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REVISITING KANTOROVICH OPERATORS IN LEBESGUE SPACES

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Abstract. According to the Weierstrass Approximation Theorem, any continuous function on the closed and bounded interval can be approximated by polynomials. A constructive proof of this theorem uses the so-called Bernstein polynomials. For the approximation of integrable functions, we may consider Kantorovich operators as certain modifications for Bernstein polynomials. In this paper, we investigate the behaviour of Kantorovich operators in Lebesgue spaces. We first give an alternative proof of the uniform boundedness of Kantorovich operators in Lebesgue spaces by using the Riesz-Thorin Interpolation Theorem. In addition, we examine the convergence of Kantorovich operators in the space of essentially bounded functions. We also give an example related to the rate of convergence of Kantorovich operators in a subspace of Lebesgue spaces.

Key words and Phrases: Kantorovich operators, Bernstein Polynomials, Lebesgue spaces, interpolation of linear operators

1. INTRODUCTION

According to the Weierstrass Approximation Theorem, any continuous function on the closed and bounded interval can be approximated by polynomials. Bernstein [2] provided a constructive proof for this theorem by using the Bernstein polynomials.

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Definition 1.1 (Bernstein Polynomials). *Let $f : [0,1] \to \mathbb{R}$ be a bounded function and $n \in \mathbb{N}$. The Bernstein polynomial of order n of f is defined by*

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x), \quad x \in [0,1] \tag{1}$$

with $b_{n,k} = \binom{n}{k} x^k (1-x)^{n-k}$.

The uniform convergence of Bernstein polynomials of continuous functions is stated as follows.

Theorem 1.2 (Bernstein [2]). *Let f be continuous on $[0,1]$, then $\{B_n f\}$ converges uniformly to f on $[0,1]$.*

The reader may notice that the term

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

is none other than the probability of k successes from n Bernoulli trials with probability of success x. That is, $b_{n,k}(x)$ is the probability mass function of the binomial distribution. As such, we get the following proposition.

Proposition 1.3. *For $n \in \mathbb{N}$, $k = 0, 1, 2, \ldots, n$, and $x \in [0,1]$ we have*

$$\sum_{k=0}^{n} b_{n,k}(x) = 1, \quad \sum_{k=0}^{n} k b_{n,k}(x) = nx, \quad \sum_{k=0}^{n} k^2 b_{n,k}(x) = n(n-1)x^2 + nx.$$

However, the Bernstein polynomials are not suitable for approximating discontinuous functions. Consider the Dirichlet function $f(x) = \chi_{[0,1]\cap\mathbb{Q}}$. This function is discontinuous for all $x \in [0,1]$. In this case, we have $f\left(\frac{k}{n}\right) = 1$ for all $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$ since $\frac{k}{n} \in [0,1] \cap \mathbb{Q}$. As a result,

$$B_n f(x) = \sum_{k=0}^{n} b_{n,k}(x) = 1, \quad x \in [0,1].$$

Therefore, $|B_n f(x) - f(x)| = 1$ for all $x \in [0,1] \setminus \mathbb{Q}$ and $\{B_n f(x)\}$ does not converge to $f(x)$ for all $x \in [0,1] \setminus \mathbb{Q}$. To address this problem, we can use an alternative to Bernstein polynomials for Lebesgue integrable functions on $[0,1]$ that is the Kantorovich operators provided by Kantorovich [4].

Definition 1.4 (Kantorovich Operators). *Let $f : [0,1] \to \mathbb{R}$ be a Lebesgue integrable function and $n \in \mathbb{N}$. The Kantorovich operator of order n of f is defined by*

$$K_n f(x) := \sum_{k=0}^{n} b_{n,k}(x)(n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\, dt, \quad x \in [0,1]$$

with $b_{n,k} = \binom{n}{k} x^k (1-x)^{n-k}$.

Note that the Kantorovich operators and the Bernstein polynomials are related by the identity

$$K_nf(x) = \frac{d}{dx}\, B_{n+1}F(x)$$

where $F(x) := \int_0^x f(t)\,dt$ is the antiderivative of some function $f \in L^1([0,1])$.

