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Generalized Implicit Function Theorem and General Fundamental Theorem of Calculus

Ashish Dhara[1], Anil Pedgaonkar[1], and Narendrakumar Ramchandra Dasre[2,*]

[1]Department of Mathematics, Institute of Science, India

[2] Ramrao Adik Institute of Technology, D. Y. Patil deemed to be University, India

Abstract. We present the notion of Henstock-Kurzweil integral for mappings assuming values in Hausdorff topological vector spaces using the direct set of gauges and derive a version of Mean Value Theorem. We use the definition of Frechet derivative and obtain a general version of *Implicit Function Theorem* for mappings from $X \times Y \to Z$ where, for existence and continuity of the function, X needs to be merely a topological space and for differentiability, X can be a Topological Vector Space (TVS) while Z is a Hausdorff topological vector space and Y is a Banach space. The implicit function theorem is proved in 3 parts as existence, continuity of the partial derivative and invertibility of the partial derivative. The proof is very similar to the classical proof.

Key words and Phrases: Frechet Derivative, Henstock-Kurzweil Integral, Topological Vector Space (TVS), Gauge.

1. INTRODUCTION

This paper will discuss the generalized *Implicit Mapping Theorem* for mappings $X \times Y \to Z$. Only Y is required to be a Banach space. Z can be a Hausdorff topological vector space. For the existence and continuity, it suffices that X is a topological space and for differentiability, X should be a topological vector space. Our proof is modelled on the classical proof given in [1]. With minor modifications, a definition of derivative for a mapping from a topological space to a topological group and the theorem holds when X is a topological space and Z is a Hausdorff topological group can be coined. In the final section, the Fundamental Theorem of Calculus for Henstock-Kurzeweil Integral for mappings assuming values in a Hausdorff topological vector space is obtained.

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*Corresponding author : narendasre@rait.ac.in

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2. PRELIMINARY

The abbreviation L.T. for a linear transformation is used and nbd for neighbourhood. The following facts about topological vector space from [2] are recalled. Topological Vector Space (TVS): A vector space $\mathbb{X}$ over a field $\mathcal{F}(\mathcal{F}$ can be $\mathbb{R}$ or $\mathbb{C})$ is called a topological vector space, abbreviated as TVS, if $\mathbb{X}$ is vector space and there is a topology on $\mathbb{X}$ such that the addition $+ : \mathcal{F} \times \mathcal{X} \to \mathcal{F}$ as well as scalar multiplication, $\cdot : \mathbb{R} \times \mathbb{X} \to \mathbb{R}$ is continuous.

Remark 2.1. *Any normed linear space is a TVS.*

Definition 2.1. *In a vector space V over $\mathcal{F}$, we define the following concepts. For any set $A \subset V, -A = \{-a : a \in A\}$. A set A is called **symmetric** if $-A = A$. For any subset $D \subset \mathcal{F}$ and $A \subset V, Da = \{d \cdot a : d \in D, a \in A\}$. $A + b$ is translate of A by the vector b and $A + b = \{a + b : b \in B\}$. A set A is called **balanced** if $tA \subset A$, for all scalars t with $|t| \leq 1$.*

Remark 2.2. *Choosing $t = -1$ shows that a balanced set is symmetric.*

Example 1. *Let A be a non-square plane rectangle. A is balanced as a subset of $\mathbb{R}^2$. But A is not balanced in the complex vector space $\mathbb{C}$ over $\mathbb{C}$ as $iA \not\subset A$.*

A set A is a TVS, is called bounded if for each nbd U of 0, if there exist $\delta > 0$ such that $tA \subset A$, for all scalars t with $|t| < \delta$. This is equivalent to any one of the following properties:

(1) There exists $\epsilon > 0$ such that $\epsilon A \subset U$, for any nbd U of 0.
(2) There exists a scalar s such that $A \subset sU$.
(3) There exists an integer n such that $A \subset nU$.

Clearly finite union of bounded sets is bounded and each finite set is bounded. It can be noted that translation and multiplication by a non-zero scalar are homeomorphisms and topology on a TVS can be defined using only neighbourhoods of 0. Also, it can be noted that each neighbourhood (nbd) of 0 contains a balanced closed nbd of 0. Scalar multiple or translate of a bounded set is bounded. Given a nbd U of 0, there exists a nbd V of 0 such that $V + V \subset U$. Our vector spaces are over the field $\mathbb{R}$.

