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ON CONDITIONS FOR CONTROLLABILITY AND LOCAL REGULARITY OF A SYSTEM OF DIFFERENTIAL EQUATIONS

Ahmad Hadra Zuhri[1,a], Yudi Soeharyadi[1,b], Jalina Widjaja [1,c]

[1]**Faculty of Mathematics and Natural Sciences, Bandung Institute of Technology Jalan Ganesha 10, Bandung 40132, Indonesia**
[a]**ahadrazuhri@gmail.com**
[b]**corresponding author: yudish@math.itb.ac.id**
[c]**jalina@math.itb.ac.id**

Abstract. We consider a system of differential equation on a Banach space X given by: $x'(t) = Ax(t) + u(t)f(t, x(t))$, $x(0) = x_0$, where A is an infinitesimal generator of a C_0-semigroup, $f : \mathbb{R}_0^+ \times X \to X$ is a locally Lipschitz function, and $u \in L^p([0,T], \mathbb{R})$ is a control defined on $[0,T]$ with $1 < p \leq \infty$. Using the Compactness Principle and the generalization of Gronwalls Lemma, the system is shown to be controllable for a γ-bounded function f. Another result of this study is the local existence and the uniqueness of the solution of the system for locally bounded function f through weighted ω-norm.

Key words and Phrases: differential equation, compactness principle, controllability, local existence

1. INTRODUCTION

We consider a control problem described as an Abstract Cauchy Problem (ACP) on a Banach space X given by:

$$\begin{cases} x'(t) = Ax(t) + u(t)f(t, x(t)), & t > 0 \\ x(0) = x_0 \in X. \end{cases} \tag{1}$$

where $A : D(A) \subset X \to X$ is an infinitesimal generator of an m-dissipative operator, $f : \mathbb{R}_0^+ \times X \to X$ describes the external factor of the system, and $u \in L^p([0,T], \mathbb{R})$ is the control term defined on $[0,T]$ with $1 < p \leq \infty$. A particular model described by this ACP is a control problem of heat distribution on a rod with the presence of heat source f and Dirichlet boundary conditions

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$$\begin{cases} \partial_t w = \partial_{xx} w + u(x,t) f(x,w), & t > 0,\ x \in I = [0,L] \subset \mathbb{R} \\ w(x,0) = w_0(x), \\ w(0,t) = w(L,t) = 0. \end{cases}$$

The m-dissipativity of A ensures the well-posedness of the ACP (1) with mild solution

$$x(t) = e^{At} x_0 + \int_0^t e^{(t-s)A} u(s) f(s, x(s))\, ds. \tag{2}$$

Here $(e^{At})_{t \geq 0}$ denotes the semigroup generated by A.

Many studies related to the control problem (1) has been developed by researchers in recent years, both in the development of its controllability and its application to some specific equations such as Schrödinger equation and Gross-Pitaevskii equation, see J. M. Ball et al. [1], Nabile Boussaïd. et al. [2, 3], Thomas Chambrion and Laurent Thomann [4], Gunther Dirr [5], and Jonas Lampart [6].

Our particular interest is the results of Gunther Dirr [5] in 2021 on the controllability of (1) with linearly bounded source f. He showed that the reachable set as the set of all the final states that can be obtained from the initial state is compact on the interval $[0,T]$ for $1 < p \leq \infty$. His study was motivated by Boussaïd et al. [2] who proved that the reachable set of (1) is contained in a countable union of compact subsets of X for a control $u \in L^1_{loc}([0,\infty), \mathbb{R})$ and a bounded source f. They also obtained other results in the form of fixed point iterations for solving (2). Analyzing those results, Gunther Dirr [5] then raised the idea using the compactness principle of fixed point maps to prove the controllability of (1) under certain assumptions as follows:

A1. Let X be a Banach space and $f : \mathbb{R}_0^+ \times X \to X$ be Lipschitz on a bounded sets of X with L^∞_{loc}-Lipschitz rate, i.e. for all bounded sets $B \subset X$ there exist $L \in L^\infty_{loc}(\mathbb{R}_0^+, \mathbb{R}_0^+)$ such that

$$\|f(t,\xi) - f(t,\eta)\| \leq L(t)\|\xi - \eta\|$$

for all $\xi, \eta \in B$ and $t \in \mathbb{R}_0^+$.