It is known that Kantorovich operators are uniformly bounded on Lebesgue spaces with norm 1 (see Lorentz [5, pages 30-33] and Theorem 3.1). Moreover, for finite values of p, $\{K_nf\}$ converges to f in $L^p([0,1])$ (see Lorentz [5]). Here, $L^p([0,1])$ denotes the set of all measurable functions f for which $|f|^p$ is integrable on $[0,1]$. In this paper, we are interested in the study of uniform boundedness of Kantorovich operators and their convergence in Lebesgue spaces. In particular, we will give an alternative proof of the uniform boundedness of these operators in Lebesgue spaces. In addition, we will disprove the convergence of Kantorovich operators in the spaces of essentially bounded functions. We also discuss the rate of convergence of Kantorovich operators in $L^p([0,1])$ for $1 < p < \infty$.

The rest of the paper is organized as follows. Some basic properties of Kantorovich operators is given in the next section. The main results of this paper are given in Section 3. The first result is an alternative proof of the uniform boundedness of Kantorovich operators in Lebesgue spaces by using interpolation of linear operators (see Theorem 3.1). The second result is a counterexample for the convergence of Kantorovich operators in $L^\infty([0,1])$ (see Proposition 3.4). Our last result is an example of the exact rate of convergence of Kantorovich operators in Lebesgue spaces (see Example 3.7). We conclude the paper by the summary of our results and future works.

2. PRELIMINARIES

Let us recall several basic properties of Kantorovich operators. The following proposition gives a few examples of Kantorovich operators of elementary functions.

Proposition 2.1. *Let $f_0(x) = 1, f_1(x) = x, f_2(x) = x^2$ for $x \in [0,1]$. We have*

$$\begin{aligned}
K_nf_0(x) &= 1,\\
K_nf_1(x) &= \frac{1}{n+1}\left(nx + \frac{1}{2}\right),\\
K_nf_2(x) &= \frac{1}{(n+1)^2}\left(n(n-1)x^2 + 2nx + \frac{1}{3}\right).
\end{aligned}$$

PROOF. Let $n \in \mathbb{N}$ and $x \in [0,1]$. For $k = 0, 1, \ldots, n$ we have

$$\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_0(t)\,dt = \frac{k+1}{n+1} - \frac{k}{n+1} = \frac{1}{n+1},$$
$$\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_1(t)\,dt = \frac{1}{2}\left[\left(\frac{k+1}{n+1}\right)^2 - \left(\frac{k}{n+1}\right)^2\right] = \frac{1}{(n+1)^2}\left(k + \frac{1}{2}\right),$$
$$\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_2(t)\,dt = \frac{1}{3}\left[\left(\frac{k+1}{n+1}\right)^3 - \left(\frac{k}{n+1}\right)^3\right] = \frac{1}{(n+1)^3}\left(k^2 + k + \frac{1}{3}\right).$$

By Proposition 1.3, we get

$$K_n f_0(x) = \sum_{k=0}^{n} b_{n,k}(x) = 1,$$
$$K_n f_1(x) = \frac{1}{n+1}\sum_{k=0}^{n}\left(k + \frac{1}{2}\right) b_{n,k}(x) = \frac{1}{n+1}\left(nx + \frac{1}{2}\right),$$
$$\begin{aligned} K_n f_2(x) &= \frac{1}{(n+1)^2}\sum_{k=0}^{n}\left(k^2 + k + \frac{1}{3}\right) b_{n,k}(x) \\ &= \frac{1}{(n+1)^2}\left(n(n-1)x^2 + 2nx + \frac{1}{3}\right). \end{aligned}$$

□

Observe that K_n can be written as

$$K_n f(x) = \int_0^1 \sigma_n(x,t) f(t)\,dt$$

where f is a Lebesgue integrable function on $[0,1]$ and

$$\sigma_n(x,t) := \sum_{k=0}^{n} (n+1) b_{n,k}(x) \chi_{I_k}(t), \quad x, t \in [0,1]$$

with $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $I_k = \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right)$ for $k = 0, 1, 2, \ldots, n-1$, and $I_n = \left[\frac{n}{n+1}, 1\right]$. In other words, K_n is an integral operator with kernel function σ_n. Note that clearly $\sigma_n(x,t) \geq 0$ for all $x, t \in [0,1]$. This fact, together with the following lemma, will be used to prove our first main result.

