{T}_2)(x)\| &= \|\mathcal{T}_1(x) + \mathcal{T}_2(x)\| \\ &\leq \|\mathcal{T}_1(x)\| + \|\mathcal{T}_2(x)\| \\ &\leq c_1\|x\| + c_2\|x\| \\ &\leq (c_1 + c_2)\|x\|.\end{aligned}$$

Thus, $\mathcal{B}(U)$ is closed under addition. Also, for all $\alpha \in \mathbb{C}$

$$\begin{aligned}\|(\alpha\mathcal{T}_1)(x)\| &= |\alpha|\|\mathcal{T}_1(x)\| \\ &\leq |\alpha|c_1\|\mathcal{T}_1(x)\|,\end{aligned}$$

so $\mathcal{B}(U)$ is closed under scalar multiplication. It is easy to see that for every $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \in \mathcal{B}(U)$, we have for every $x \in U$

$$(\mathcal{T}_1 + \mathcal{T}_2)(x) = (\mathcal{T}_2 + \mathcal{T}_1)(x)$$

and

$$((\mathcal{T}_1 + \mathcal{T}_2) + \mathcal{T}_3)(x) = (\mathcal{T}_1 + (\mathcal{T}_2 + \mathcal{T}_3))(x).$$

So, $\mathcal{T}_1 + \mathcal{T}_2 = \mathcal{T}_2 + \mathcal{T}_1$ and $(\mathcal{T}_1 + \mathcal{T}_2) + \mathcal{T} = \mathcal{T}_1 + (\mathcal{T}_2 + \mathcal{T}_3)$. We also have the zero operator $\mathcal{O}$ acting as identity element under addition in $\mathcal{B}(U)$ and for every $\mathcal{T}_1 \in \mathcal{B}(U)$ there is $-\mathcal{T}_1$ such that $\mathcal{T}_1 + (-\mathcal{T}_1) = \mathcal{O} = -\mathcal{T}_1 + \mathcal{T}_1$.

It is also clear that for all $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{B}(U)$ and $\alpha, \beta \in \mathbb{C}$, we have

- $\alpha(\mathcal{T}_1 + \mathcal{T}_2) = \alpha\mathcal{T}_1 + \alpha\mathcal{T}_1 \in \mathcal{B}(U)$;
- $(\alpha + \beta)\mathcal{T}_1 = \alpha\mathcal{T}_1 + \beta\mathcal{T}_1 \in \mathcal{B}(U)$;
- $(\alpha\beta)\mathcal{T}_1 = \alpha(\beta\mathcal{T}_1) \in \mathcal{B}(U)$.

Also, $1\mathcal{T}_1 = \mathcal{T}_1 \in \mathcal{B}(U)$. Hence, $\mathcal{B}(U)$ is a vector space and a dual space of U.

Next, we show that the bounded operators of SIP space is closed under composition.

Corollary 3.2. *Let U be finite dimensional SIP space over $\mathbb{C}$ and let $\mathcal{B}(U)$ as defined on (5). If $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{B}(U)$, then $\mathcal{T}_1 \circ \mathcal{T}_2$ is also in $\mathcal{B}(U)$.*

Proof. Since $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{B}(U)$, then for all $x \in U$ there are $c_1, c_2 \in \mathbb{C}$ such that

$$\|\mathcal{T}_1(x)\| \leq |c_1|\|x\|$$

and

$$\|\mathcal{T}_2(x)\| \leq |c_2|\|x\|.$$

Now take $x \in U$, then we have

$$\begin{aligned}\|\mathcal{T}_1 \circ \mathcal{T}_2(x)\| &= \|\mathcal{T}_1(\mathcal{T}_2(x))\| \\ &\leq |c_1|\|\mathcal{T}_2(x)\| \\ &\leq |c|\|x\|,\end{aligned}$$

where $c = c_1c_2$ Using composition of operators as multiplication in $\mathcal{B}(U)$ and identity operator $\mathcal{T}(x) = x$, $\forall x \in U$, it is easy to see that $\mathcal{B}(U)$ forms an algebra.

Lemma 3.3. *Let U be a finite dimensional SIP space with U_0 be the isotropic part of U. Let $\mathcal{N}(U) = \{\mathcal{T} \in \mathcal{B}(U) | R(\mathcal{T}) \subseteq U_0\}$. Then $\mathcal{N}(U)$ is a subspace of $\mathcal{B}(U)$.*

The proof of this lemma is quite straightforward using Cauchy-Schwarz inequality.

3.2. **Adjoint operators in SIP.** We start with the following lemma.