A2. Let X be a Banach space and $f : \mathbb{R}_0^+ \times X \to X$ be linearly bounded, i.e. there exist $\alpha, \beta \in L^\infty_{loc}(\mathbb{R}_0^+, \mathbb{R}_0^+)$ such that

$$\|f(t,\xi)\| \leq \alpha(t)\|\xi\| + \beta(t)$$

for all $\xi \in X$ and $t \in \mathbb{R}_0^+$.

The purpose of this study is to examine the controllability of control system (1) for γ-bounded source f and its local regularity for locally bounded source f. First, we will review some of the main results of Gunther Dirr [5] and L. Younes et al. [7] in Section 2 that is instrumentals in this study. In Section 3, we will show that every mild solution of (1) for f which is Lipschitz on bounded sets is also a mild solution of (1) with a globally Lipschitz f. These results then lead us to the controllability of (1) for a γ-bounded source term f that refers to the compactness of the reachable set of the system. Next, the local regularity of (1) with a locally bounded source f will be proved in Section 4. Here, regularity refers to the temporal continuity of the solution of a PDE. In proving regularity, we will prove the local existence and the uniqueness of the solution using weighted norms and Banach Fixed Point Theorem. Furthermore, the results in Section 4 will be simulated through some particular cases in Section 5.

2. PRELIMINARIES

The semigroup that will be used in this study is a strongly continuous one, $(e^{At})_{t\geq 0}$. Here, A generates an m-dissipative operator that is a linear operator on a Banach space that satisfies a certain dissipativity condition, which ensures that the solutions to the associated system are exponentially bounded. One important result in this area is the Hille-Yosida theorem, which states that a densely defined, closed linear operator on a Banach space generates a strongly continuous semigroup of contractions if and only if it is m-dissipative for some $m > 0$. Moreover, if the linear operator A generating the strongly continuous semigroup is m-dissipative, then the semigroup, $(e^{At})_{t\geq 0}$, satisfies a bound of the form

$$\left\|e^{At}\right\| \leq Me^{\mu t},$$

for some $M, \mu > 0$ and for all $t \geq 0$.

Let X, P, and Z be metric spaces and $F : X \times P \to Z$ be an arbitrary map. For $x \in X$ and $u \in P$, define the partial maps F_x and F_u as follows:

$$\begin{aligned} F_x &: P \to Z, \quad u \mapsto F_x(u) = F(x, u) = F(x, \cdot), \\ F_u &: X \to Z, \quad x \mapsto F_u(x) = F(x, u) = F(\cdot, u). \end{aligned}$$

Gunther Dirr [5] proved the controllability of a differential system through the Compactness Principle Theorem (Theorem 2.4 [5]) which is one of the useful tools for the application in the area of ODE or PDE. There are several corollaries of the theorem but the necessary one that is used to prove the controllability of the system (1) for $1 < p \leq \infty$ is Corollary 2.9 [5] which will be stated as Theorem 2.1 as follow:

Theorem 2.1. *Let X and P be complete metric spaces and $F : X \times P \to X$ satisfy the following conditions:*

(a) For every bounded set $B \subset P$ there exists a strongly equivalent metric d_B on X and a constant $0 \leq C < 1$ such that

$$d_B(F_u(x), F_u(y)) \leq Cd_B(x, y)$$

for all $x, y \in X$ and $u \in B$.

(b) $F_x : P \to X$ *is continuous and compact for all* $x \in X$.
Then $\Phi : P \to X$ *which assigns each* $u \in P$ *to the unique fixed point of* $F_u(\cdot)$ *is continuous and compact. If additionally*

(c) $F : P \to X$ *is Lipschitz in* u *locally uniformly in* x *then* $\Phi : P \to X$ *is also locally Lipschitz in* u.