Lemma 2.2 (Lorentz [5])**.** *For all $n \in \mathbb{N}$ we have*

$$\int_0^1 \sigma_n(x,t)\,dt = 1, \quad x \in [0,1],$$
$$\int_0^1 \sigma_n(x,t)\,dx = 1, \quad t \in [0,1].$$

Proof. Let $x \in [0,1]$. Since $\displaystyle\int_0^1 \chi_{I_k}(t)\,dt = \frac{1}{n+1}$ for $k = 0,1,2,\dots,n$, we have

$$\begin{aligned}\int_0^1 \sigma_n(x,t)\,dt &= \sum_{k=0}^{n}(n+1)b_{n,k}(x)\int_0^1 \chi_{I_k}(t)\,dt\\ &= \sum_{k=0}^{n} b_{n,k}(x) = 1.\end{aligned}$$

On the other hand, let $t \in [0,1]$. By using integration by parts and mathematical induction, we get $\displaystyle\int_0^1 b_{n,k}(x)\,dx = \frac{1}{n+1}$, hence

$$\begin{aligned}\int_0^1 \sigma_n(x,t)\,dx &= \sum_{k=0}^{n}(n+1)\chi_{I_k}\int_0^1 b_{n,k}(x)\,dx\\ &= \sum_{k=0}^{n}\chi_{I_k}(t)\\ &= \chi_{[0,1]}(t) = 1.\end{aligned}$$

Thus, $\displaystyle\int_0^1 \sigma_n(x,t)\,dx = 1$ for $t \in [0,1]$. □

3. MAIN RESULTS

We first state the uniform boundedness of Kantorovich operators in Lebesgue spaces.

Theorem 3.1 (Lorentz [5]). *Let $1 \le p \le \infty$ and $n \in \mathbb{N}$. If $f \in L^p([0,1])$, then $\|K_n f\|_{L^p([0,1])} \le \|f\|_{L^p([0,1])}$.*

We will give an alternative proof of this theorem by using the following interpolation theorem.

Theorem 3.2 (Riesz-Thorin Interpolation Theorem). *Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ and let E be a measurable subset of $\mathbb{R}$. Suppose that T is a linear mapping from $L^{p_0}(E) + L^{p_1}(E)$ to $L^{q_0}(E) + L^{q_1}(E)$. Assume that there exist $\alpha_0, \alpha_1 > 0$ such that*

$$\|T(f)\|_{L^{q_0}([0,1])} \le \alpha_0 \|f\|_{L^{p_0}([0,1])} \quad \textit{and} \quad \|T(f)\|_{L^{q_1}([0,1])} \le \alpha_1 \|f\|_{L^{p_1}([0,1])}.$$

Then there exists $\alpha > 0$ such that

$$\|T(f)\|_{L^{q}([0,1])} \le \alpha \|f\|_{L^{p}([0,1])}$$

with

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

for some $t \in [0,1]$. Moreover, we have $\alpha \le \alpha_0^{1-t}\alpha_1^{t}$.

We are now ready to prove Theorem 3.1.
Proof of Theorem 3.1. For $f \in L^p([0,1])$ and $x \in [0,1]$ we have

$$|K_n f(x)| = \left| \int_0^1 \sigma_n(x,t) f(t)\, dt \right| \le \int_0^1 \sigma_n(x,t)|f(t)|\, dt.$$

As a result, $|K_n f(x)| \le \|f\|_{L^\infty([0,1])} \displaystyle\int_0^1 \sigma_n(x,t)\, dt = \|f\|_{L^\infty([0,1])}$, hence the boundedness holds for $p = \infty$. Next, the boundedness for $p = 1$ is obtained by switching order of integration as follows:

$$\begin{aligned} \|K_n f\|_{L^1([0,1])} &= \int_0^1 |K_n f(x)|\, dx \le \int_0^1 \int_0^1 \sigma_n(x,t)|f(t)|\, dt\, dx \\ &= \int_0^1 |f(t)| \left(\int_0^1 \sigma_n(x,t)\, dx \right) dt = \|f\|_{L^1([0,1])}. \end{aligned}$$

Finally, the boundedness for $1 < p < \infty$ is obtained by choosing $p_0 = q_0 = \infty, p_1 = q_1 = 1$, and $t = \frac{1}{p}$. Therefore, according to Theorem 3.2 we have

$$\|K_n f\|_{L^p([0,1])} \le \|f\|_{L^p([0,1])}.$$

$\square$

Remark. By Theorem 3.1, for $1 \le p \le \infty$ and $n \in \mathbb{N}$, we have the operator norm of Kantorovich operators satisfy the inequality

$$\|K_n\|_{L^p([0,1]) \to L^p([0,1])} = \sup_{f \neq 0} \frac{\|K_n f\|_{L^p([0,1])}}{\|f\|_{L^p([0,1])}} \le 1.$$

Recall that, for $f_0(x) = 1$, we have $K_n f(x) = 1$ for all $n \in \mathbb{N}$ (Proposition 2.1), hence $\dfrac{\|K_n f_0\|_{L^p([0,1])}}{\|f_0\|_{L^p([0,1])}} = 1$. Therefore, we obtain the operator norm of Kantorovich operators,

$$\|K_n\|_{L^p([0,1]) \to L^p([0,1])} = \sup_{f \neq 0} \frac{\|K_n f\|_{L^p([0,1])}}{\|f\|_{L^p([0,1])}} = 1.$$

Note that, the classical proof of Theorem 3.1 relies on the definition of the norm of Lebesgue spaces. The method used in this alternative proof might be of use when investigating the boundedness of Kantorovich operators in other function spaces.