3. Definition of Frechet derivative

Definition 3.1. *Let X, Y be a TVS. A mapping $f : X \to Y$ is called differentiable at a point x in X if there exist a continuous L.T. $f'(x) = D$ such that,*

$$\lim_{h \to 0} \frac{f(x + th) - f(x)}{t} - D \cdot h \to 0$$

uniformly for any h in a bounded set, that is, $f(x+th) - f(x) - D \cdot th = R(h)$ and given a nbd U of 0 in Y, there exist $\delta > 0$ such that $R(h) \in U$ when $0 < |t| < \delta$. We say $R(h)$ is a δ-tangent or simply tangent.

Remark 3.1. *The definition is clearly equivalent to the usual definition in Frechet spaces as shown in* [3, 4] *that the usual laws of differentiation including the chain rule holds but a differentiable mapping need not be continuous, however merely sequentially continuous.*

Definition 3.2. *A topological space X is called* ***sequential*** *if for any subset A, each cluster point a of A is the limit of a sequence of points in A. When X is a sequential space, each mapping differentiable at a is continuous at a.*

Definition 3.3 (Topology on L(X,Y))**.** *L(X,Y) is the space of continuous linear maps with the topology of uniform convergence on bounded subsets of E.*

$(B, V) = \{T : X \to L(X, Y)|T(B) \subset V\}$ denotes a basic $0 - \textit{neighbourhood}$ in the space $L(X, Y)$. B is a bounded subset of X, V is a $0 - \textit{neighbourhood}$ in Y.

Definition 3.4. *A mapping $f : \mathbb{X} \to \mathbb{Y}$ is called a C^1 mapping in an open set E if the mapping $x \to f'(x)$ is a sequentially continuous mapping $\mathbb{X} \to L(\mathbb{X}, \mathbb{Y})$, the space of continuous linear maps: $\mathbb{X} \to \mathbb{Y}$.*

Definition 3.5 (Norm of a linear mapping)**.** *In the case when $\mathbb{X}$ and $\mathbb{Y}$ are normed spaces, the Norm of a continuous L.T. $T \in L(\mathbb{X}, \mathbb{Y})$ is $sup\{|Tx| : |x| \leq 1\}$. It is denoted by $||T||$. It satisfies $|Tx| \leq ||T||\ |x|, \forall x \in \mathbb{X}$.*

Definition 3.6 (Derivative for mappings of real variable)**.** *We note that when we consider mappings $\mathbb{R} \to X$, where X is Hausdorff, the derivative can be identified with a vector in X and can be equivalently defined as in $F : \mathbb{R} \to X$, we define derivative*

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}.$$

Definition 3.7 (Primitive of a mapping f)**.** *Let $K = [a, b]$ be a closed cell in $\mathbb{R}$. Let F and f be mappings defined on K with values in a Hausdorff TVS space Y. Let $D = (a, b)$. A continuous mapping F on K is said to be a primitive of f on K if F is differentiable on D, with $F'(t) = f(t)$.*

Definition 3.8. *A division of cell $[a, b]$ into mutually separated cells $K_k = [u_k, v_k], 1 \leq k \leq \kappa$, for some $\kappa \in \mathbb{N}, u_k < v_k, u_1 = a, v_\kappa = b, K_k$ is called a block in the partition of cell $[a, b]$.*

We briefly discuss Henstock-Kurzweil Integral. The details can be found in [5, 6],[7, 8], [9], [10] and [11].

Definition 3.9 (Gauge)**.** *A gauge is a strictly positive function δ, defined on $\mathbb{R}^*$, taking values in $\mathbb{R}^*, \delta(\infty) = \delta(-\infty)$.*

(1) $\forall x \neq \pm\infty$, the gauge determines a closed interval $\delta[x]$ often denoted by δ_x and we set $\delta[x] = [x - \delta(x), x + \delta(x)]$.
The intervals $\delta[\infty] = \left[\frac{1}{\delta(\infty)}, \infty\right]$, $\delta[-\infty] = \left[-\infty, \frac{1}{\delta(-\infty)}\right]$.

(2) If for any $x, \delta(x) = \infty$, we set $\delta[x] = \delta_x = \mathbb{R}^* = \mathbb{X}$ for that point x.

A gauge allows the selection of tag points from an interval at which the Riemann sum is evaluated. It controls the size of the blocks in the partition. If δ and λ are two gauges on $\mathbb{X}$, then there exist a gauge denoted by $\delta \cap \lambda$, defined as $(\delta \cap \lambda)(x) = min\{\delta(x), \lambda(x)\}$, for all $x \in \mathbb{R}^*$. Note: $(\delta \cap \lambda)[x] \subset \delta[x] \cap \lambda[x]$.