Using this theorem, Gunther Dirr proved the controllability of the system (1) under continuous vector field $f_i : \mathbb{R}_0^+ \times X \to X$ and $u = (u_1, u_2, \ldots, u_m) \in L^p([0,T], \mathbb{R}^m)$ for $i = 1, 2, \ldots, m$ and $p > 1$, by proving the compactness of the reachable set of the system through a solution operator which is also a fixed point operator. Furthermore, the main results of Gunther Dirr's can be seen in the Theorem 2.2 below.

Theorem 2.2. *Let* X *be a Banach space,* A *be the infinitesimal generator of a* C_0*-semigroup* $(e^{tA})_{t\geq 0}$ *of bounded operator on* X*, and* $p > 1$*. Moreover, let* $f_i : \mathbb{R}_0^+ \times X \to X$*,* $i = 1, \ldots, m$ *be continuous vector field which satisfy (A1) and (A2). Then for all* $T \geq 0$*,* $\xi_0 \in X$*, and* $u = (u_1, u_2, \ldots, u_m) \in L^p([0,T], \mathbb{R}^m)$ *the equation*

$$x'(t) = Ax(t) + \sum_{i=1}^{m} u_i(s) f_i(t, x(t)), \quad x(0) = \xi_0$$

has unique mild solution and the solution operator $\Phi : L^p([0,T], \mathbb{R}^m) \to C([0,T], X)$ *is compact and locally Lipschitz continuous.*

The following generalization of Gronwalls Lemma is proved by L. Younes et al. [7].

Theorem 2.3. *Let* $u(t)$ *be a positive differentiable function on some interval* $I = [a, b]$ *which satisfies*

$$u(t) \leq C + \int_a^t \sum_{i=1}^{n} (f_i(s) u^i(s))\, ds, \quad t \in I = [a, b]$$

for some nonnegative constant C *and some continuous function* $f_i(t)$ *on* I*, for* $i = 1, 2, \ldots, n$*. Then*

$$u(t) \leq \frac{C e^{\int_a^t f_1(s)\, ds}}{(1 - (n-1) \int_a^t \sum_{i=2}^{n} C^{i-1} f_i(s) e^{\int_a^s (n-1) f_1(\sigma)\, d\sigma}\, ds)^{\frac{1}{n-1}}}$$

for all $t, s \in I$ *provided*

$$1 - (n-1) \int_a^t \sum_{i=2}^{n} C^{i-1} f_i(s) e^{\int_a^s (n-1) f_1(\sigma)\, d\sigma}\, ds > 0.$$

3. CONDITIONS FOR CONTROLLABILITY OF A SYSTEM OF DIFFERENTIAL EQUATION

In this section, we prove one of our main results that is finding a mild solution of the system (1) by solving its corresponding auxiliary problem. In this case, the assumptions (A1) and (A2) will be replaced by (A3) and (A4) below:

A3. $f : \mathbb{R}_0^+ \times X \to X$ be Lipschitz on a bounded sets of X with L_{loc}^{∞}-Lipschitz rate, i.e. for all bounded sets $B \subset X$ there exist $\gamma > 1$ and $L \in L_{loc}^{\infty}(\mathbb{R}_0^+, \mathbb{R}_0^+)$ such that

$$\|f(t,\xi) - f(t,\eta)\| \le L(t)\|\xi - \eta\|(\|\xi\|^{\gamma} + \|\eta\|^{\gamma})$$

for all $\xi, \eta \in B$ and $t \in \mathbb{R}_0^+$.

A4. $f : \mathbb{R}_0^+ \times X \to X$ be a γ-bounded function, i.e. there exist $\gamma > 1$ such that for all $t \in \mathbb{R}_0^+$ exist an $\alpha \in L_{loc}^{\infty}(\mathbb{R}_0^+, \mathbb{R}_0^+)$ so that

$$\|f(t,\xi)\| \le \alpha(t)\|\xi\|^{\gamma}$$

for all $\xi \in X$.

This theorem is needed to obtain the uniqueness property of the solutions of system (1) through Banach Fixed Point Theorem.