We now consider the convergence of Kantorovich operators in Lebesgue spaces. It is known that Kantorovich polynomials $\{K_n f\}$ converge to f in $L^p([0,1])$ for every $1 \le p < \infty$ [5, pages 30-33]. We show that this convergence result does not hold in $L^\infty([0,1])$ by providing the following counterexample. We first prove the following lemma.

Lemma 3.3. *Let* $f(x) = \chi_{\left[\frac{1}{2},1\right]}(x)$ *for* $x \in [0,1]$. *Then* $K_n f\left(\frac{1}{2}\right) = \frac{1}{2}$ *for all* $n \in \mathbb{N}$.

Proof. Observe that $b_{n,k}\left(\frac{1}{2}\right) = \binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k} = \binom{n}{k}\frac{1}{2^n}$. Then,

$$\begin{aligned} K_n f\left(\frac{1}{2}\right) &= (n+1)\sum_{k=0}^{n} b_{n,k}\left(\frac{1}{2}\right)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt \\ &= \frac{n+1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt. \end{aligned}$$

If n is odd, for $k = 0, 1, \ldots, \frac{1}{2}n - \frac{1}{2}$ we have

$$\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt = \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} 0\,dt = 0,$$

while for $k = \frac{1}{2}n + \frac{1}{2}, \ldots, n$ we have

$$\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt = \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} 1\,dt = \frac{1}{n+1}.$$

As a result,

$$\begin{aligned} K_n f\left(\frac{1}{2}\right) &= \frac{n+1}{2^n}\sum_{k=\frac{n+1}{2}}^{n}\binom{n}{k}\frac{1}{n+1} \\ &= \frac{1}{2^n}\sum_{k=\frac{n+1}{2}}^{n}\binom{n}{k} \\ &= \frac{1}{2^n}\left(\frac{1}{2}\cdot 2^n\right) = \frac{1}{2}. \end{aligned}$$

On the other hand, if n is even, for $k = 0, 1, \ldots, \frac{n}{2} - 1$ we have

$$\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt = \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} 0\,dt = 0,$$

for $k = \frac{n}{2} + 1, \ldots, n$ we have

$$\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt = \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} 1\,dt = \frac{1}{n+1},$$

and for $k = \frac{n}{2}$, we have $\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt = \dfrac{1}{2(n+1)}$. As a result,

$$\begin{aligned} K_n f\left(\frac{1}{2}\right) &= \frac{n+1}{2^n}\left[\frac{1}{2(n+1)}\binom{n}{n/2} + \sum_{k=\frac{n}{2}+1}^{n}\binom{n}{k}\frac{1}{n+1}\right] \\ &= \frac{1}{2^n}\left[\frac{1}{2}\binom{n}{n/2} + \sum_{k=\frac{n}{2}+1}^{n}\binom{n}{k}\right] \\ &= \frac{1}{2^n}\left(\frac{1}{2}\cdot 2^n\right) = \frac{1}{2}. \end{aligned}$$

Hence, $K_n f\left(\frac{1}{2}\right) = \frac{1}{2}$ for all $n \in \mathbb{N}$. □

Our counterexample for the convergence of $\{K_n f\}$ in $L^\infty([0,1])$ is given as follows.

Proposition 3.4. *Let $f(x) = \chi_{[\frac{1}{2},1]}(x)$ for $x \in [0,1]$. Then $\{K_n f\}$ does not converge to f in $L^\infty([0,1])$.*

PROOF. Let n be a positive integer. By the continuity of $K_n f$, we can choose $0 < \delta_n < \frac{1}{2}$ such that for all $x \in \left(\frac{1}{2} - \delta_n, \frac{1}{2} + \delta_n\right)$ we have

$$\left| K_n f(x) - \frac{1}{2} \right| = \left| K_n f(x) - K_n f\left(\frac{1}{2}\right) \right| < \frac{1}{4}.$$