Definition 3.10 (Tagged Partition of a cell K)**.** *By a tagged partition P of a cell K, we mean a finite collection $P = \{(x_k, K_k) : k = 1, 2, ..\kappa\}$, where the collection $K_k : k = 1, 2, .., \kappa$ is a partition of K, x_k is a tag of K_k, $x_k \in {K_k}^*(= K_k$ on the real line). The pair (x_k, K_k) is called as a tagged block.*

Definition 3.11 ($\delta-$ fine partitions)**.** *For a gauge δ, a block (x, J) of a tagged partition P, is said to be $\delta-$ fine, if $J \subset \delta_x$. The tagged partition P is called $\delta-$ fine if each tagged block is $\delta-$ fine. We say $P \ll \delta$.*

Definition 3.12 (Riemann sum)**.** *Let $P = \{(x_k, K_k), k = 1, 2, \dots, n\}$ be a $\delta-$ fine tagged partition for some gauge δ. We evaluate the mapping f at the tag points to form the Riemann sums $S(P, \delta, f) = \sum_{k=1}^{n} f(x_k)|K_k|$, where $|K_k|$ is the length of the sub-interval K_k.*

The gauges form a directed set and $\delta \to 0$ implies $\delta(t) \to 0$. The Riemann sums form a net in the TVS $\mathbb{X}$.

Definition 3.13. *The mapping f with values in a TVS X is said to be Henstock–Kurzweil integrable over a cell $K = [a, b]$, if there exist $I \in X$, such that $\lim_{\delta \to 0} S(P, \delta, f) = I$, that is with the property, given a balanced nbd U of 0, a gauge δ such that, for $P \ll \delta$, $S(P, \delta, f) - I \in U$.*

4. IMPLICIT FUNCTION THEOREM

Theorem 4.1 (Implicit Function Theorem)**.** *Let X be a topological space, Y a Banach space and Z a Hausdorff TVS.*

(1) Let F be a continuous mapping of an open set $E = D \times V \subset X \times Y \to Z$ such that $F(a, b) = 0$ for some point $(a, b) \in E$.
(2) $\forall x \in D$, let the mapping $F_x(y) = F(x, y)$ on V be differentiable with respect to y and the derivative $\frac{\partial F}{\partial y}$ is continuous on E.
(3) Let $A = \frac{\partial F}{\partial y}(a, b)$ be an invertible continuous L.T.: $Y \to Z$, with continuous inverse.

Part I: There exist a open nbd $U \subset D, W \subset Y$ with $a \in U, b \in W$, having the following property: there exist a continuous mapping $\phi : U \to W$, such that $y = \phi(x), F(x, \phi(x)) = 0, b = \phi(a)$.
Part II: When X is TVS and the map $h : X \to F(x, b)$ is differentiable with respect to x at the point $x = a$ with the derivative as $T = \frac{\partial F}{\partial x}(a, b)$, then ϕ is differentiable at a, $\phi'(a) = \frac{dy}{dx}(x = b) = -A^{-1} \circ T$.$'\circ'$ stands for composite of linear maps. The assertion holds in a nbd of a and when F is C^1, ϕ is C^1.

Proof. The proof works even if $F(a,b)$ is a constant $c \neq 0$. $F(x,y)$ is replaced with $F(x,y) - c$ which now satisfies the conditions in the theorem.

Proof of the First Part

Step 1: As A^{-1}, F and $\frac{\partial F}{\partial y}$ are continuous on E and $F(a,b) = 0$, select a nbd $U \subset D$ of a and $r > 0, W = B(b,r) \subset B[b,r] \subset V$, such that, for $(x,y) \in D$,

$$\left|\left|A^{-1}\left(\frac{\partial F}{\partial y} - A\right)\right|\right| = \left|\left|A^{-1}\frac{\partial F}{\partial y} - I\right|\right| < \frac{1}{2},\ |A^{-1}F(x,b)| < \frac{r}{2} \tag{4.1}$$

Consider the continuous mapping $K(x,y) = y - A^{-1}F(x,y)\ \forall x \in D$.
Define the mapping $K_x : W \to Y,$ as $K_x(y) = y - A^{-1}F(x,y)$
Then $F(x,y) = 0$ if and only if $K_x(y) = y,$ that is y is a fixed point of K.
Step 2: K_x is shown as a contraction mapping for all $x \in D$. Consider the mapping $H = A^{-1}F - I$. By chain rule, noting that as I and A^{-1} are linear mappings, derivative of A^{-1} is A^{-1} and derivative of I is I. We have,

$$\frac{\partial H}{\partial y} = A^{-1}\frac{\partial F}{\partial y} - I \tag{4.2}$$