Theorem 3.1. *Let $p \ge 1$ and $f : \mathbb{R}_0^+ \times X \to X$ satisfies (A3) and (A4). Then for all bounded sets $B \subset L^p([0,T],\mathbb{R})$ and $\xi_0 \in X$, there exists a Lipschitz function $\hat{f} : \mathbb{R}_0^+ \times X \to X$ in ξ with L_{loc}^{∞}-Lipschitz rate so that, every mild solution of*

$$x'(t) = Ax(t) + u(s)f(t,x(t)), \quad x(0) = \xi_0, t \in [0,T], \tag{3}$$

is also a mild solution of

$$x'(t) = Ax(t) + u(s)\hat{f}(t,x(t)), \quad x(0) = \xi_0, t \in [0,T], \tag{4}$$

and vice versa.

Proof. Let $x : [0,T] \to X$ be the mild solution of (3) and $B \subset L^p([0,T],\mathbb{R})$ is bounded. For $\mu > 0$, define $\phi(t) = e^{-\mu t}\|x(t) - e^{At}\xi_0\|$ on X. Then for every $u \in B$, we have

$$
\begin{aligned}
\phi(t) &= e^{-\mu t}\left\|\int_0^t e^{A(t-s)}u(s)f(s,x(s))\,ds\right\| \\
&\le M\int_0^t e^{-\mu s}|u(s)|\alpha(s)\|x(s)\|^{\gamma}\,ds \\
&\le K\int_0^t |u(s)|\phi(s)\|x(s)\|^{\gamma-1}+M|u(s)|\|\xi_0\|\|x(s)\|^{\gamma-1}ds \\
&\le K\int_0^t |u(s)|\phi^2(s)e^{\mu s}\|x(s)\|^{\gamma-2}+M|u(s)|\phi(s)e^{\mu s}\|\xi_0\|\|x(s)\|^{\gamma-2}+ \\
&\quad M|u(s)|\phi(s)e^{\mu s}\|\xi_0\|\|x(s)\|^{\gamma-2}+M^2e^{\mu s}\|\xi_0\|\|x(s)\|^{\gamma-2}ds \\
&\vdots \\
\phi(t) &\le C+K\int_a^t\sum_{i=1}^{n}(f_i(s)\phi^i(s))ds,
\end{aligned}
$$

where $K=M\|\alpha\|_\infty$ and $C=K\int_a^t M^{\gamma}e^{(\gamma-1)\mu s}\|\xi_0\|^{\gamma-1}ds$. Application of Theorem 2.3 yields

$$
\phi(t)\le \frac{Ce^{\int_a^t f_1(s)\,ds}}{(1-(n-1)\int_a^t\sum_{i=2}^{n}C^{i-1}f_i(s)e^{(n-1)\int_a^s f_1(\sigma)\,d\sigma}\,ds)^{\frac{1}{n-1}}}=P.
$$

Thus, $\|x(t)-\xi_0\|\le e^{\mu T}P+(Me^{\mu T}+1)\|\xi_0\|=Q$.
Next, for every $N>Q$, define $\hat{f_N}(t,\eta)=\rho_N(\eta-\xi_0)f(t,\eta)$, where $\rho_N: X\to[0,1]$ is a Lipschitz cut-off function given by

$$
\rho_N(\eta-\xi_0)=\begin{cases}1, & \|\eta-\xi_0\|\le N\\ 0, & \|\eta-\xi_0\|\ge N+1\\ N+1-\|\eta\|, & N\le\|\eta-\xi_0\|\le N+1.\end{cases}
$$

Then for every $t\in[0,T]$ and $\eta,\zeta\in X$, one has

$$
\begin{aligned}
\|\hat{f_N}(t,\eta)-\hat{f_N}(t,\zeta)\| &\le |\rho_N(\eta-\xi_0)|\|f(t,\eta)-f(t,\zeta)\|+ \\
&\quad |\rho_N(\eta-\xi_0)-\rho_N(\zeta-\xi_0)|\|f(t,\zeta)\| \\
&\le |\rho_N(\eta-\xi_0)|L(t)\|\zeta-\eta\|(\|\zeta\|^{\gamma}+\|\eta\|^{\gamma})+ \\
&\quad |\rho_N(\eta-\xi_0)-\rho_N(\zeta-\xi_0)||\alpha(t)|\|\zeta\|^{\gamma}.
\end{aligned}
$$