Observe that if $x \in \left(\frac{1}{2} - \delta_n, \frac{1}{2}\right)$, then $f(x) = 0$ and $\frac{1}{4} < K_n f(x) < \frac{3}{4}$, hence

$$|K_n f(x) - f(x)| = |K_n f(x) - 0| = |K_n f(x)| > \frac{1}{4}.$$

As a result, we have

$$\begin{aligned} \|K_n f - f\|_{L^\infty([0,1])} &= \operatorname*{ess\,sup}_{x\in[0,1]} |K_n f(x) - f(x)| \\ &\geq \operatorname*{ess\,sup}_{x\in\left(\frac{1}{2}-\delta_n,\frac{1}{2}\right)} |K_n f(x) - f(x)| \\ &> \frac{1}{4}. \end{aligned}$$

Since $\|K_n f - f\|_{L^\infty([0,1])} > \frac{1}{4}$ for all $n \in \mathbb{N}$, $\{K_n f\}$ does not converge to f in $L^\infty([0,1])$. □

REMARK. Recall that $L^{p_1}([0,1]) \subseteq L^{p_2}([0,1])$ whenever $p_1 \geq p_2$. In particular, $L^\infty([0,1]) \subseteq L^p([0,1])$ for every finite p. Therefore, the convergence of a sequence of functions in $L^\infty([0,1])$ implies the convergence in $L^p([0,1])$. However, the converse of this fact does not hold in general. Thus, Proposition 3.4 is not a consequence of [5, pages 30-33].

For the rate of convergence of Kantorovich operators in Lebesgue spaces, let us recall the following result.

Theorem 3.5 (Maier, [6]). *Let $f : [0,1] \to \mathbb{R}$ and $1 < p < \infty$. If f' is absolutely continuous on $[0,1]$ and $f'' \in L^p([0,1])$, then there exists $C > 0$ such that*

$$\|K_n f - f\|_{L^p([0,1])} \leq \frac{C}{n} (\|f'\|_{L^p([0,1])} + \|f''\|_{L^p([0,1])}).$$

Corollary 3.6. *The rate of convergence of Kantorovich Operators is of order $\frac{1}{n} = n^{-1}$ or faster, written as $\|K_n f - f\|_{L^p([0,1])} = O(n^{-1})$.*

Other results related to the rate of convergence of Kantorovich operators in Lebesgue spaces can be found in [1, 3, 7]. we give an example for Theorem 3.5 where the exact rate of convergence of Kantorovich operators in Lebesgue spaces is attained.

Example 3.7. *Consider the function $f_1(x) = x$. Then the rate of convergence of $\{K_n f_1\}$ to f_1 in $L^p([0,1])$ is exactly of order $O(n^{-1})$ for $1 \leq p < \infty$.*

Proof. Clearly f_1 is differentiable on $[0,1]$, $f_1'(x) = 1$ is absolutely continuous on $[0,1]$, and $f_1''(x) = 0 \in L^p([0,1])$ for $1 \leq p < \infty$. By Proposition 2.1, we have

$$K_n f_1(x) - f_1(x) = \frac{1}{n+1} \left(nx + \frac{1}{2} \right) - x = \frac{1}{n+1} \left(\frac{1}{2} - x \right).$$

Hence, by writing $g_1(x) = \dfrac{1}{2} - x$ for $x \in [0,1]$,

$$\|K_n f_1 - f_1\|_{L^p([0,1])} = \frac{1}{n+1} \|g_1\|_{L^p([0,1])} \geq \frac{1}{2n} \|g_1\|_{L^p([0,1])} = \frac{1}{4n(p+1)^{1/p}}$$

for all $n \in \mathbb{N}$ and $1 \leq p < \infty$. Thus, the rate of convergence of $\{K_n f_1\}$ to f_1 in $L^p([0,1])$ is exactly of order $O(n^{-1})$ for $1 \leq p < \infty$. □

4. CONCLUDING REMARKS

We have given an alternative proof of the boundedness of Kantorovich operators in $L^p([0,1])$ for $1 \leq p \leq \infty$. In addition, we give a counterexample in order to disprove the convergence of Kantorovich operators in $L^\infty([0,1])$. We also provide an example of the exact rate of convergence of Kantorovich operators in Lebesgue spaces. The continuity of first derivative assumption for the class of functions that were being used to approximate the rate of convergence in this paper is still rather strong. For future research, it is of our interest to investigate the rate of convergence of Kantorovich operators in subspaces of $L^p([0,1])$ with weaker assumptions or in another function space altogether such as the weighted Lebesgue spaces.

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