So by Mean Value Theorem applied to the line segment joining y_1 and y_2, we have,

$$|H(y_1) - H(y_2)| \leq sup\left|\left|\frac{\partial H}{\partial y}\right|\right| \cdot |y_1 - y_2| < \frac{|y_1 - y_2|}{2} \tag{4.3}$$

$$\begin{aligned}|K_x(y_2) - K_x(y_1)| =& |y_1 - y_2 - A^{-1}[F(x,y_1) - F(x,y_2)]|\\ =& |A^{-1}[A(y_1 - y_2) - F(x,y_1) - F(x,y_2)]|. \text{ Thus,}\end{aligned}$$

$$|K_x(y_2) - K_x(y_1)| = |H(y_1) - H(y_2)| < \frac{1}{2}|y_1 - y_2| \tag{4.4}$$

Thus K_x is a contraction mapping with contracting factor $\frac{1}{2}$.
Step 3: To show K_x maps $B[b,r] \to B[b,r]$ and $B(b,r) \to B(b,r)$.
Consider, for $y \in B[b,r]$,

$$\begin{aligned}|K_x(y) - b| \leq& |K_x(y) - K_x(b)| + |K_x(b) - b|\\ \leq& \frac{1}{2}r + |A^{-1}F(x,b)|\\ =& \frac{r}{2} + \frac{r}{2}\\ =& r, \qquad \text{by using (4.1).}\end{aligned}$$

So given x, K_x has unique fixed point say, y_x in $B[b,r]$. To find $F(x,\phi(x))$, let $y_x = \phi(x)$. Consider $F(x,y_x) = A(y_x - K_x(y_x)) = A(y_x - y_x) = 0$. Since the fixed point is unique, $\phi(a) = b$, as $F(a,b) = 0$.
Step 4: To show ϕ is continuous. Consider,

$$|\phi(x) - \phi(x')| = |K(x,\phi(x)) - K(x',\phi(x'))|$$

$$\leq |K(x,\phi(x)) - K(x,\phi(x')| + |K(x,\phi(x') - K(x',\phi(x')|$$
$$\leq |K_x(\phi(x)) - K_x(\phi(x'))| + |K(x,\phi(x')) - K(x',\phi(x'))|$$
$$\leq \frac{1}{2}|\phi(x) - \phi(x')| + |K(x,\phi(x')) - K(x',\phi(x'))|, \qquad \text{by using (4.4)}$$
$$|\phi(x) - \phi(x')| \leq 2|K(x,\phi(x')) - K(x',\phi(x'))| \tag{4.5}$$

Therefore, ϕ is continuous as K is continuous.

Proof of the Second Part

Step 5: Without loss of generality, let $\phi(a) = b = 0$. The map $g : U \to Z$ where, $g(x) = F(x,0)$ is differentiable at $x = a$, with derivative $\frac{\partial F}{\partial x}(a,0) = T$. Consider $\frac{\phi(a+tv) - \phi(a)}{t} + A^{-1}T \cdot v = \frac{\phi(a+tv) - \phi(a) + A^{-1}T \cdot tv}{t} = \frac{\phi(x) - \phi(a) + A^{-1}T \cdot h}{t}$ where, $x = a + tv, h = tv$. Given $\epsilon > 0$, as $\frac{\partial F}{\partial y}$ is continuous on $D \times V$, there exist a nbd U' of a, $U' \subset U$ and $0 < s < r$ such that, if we set $W' = B(0,s)$ then on $U' \times W'$, $\left\|A^{-1}\left[\frac{\partial F}{\partial y}(x,y) - \frac{\partial F}{\partial y}(a,0)\right]\right\| = \left\|\frac{\partial h}{\partial y}\right\| < \epsilon$. We now use (4.3) and then
$A^{-1}[F(x,y) - F(x,0) - A(y-0)]$
$= A^{-1}[F(x,y) - Ay] - A^{-1}[F(x,0) - A0]$. Hence we have,

$$A^{-1}[F(x,y) - F(x,0) - A(y-0)] = Hy - H0 \leq \epsilon|y| \tag{4.6}$$

Given $y \in W'$ we write $y = \phi(x)$ and y is a fixed point of K_x. Also $\phi(a) = b = 0$.