Therefore,
$\|\hat{f_N}(t,\eta)-\hat{f_N}(t,\zeta)\|$

$$
\le\begin{cases}L(t)\|\zeta-\eta\|(\|\zeta\|^{\gamma}+\|\eta\|^{\gamma}), & \|\eta-\xi_0\|,\|\zeta-\xi_0\|\le N\\ 0, & \|\eta-\xi_0\|,\|\zeta-\xi_0\|\ge N+1\\ (L(t)(N+1)+\alpha(t))\|\eta-\zeta\|(\|\zeta\|^{\gamma}+\|\eta\|^{\gamma}), & N\le\|\eta-\xi_0\|,\|\zeta-\xi_0\|\le N+1.\end{cases}
$$

Choosing $\hat{f}(t,\eta) = \lim_N \hat{f}_N(t,\eta)$ yields $x(t)$ is also the mild solution of (4). Conversely, since $\|\hat{f}(t,\zeta_0)\| \leq \alpha(t)\|\xi_0\|^\gamma$, then similarly, $\hat{x}(t)$ is the mild solution of (3). ∎

This theorem leads us to the controllability of system (1), that is a version of Theorem 2.2 for a distinct condition of a source f i.e. (A3) and (A4).

Corollary 3.2. *Let X be a Banach space, $p > 1$, A be the infinitesimal generator of a C_0-semigroup $(e^{tA})_{t\geq 0}$. Moreover, let $f_i : \mathbb{R}_0^+ \times X \to X$, $i = 1,\ldots,m$ be continuous vector field which satisfy (A3) and (A4). Then for all $T \geq 0$, all $\xi_0 \in X$, and all $u = (u_1, u_2, \ldots, u_m) \in L^p([0,T],\mathbb{R}^m)$ the equation*

$$x'(t) = Ax(t) + \sum_{i=1}^{m} u_i(s) f_i(t, x(t)), \quad x(0) = \xi_0,$$

has a unique mild solution. The solution operator $\Phi : L^p([0,T],\mathbb{R}^m) \to C([0,T],X)$ is compact and locally Lipschitz continuous.

4. CONDITIONS FOR LOCAL REGULARITY OF A SYSTEM OF DIFFERENTIAL EQUATION

Our other result is related to the regularity of (1). We use weighted ω-norms defined by

$$\|\cdot\|_\omega = \max_{t\in[0,T]} e^{-\omega t}\|\cdot\|,$$

which are equivalent to the standard maximum norm on $C([0,T],X)$. We establish the continuity of the solution (2). In particular, we prove that the system has a local unique solution for a locally bounded function f.

Theorem 4.1. *Suppose that X is a Banach space and $x_0 \in X$. For some $\alpha, \rho > 0$, let $I_\alpha = [0,\alpha] \subset \mathbb{R}_0^+$, and $J = \overline{B_\rho(x_0)} \subset X$. Moreover, let $f : \mathbb{R}_0^+ \times X \to X$ be Lipschitz and locally bounded function. Then there exists $\beta \in (0,\alpha)$ such that the integral equation given by*

$$x(t) = e^{At}x_0 + \int_0^t e^{(t-s)A} u(s) f(s, x(s))\, ds,$$

has a unique local solution on $C(I_\beta, X)$, where $I_\beta = [0,\beta]$.

Proof. As a priori assumption, let $\beta \in (0,\alpha)$. For $\omega \geq 0$, define the weighted ω-norms by:

$$\|\cdot\|_\omega = \max_{t\in I_\beta} e^{-\omega t}\|\cdot\|_X.$$

Moreover, let $F : C(I_\beta, J) \times L^p(I_\beta,\mathbb{R}) \to C(I_\beta, J)$ be a partial map given by

$$F(x,u)(t) = e^{At}x_0 + \int_0^t e^{(t-s)A} u(s) f(s, x(s))\, ds.$$

Then for any $(x,u) \in C(I_\beta, J) \times L^p(I_\beta, \mathbb{R})$ and $t \in I_\beta$, we have