Lemma 3.4. *Let U be a SIP space with the SIP $[-,-]$ and let U_0 be the isotropic part of U, then U/U_0 can be formed into inner product space induced by the SIP $[-,-]$.*

Proof. Let us define $\langle \hat{x}, \hat{y} \rangle = [x, y]$, $\forall \hat{x}, \hat{y} \in U/U_0$, where

$$\hat{x} = x + U_0 \text{ and } \hat{y} = y + U_0,\ x, y \in U.$$

First, we show that $\langle \hat{x}, \hat{y} \rangle$ is well-defined. Take any $\hat{x_1}, \hat{x_2}, \hat{y_2}, \hat{y_2} \in U/U_0$, where $\hat{x_1} = \hat{x_2}$ and $\hat{y_1} = \hat{y_2}$. Let $x_1 - x_2 = u_1 \in U_0$ and $y_1 - y_2 = u_2 \in U_0$, then we have

$$\begin{aligned}
\langle \hat{x_1}, \hat{y_1} \rangle &= [x_1, y_1] = [x_2 + u_1, y_2 + u_2] \\
&= [x_2, y_2] + [u_1, y_2] + [x_2, u_2] + [u_1, u_2] \\
&= [x_2, y_2] + 0 + 0 + 0 \\
&= [x_2, y_2] = \langle \hat{x_2}, \hat{y_2} \rangle.
\end{aligned}$$

Next, we show that $\langle \hat{x}, \hat{y} \rangle$ is an inner product.
From the definition of SIP, $\langle \hat{x}, \hat{y} \rangle = [x, y]$ will satisfy (SIP1), (SIP2), and (SIP3). Since $\hat{0} = U_0$, for $x \in U_0$, we have $\langle \hat{0}, \hat{0} \rangle = [x, x] = 0$. Thus, we need to show that $\langle \hat{x}, \hat{x} \rangle = 0$ is causing $\hat{x} = \hat{0}$.
Let $\hat{x} = x + U_0$ for some $x \in U$ satisfying $\langle \hat{x}, \hat{x} \rangle = 0$. Then, we have

$$0 = \langle \hat{x}, \hat{x} \rangle = [x, x].$$

Therefore, x must be an element of U_0, so that $\hat{x} = U_0 = \hat{0}$. Hence, it has been proven that $\langle \hat{x}, \hat{y} \rangle$ is an inner product and U/U_0 is inner product space.

One method of constructing the adjoint operator on IP space is by employing Riesz Representation Theorem. Here is an extension of Riesz Representation Theorem in the context of SIP space.

Theorem 3.5. *Let U be finite dimensional SIP space over $\mathbb{C}$ and let $\mathcal{F} : U \to \mathbb{C}$ be a linear functional. The linear functional $\mathcal{F}$ is bounded if and only if there is $a \in U$ so that $\mathcal{F}(x) = [x, a]$ for all $x \in U$.*

Proof. It is easy to show that the mapping that is defined by $\mathcal{F}(x) = [x, a]$ for all $x \in U$, for some $a \in U$, is a bounded linear functional. Hence, we obtain the theorem if we can show the implication from right to left. Let $\mathcal{F}$ be any bounded linear functional on U and U_0 is the isotropic part of U. Let $c > 0$ such that $|\mathcal{F}(x)| \leq c\|x\|$ for all $x \in U$. First, we show $U_0 \subseteq N(\mathcal{F})$. Let $x \in U_0$, then we have $|\mathcal{F}(x)| \leq c\|x\| = 0$, since $x \in U_0$. Hence, $|\mathcal{F}(x)| = 0$, which implies $\mathcal{F} = 0$

or $x \in N(\mathcal{F})$. Thus, we have $U_0 \subseteq N(\mathcal{F})$. As a result, $\mathcal{F}$ induces the well defined mapping

$$\begin{aligned}\hat{\mathcal{F}} : U/U_0 &\to \mathbb{C}\\ \hat{x} &\mapsto \mathcal{F}(x).\end{aligned}$$

Linearity property of $\hat{\mathcal{F}}$ implies $\hat{\mathcal{F}}$ is a linear functional on U/U_0 which is a finite dimensional IP space, a finite dimensional Hilbert space. According to Riesz Representation Theorem, there exists $\hat{a} \in U/U_0$ such that $\hat{\mathcal{F}}(\hat{x}) = \langle \hat{x}, \hat{a} \rangle$ for all $\hat{x} \in U/U_0$. As a result, there exists $a \in U$ such that for all $x \in U$

$$\mathcal{F}(x) = \hat{\mathcal{F}}(\hat{x}) = \langle \hat{x}, \hat{a} \rangle = [x, a].$$

Remark: Theorem 3.5 guarantees the existence of a vector $a \in U$ that is representing the bounded functional $\mathcal{F}$. The theorem does not guarantee the uniqueness of such representation. It is indeed, in the context of SIP spaces, Riesz vectors that representing a linear functional is not unique. The following explains a counter example of such claim.