$$\begin{aligned}
\frac{\phi(x) + A^{-1}T \cdot h}{t} &= \frac{y - A^{-1}[F(x,y) - T \cdot h]}{t} \\
&= \frac{y - A^{-1}[F(x,y) - F(x,0) - Ay + Ay] - A^{-1}[F(x,0) - T \cdot h]}{t} \\
&= \frac{y - A^{-1}[F(x,y) - F(x,0) - A(y-0)] - A^{-1}Ay + A^{-1}[F(x,0) - T \cdot h]}{t} \\
&\leq \epsilon|y| + A^{-1}R(h) \text{ where, } R(h) \text{ is tangent in } X \text{ as } \frac{\partial f}{\partial x} \text{ exists at } x = a.
\end{aligned}$$

Now, $|y| = |\phi(x)| = |K_x(y)| = |K_x(y) - K_x(0) + K_x(0)| \leq |y - K_x(0)| + |K_x(0)|$. Thus, $|y| \leq \frac{1}{2}|y| + |K(x,0)| = \frac{1}{2}|y| + |-A^{-1}F(x,0)|$, by the definition of the mapping K. So, $|y| \leq 2|A^{-1}[F(a,0) + T \cdot h + tR(h)]|$, where $R(h)$ is tangent in X. But $F(a,0) = 0$. Therefore, $|y| \leq |2A^{-1}tR(h)| + |2A^{-1}T(h)|$. As $A^{-1}T$ is continuous, given $\epsilon > 0$, we can select the nbd of x, so that $|A^{-1}T(h)| < \epsilon$. Since $R(h)$ is tangent and A^{-1} is continuous $|y|$ is tangent in X.

Step 6: Now suppose $g(x)$ is C^1 mapping from $X \to Z$. We see (by the lemma which follows) that, as A is invertible at $(a,0)$, there exist a nbd $U = G \times W$ of $(a,0)$ in which $\frac{\partial F}{\partial y} f(x,y)$ is invertible. So there is nothing special about the point $(a,0)$. So ϕ is differentiable at x, in a small neighbourhood around x. Now as

$\frac{dy}{dx} = -\left(\frac{\partial F}{\partial y}\right)^{-1} \cdot \frac{\partial F}{\partial x}, \frac{dy}{dx}$ will be sequentially continuous when $\frac{\partial F}{\partial x}$ is sequentially continuous. Thus ϕ is C^1. □

Lemma 4.2. *A nbd of the point $(a, 0)$ exists in which $\frac{\partial F}{\partial y}(x, y)$ is invertible.*

Proof. Let $\frac{\partial F}{\partial y}(x, y) = P$. It can be noted that L.T. B such that $||B|| < 1$ is invertible, as it is the sum of the convergent geometric series $(I + B + B^2 + \cdots)$. So any L. T. M such that $||M - I|| < 1$ is invertible, as $||M|| = ||I - (I - M)||$. So $||A^{-1}P - I|| < \frac{1}{2}$. Hence $A^{-1}P$ is invertible. Thus, $P = AA^{-1}P$ is invertible as $P^{-1} = (A^{-1}P)^{-1}A^{-1}$. □

Remark 4.1. *An Inverse Function Theorem can be deduced as in* [12].

5. Henstock-Kurzweil Integration

Theorem 5.1. *The integral is well defined.*

Proof. Cousin's lemma [9, 11, 13, 14] ensures that given any gauge $\delta-$ fine tagged partition P exists. Suppose I and I' are two values of the integral. Given a balanced nbd U of 0 select a balanced nbd V of 0 such that $v + V \subset U$. As I is a value of the integral given V, there exist a gauge δ_1 such that $S(P, \delta_1, f) - I \in V$. As I' is a value of the integral given V, a gauge δ_2 such that $S(P, \delta_2, f) - I' \in V$. Consider the gauge $\delta = \delta_1 \cap \delta_2$.
For $P \ll \delta$ we have $S(P, f) - I + I' - S(P, f) \in V + V \subset U$, that is $I - I' \in U$. Since X is Hausdorff, $I = I'$. If $I \neq I'$ they should have disjoint nbds. So we arrive at a contradiction unless $I = I'$ □

It is now shown that every derivative of a continuous function is integrable over a closed interval. This is not true for Riemann or Lebesgue integral.