$$\begin{aligned} e^{-\omega t}\|F(x,u)(t) - x_0\| &= e^{-\omega t}\left\| e^{At}x_0 + \int_0^t e^{(t-s)A}u(s)f(s,x(s))\,ds - x_0 \right\| \\ &\leq e^{-\omega t}\left(M\|x_0\|e^{t\mu} + MNe^{t\mu}\int_0^t e^{-s\mu}|u(s)|\,ds \right) \\ &= M\|x_0\|e^{(\mu-\omega)t} + MNe^{(\mu-\omega)t}\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}} [1 - e^{-qt\mu}]^{\frac{1}{q}}, \end{aligned}$$

where $N = \max_{t \in I_\beta} \|f(t, x(t))\|$.
Taking the maximum value of the both side of the last inequality on I_β yields

$$\|F(x,u)(t) - x_0\|_\omega \leq \max_{t \in I_\beta}\left(M\|x_0\|e^{(\mu-\omega)t} + MNe^{(\mu-\omega)t}\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}} [1 - e^{-qt\mu}]^{\frac{1}{q}} \right).$$

Case 1: $\omega = \mu$

$$\begin{aligned} \|F(x,u)(t) - x_0\|_\omega &\leq \max_{t \in I_\beta}\left(M\|x_0\| + MN\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}} [1 - e^{-qt\mu}]^{\frac{1}{q}} \right) \\ &\leq M\|x_0\| + MN\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}} [1 - e^{-q\beta\mu}]^{\frac{1}{q}}. \end{aligned}$$

If $M\|x_0\| + MN\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}} [1 - e^{-q\beta\mu}]^{\frac{1}{q}} \leq \rho$, then

$$\beta \leq \left[\frac{1}{q\mu}\right] \left[\ln\left(1 - \left[\frac{\rho - N\|x_0\|}{MN\|u\|_p} (q\mu)^{\frac{1}{q}} \right]^q \right)^{-1} \right].$$

Case 2: $\omega > \mu$

$$\|F(x,u)(t) - x_0\|_\omega \leq \max_{t \in I_\beta}\left(M\|x_0\|e^{(\mu-\omega)t} + MNe^{(\mu-\omega)t}\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}} [1 - e^{-qt\mu}]^{\frac{1}{q}} \right).$$

Assume $M\|x_0\|e^{(\mu-\omega)t} + MNe^{(\mu-\omega)t}\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}} [1 - e^{-qt\mu}]^{\frac{1}{q}}$ attains its maximum value at $\gamma \in I_\beta$, then for $\delta = \mu - \omega$

$$\begin{aligned} \|F(x,u)(t) - x_0\|_\omega &\leq M\|x_0\|e^{\delta\gamma} + MNe^{\delta\gamma}\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}} [1 - e^{-q\gamma\mu}]^{\frac{1}{q}} \\ &\leq M\|x_0\|e^{\delta\gamma} + MNe^{\delta\gamma}\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}}. \end{aligned}$$

If $M\|x_0\|e^{\delta\gamma} + MNe^{\delta\gamma}\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}} \leq \rho$ and in particular for $\beta = \gamma$

$$\beta \geq \left[\frac{1}{-\delta}\right] \ln\left[\frac{1}{\rho}\left(M\|x_0\| + MN\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}} \right) \right].$$

Case 3: $\omega < \mu$

$$\|F(x,u)(t) - x_0\|_\omega \leq \max_{t \in I_\beta} \left(M\|x_0\| e^{(\mu-\omega)t} + MNe^{(\mu-\omega)t}\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}} [1 - e^{-qt\mu}]^{\frac{1}{q}} \right)$$
$$\leq M\|x_0\| e^{(\mu-\omega)\beta} + MNe^{(\mu-\omega)\beta}\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}}.$$

If $M\|x_0\| e^{(\mu-\omega)\beta} + MNe^{(\mu-\omega)\beta}\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}} \leq \rho$ and $\delta = \mu - \omega$, then

$$\beta \leq \left[\frac{1}{\delta}\right] \ln \left[\frac{1}{\rho} \left(M\|x_0\| + MN\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}} \right) \right].$$