The following is an example of an application of the Theorem 3.5.

Example 4 Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ induced the SIP in $\mathbb{C}^2$ as shown on Example 1. Every bounded linear functional on $\mathbb{C}^2$ has the form of

$$\mathcal{F}(x) = [x, a] = \overline{\alpha} x_1.$$

for some $a = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$, where $\alpha \in \mathbb{C}$.
On the other hand, for any $u_0 \in U_0$, we also have

$$\mathcal{F}(x) = \overline{\alpha} x_1 = [x, a + u_0].$$

We can also see that for the unbounded operator $\mathcal{G}(x) = x_2$, since $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ will cause $[x, a] = 0$ for every $a \in \mathbb{C}^2$.

The next theorem states the necessary condition for the existence of adjoint operators in SIP space.

Theorem 3.6. *Let U be finite dimensional SIP space, with SIP $[-, -]_A$ induced by positive semi-definite operator A, and U_0 is the isotropic part of U. If $\mathcal{T} : U \to U$ is a bounded linear operator, then there is linear operator $\mathcal{S} : U \to U$ a A-adjoin operator of $\mathcal{T}$. Moreover, if $\mathcal{S}, \mathcal{R}$ are two adjoints of $\mathcal{T}$, then $\mathcal{S} - \mathcal{R} \in \mathcal{N}(U)$. Hence, the set of all adjoint of $\mathcal{T}$ is*

$$\mathcal{S} + \mathcal{N}(U) = \{\mathcal{S} + \mathcal{R} | \mathcal{R} \in \mathcal{N}(U)\}.$$

PROOF. First, take B a basis of U. For all $b \in B$,we define the linear functional

$$\begin{aligned}\mathcal{F}_b : U &\to \mathbb{C}\\ u &\mapsto [\mathcal{T}(u), b]_A.\end{aligned}$$

since $\mathcal{T}$ is bounded, there exists $c > 0$ such that for all $u \in U$ we have

$$\|\mathcal{T}(u)\| \leq c\|u\|$$

and for all $u \in U$, we also have

$$|\mathcal{F}_b(u)| = |[\mathcal{T}(u), b]_A| \leq \|\mathcal{T}(u)\|\|b\| = c\|u\|\|b\| = c\|b\|\|u\|.$$

Hence, $\mathcal{F}_b$ is bounded. From Theorem 3.5 there exists $a \in U$ such that $\mathcal{F}_b(u) = [u, a]_A$ for all $u \in U$. However, a is not unique, since every element of coset $a + U_0$ satisfy Theorem 3.5. Let us select one $u_b \in a + U_0$ and define

$$\begin{aligned} \lambda : & B \to U \\ & b \mapsto u_b. \end{aligned}$$

We have $[\mathcal{T}(u), b]_A = [u, u_b]_A$. Since $b \in B$ is an arbitrary element of the basis of U, it implies λ can be expanded into linear operator on U using the linear combination of the elements of B. That is,

$$\begin{aligned} \mathcal{S} : U &\to U \\ \sum_{b \in B} \alpha_b b &\mapsto \sum_{b \in B} \alpha_b u_b. \end{aligned}$$

Therefore, for each $u, v \in U$, where $v = \sum_{b \in B} \alpha_b b$, we get

$$\begin{aligned} [\mathcal{T}(u), v]_A &= [\mathcal{T}(u), \sum_{b \in B} \alpha_b b]_A \\ &= \sum_{b \in B} \overline{\alpha_b} [\mathcal{T}(u), b]_A \\ &= \sum_{b \in B} \overline{\alpha_b} [u, u_b]_A \\ &= [u, \sum_{b \in B} \alpha_b u_b]_A = [u, \mathcal{S}(v)]_A. \end{aligned}$$

We see that to construct $\mathcal{S}$, we map b into $u_b \in a + U_0$. Suppose we map b into $v_b \in a + U_0$ to construct $\mathcal{R}$ another adjoint of $\mathcal{T}$. Thus, $S - T$ will map x into a linear combination of $u_b - v_b \in U_0$. Also, it is quite easy to see that $\mathcal{S} + \mathcal{R}$ is also adjoint operator of $\mathcal{T}$, for every $\mathcal{R} \in \mathcal{N}(U)$.

Ahmed and Saddi [3] showed one distinguished adjoint of operator T on $U = \mathbb{C}^{n \times n}$, that is $T^\sharp = A^\dagger T^* A$, where $A^\dagger$ is the Moore-Penrose inverse of A and T^* is the conjugate transpose of T. However, in the proof in Theorem 3.6 we show that the construction of the adjoint operator and that the adjoint in SIP space may not unique since all elements of the coset are also adjoint operator. The adjoint operator will be unique if the SIP is also IP, since the $\mathcal{N}(U)$ only has zero operator as its element.

In the following, we show an example of adjoint construction in SIP space.

Example 5 Let $U = \mathbb{C}^2$ be a SIP space with SIP defined in Example 1 and 4.

Also, let $\mathcal{T} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ be an operator on $\mathbb{C}^2$. Then, the operator

$$\mathcal{S} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is an adjoint of $\mathcal{T}$.
Also, every $\hat{\mathcal{S}}$ defined by

$$\hat{\mathcal{S}} = \begin{bmatrix} 1 & 0 \\ \alpha & \beta \end{bmatrix}$$

is also adjoint of $\mathcal{T}$ and the matrix $\mathcal{R} = \begin{bmatrix} 0 & 0 \\ \alpha & \beta \end{bmatrix} \in \mathcal{N}(U)$

On the following theorem, we show that boundedness of the operator is the necessary and sufficient condition for the existence of the adjoint operator. Moreover, the adjoint operator is also bounded.

Theorem 3.7. *Let U be a finite dimensional SIP space and A a linear operator on U.*

(i) $\mathcal{T}$ has adjoints if and only if $\mathcal{T}$ is bounded;
(ii) If $\mathcal{T}$ is a bounded linear operator and $\mathcal{S}$ is a adjoint of $\mathcal{T}$, then $\mathcal{S}$ is also bounded.

PROOF. (i) It is obvious from theorem 3.6 that if $\mathcal{T}$ is bounded, then $\mathcal{T}$ has adjoint operators. Thus, we still need to show the implication from left to right. Suppose that $\mathcal{T}$ unbounded, then Lemma 2.6 states that U_0 is not A-invariant, i.e. there exists $x_0 \in U_0$, $x_0 \neq 0$ such that $\mathcal{T}(x_0) \notin U_0$. Let $y_0 = \mathcal{T}(x_0) \neq 0$, then $[y_0, y_0] \neq 0$. Take $\mathcal{S}$ any linear operator on U, then

$$\begin{aligned} [\mathcal{T}(x_0), y_0] &= [y_0, y_0] \neq 0 \\ &= [x_0, \mathcal{S}(y_0)], (\text{ since } x_0 \in U_0). \end{aligned}$$

Hence, there exist $x_0, y_0 \in U$ so that every linear operator $\mathcal{S}$ in U will made $[\mathcal{T}(x_0), y_0] \neq [x_0, \mathcal{S}(y_0)]$. Therefore, $\mathcal{T}$ must be bounded.

(ii) Let $\mathcal{T}$ be a bounded linear operator on U and $\mathcal{S}$ is the adjoint of $\mathcal{T}$. We will show that U_0 is $\mathcal{S}$-invariant. Since $\mathcal{T}$ is bounded, then U_0 is $\mathcal{T}$-invariant. Let $x \in U_0$ such that $\mathcal{T}(x) \in U_0$. For all $y \in U$, we have

$$[y, \mathcal{S}(x)] = [\mathcal{T}(y), x] = 0.$$

Choose $y = \mathcal{S}(x)$, so we have $[\mathcal{S}(x), \mathcal{S}(x)] = 0$ and $\mathcal{S}(x) \in U_0$. Therefore, U_0 is $\mathcal{S}$-invariant and, from Lemma 2.6, $\mathcal{S}$ is bounded.

4. CONCLUSION

In this paper, we have showed generalizations of several properties of IP spaces on SIP spaces. We have showed that the bounded operators on SIP space forms an algebra under addition and composition of operators. We also gave the Riesz representation theorem in SIP space, where the vector representing the bounded functional will be unique if the SIP is also IP. The boundedness of an operator is the necessary and sufficient condition for the existence of its adjoint operator. The adjoint operator is bounded and will be unique if the SIP is also IP. Recent studies using the adjoint operator on SIP space mostly focused on the distinguished adjoint $T^\sharp$, here we have shown the identification of all adjoint operator, which hopefully will be useful for future studies, particularly the studies concerning self-adjoint operators and normal operators on SIP spaces.

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