From the three cases above we can conclude that for sufficiently large ω, there is $\beta \in (0, \alpha)$ such that the solution exists in $\overline{B_\rho(x_0)}$, for sufficiently small ρ. Next, we will show that for $\omega > \mu$, $F(x,u)$ is a contraction map.

$$\|F(x,u)(t) - F(y,u)(t)\| = e^{-\omega t} \left\| \int_0^t e^{(t-s)A} u(s)(f(s,x(s)) - f(s,y(s)))\, ds \right\|$$
$$\leq e^{-\omega t} \left[\int_0^t Me^{(t-s)\mu} |u(s)| L e^{\omega s} e^{-\omega s} \|x(s) - y(s)\|\, ds \right]$$
$$\leq e^{(\mu-\omega)t} ML\|u\|_p \left[\frac{e^{q(\omega-\mu)t} - 1}{q(\omega-\mu)} \right]^{\frac{1}{q}} \|x - y\|_\omega$$
$$\leq ML\|u\|_p \left[\frac{1}{q(\omega - \mu})\right]^{\frac{1}{q}} \|x - y\|_\omega.$$

Taking the maximum value of the both side of the last inequality on interval I_β then we have

$$\|F(x,u)(t) - F(y,u)(t)\|_\omega \leq \max_{t \in I_\beta} ML\|u\|_p \left[\frac{1 - e^{q(\mu-\omega)t}}{q(\omega-\mu)} \right]^{\frac{1}{q}} \|x - y\|_\omega.$$

Hence $F(x,u)$ is a contraction map by choosing $\omega \geq \frac{(ML\|u\|_p)^q}{q} + \mu$. ■

5. SIMULATIONS

In Section 4, we have proved that the control system (1) has a unique local solution near the given initial state by choosing sufficiently large weight of the norm. In this section, we give some illustrations to our main results. Let let $X = \mathbb{R}$ and consider an operator $f : [0,T] \to \mathbb{R}$ as

$$f(t) = N\|x_0\| e^{(\mu-\omega)t} + NMe^{(\mu-\omega)t}\|u\|_p \left[\frac{1}{q\mu}\right]^{\frac{1}{q}} [1 - e^{-qt\mu}]^{\frac{1}{q}}.$$

where μ is the growth bound of the semigroup, ω is the weight of the norm, x_0 is the initial state of (1) in $\mathbb{R}$, N, M, p, q are positive constants.

Set $N = M = p = q = x_0 = 1$ and $\|u\|_1 = 10$.

Case 1: $\omega = \mu$

$$f(t) = 1 + \frac{10}{\omega}[1 - e^{-\omega t}]$$

The graphs of f for $\omega = 2$, $\omega = 10$, and $\omega = 100$ are the following

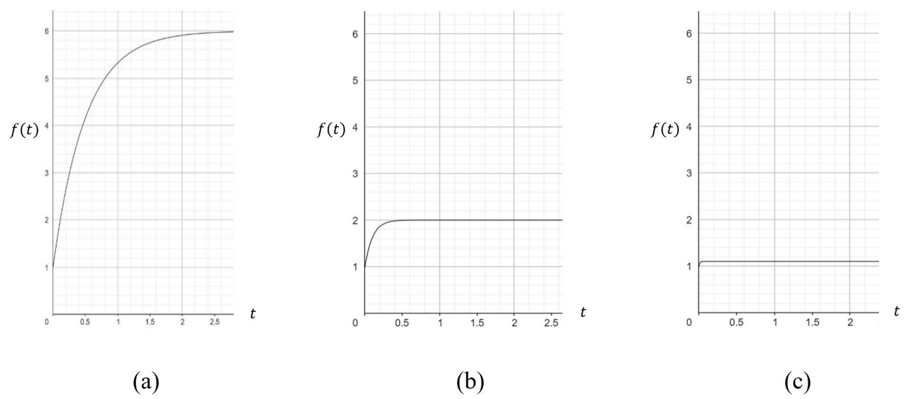

FIGURE 1. (a) $\omega = 2$, (b) $\omega = 10$, (c) $\omega = 100$

Case 2: For $\omega > \mu$ let $\mu = 10$ then

$$f(t) = e^{(10-\omega)t} + e^{(10-\omega)t}[1 - e^{-10t}]$$

The graphs of f for $\omega = 11$, $\omega = 30$, and $\omega = 100$ are the following

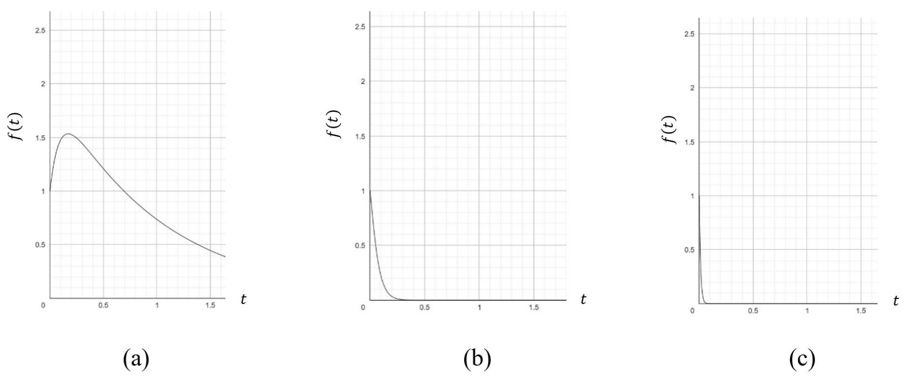

FIGURE 2. (a) $\omega = 11$, (b) $\omega = 30$, (c) $\omega = 100$

Case 3: For $\omega < \mu$ let $\mu = 10$ then

$$f(t) = e^{(10-\omega)t} + e^{(10-\omega)t}[1 - e^{-10t}]$$

The graphs of f for $\omega = 9$, $\omega = 4$, and $\omega = 0.5$ are the following

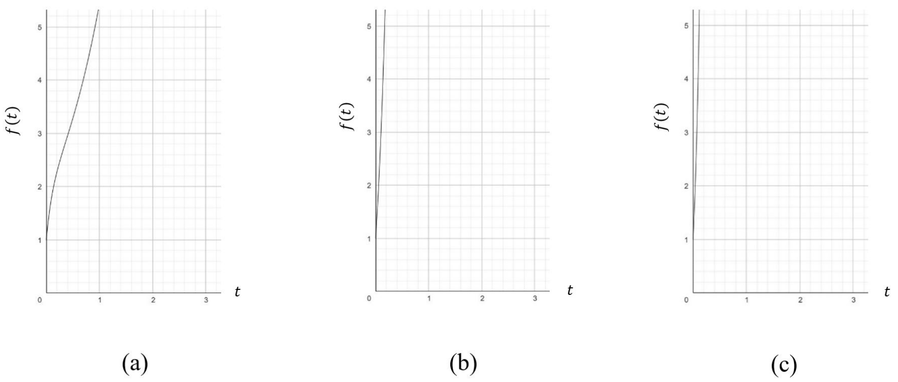

FIGURE 3. (a) $\omega = 9$, (b) $\omega = 4$, (c) $\omega = 0.5$

If we consider $f(t)$ as the distance between the solution of the system and the given initial state, then for case 1, the solution exists only within certain distance from the initial state. For case 2, the solution exists near the initial state. It means that for sufficiently small neighbourhood of the initial state, we can always find the solution of the system. For case 3, by choosing ω smaller than μ, the solution will grow indefinitely, so the solution stays near the initial state in a fairly short time.

6. CONCLUSIONS

We have considered a system of differential equation on a Banach space X given by $x'(t) = Ax(t) + u(t)f(t, x(t)), \quad x(0) = x_0$. For some γ-bounded source f, we prove that the system is controllable as a corollary of Theorem 3.1 in conjunction to the results in [5]. For a locally bounded function f and $I_\alpha = [0, \alpha]$, there exists $\beta \in (0, \alpha)$ such that the system has a unique local solution on $C(I_\beta, X)$. Furthermore, by choosing $\omega > \mu$, the solution exists in sufficiently small neighbourhood of the initial state.

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