6. Fundamental Theorem of Calculus

Theorem 6.1 (Fundamental theorem of Calculus)**.** *Let F be a primitive of a mapping f on a closed cell $J = [a, b]$ in $\mathbb{R}$. Then $\int_a^b f(x)dx = F(b) - F(a)$.*

The proof of the theorem depends upon the following simple lemma. It must be noted that only consider balanced neighbourhoods of 0 have to be considered.

Lemma 6.2 (Straddle Lemma)**.** *If F is differentiable at a point t, with the derivative $F'(t)$ denoted by $f(t)$, then for each nbd U of 0, there exist $\delta_\epsilon(t) > 0$ and a cell $T_t, = [t - \delta_\epsilon(t), t + \delta_\epsilon(t)]$ such that whenever $x \geq t \geq y$ are in T_t, that is $[y, x] \ll \delta_U$, $F(x) - F(y) - f(t)(x - y) \in U$, that is, $F(x) - F(y) - f(t)(x - y) \to 0$ as $\delta_U(t) \to 0$.*

Proof. As F is differentiable at t, the derivative $F'(t)$ being given by $f(t)$, $\frac{F(z)-F(t)}{z-t} \to f(t)$, as $z \to t$. As each nbd of origin contains a balanced nbd, given a balanced nbd U of 0, consider a balanced nbd V of 0 such that $V + V \subset U$. So corresponding to V, a number $\delta_u(t) > 0$, such that $\delta_u(t) < \frac{1}{2}$. So, $F(z) - F(t) - f(t) \cdot (z - t) \in |z - t|V, \forall\ z \in T_U = [t - \delta_\epsilon(t), t + \delta_\epsilon(t)]$. Choose $x, y \in T_U$ such that $x \geq t \geq y$, $F(x) - F(t) - f(t)(x - t) \in (x - t)V \subset V$, $F(x) - F(t) - f(t)(x - t) \in (x - t)V \subset V$. The result follows by the addition, using the triangle inequality on the real line and the order $x \geq t \geq y$ and $V + V \in U$. So $F(x) - F(y) - f(t)(x - y) \in U$. Hence we have $F(x) - F(y) - f(t)(x - y) \to 0$, as $\delta_U(t) \to 0$. □

Remark 6.1. [15] *p and q need to straddle r, that is r is between p and q. The lemma states that the slope of the chord joining the points, with ordinates p and q and the slope of the tangent at the point whose ordinate is r are approximately equal as shown in Figure 1.*

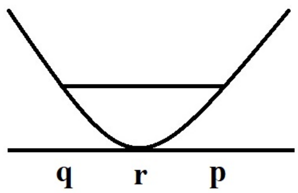

FIGURE 1. Figure showing the positions of p, q and r

Proof. of theorem 6.1. We do this for every point t. So a gauge δ_u is obtained having the property. Let the gauge $\delta(t) = \delta_u(t)$ as in the Straddle lemma. Let $T = \{(x_k, K_k)|k = 1, 2, \ldots, \kappa\}$ be a $\delta-$ fine tagged partition of J. $K_k = [v_{k-1}, v_k]$ where, $k = 1, 2, \ldots, \kappa$ so that $v_0 = a, v_\kappa = b$.

$$\sum_{x_k \notin \mathbb{Z}} F(v_k) - F(v_{k-1}) - f(x_k) \cdot (v_k - v_{k-1}) = [F(b) - F(a)] - S(P, f).$$

By Straddle lemma, $F(v_k) - F(v_{k-1}) - f(t)(v_k - v_{k-1}) \to 0$, for each k adding finitely many terms. So, $|S(P, f) - [F(b) - F(a)]| \to 0$ adding the constant mapping $F(b) - F(a)$, $S(P, f) \to F(b) - F(a)$. □

Theorem 6.3 (Mean Value theorem for Vector valued mappings)**.** [15] *Let F be a mapping continuous on $[c, d]$ assuming values in a Hausdorff TVS X and differentiable in (c, d), then $F(d) - F(c) = h \cdot \int_0^1 F'(c + \theta \cdot h)d\theta$, where $h = d - c$.*

Proof. $F(d) - F(c) = \int_c^d F'(t)dt$ by the Fundamental Theorem of Calculus. By chain rule F is a differentiable mapping of θ and for $t = c + \theta \cdot h$, derivative of F with respect to t is the derivative of F with respect to θ multiplied by h. The proof follows from Fundamental Theorem of Calculus, noting that h is constant. □

Corollary 6.4. *If X is locally convex, that is each nbd contains a convex nbd then if $F'(x)$ is 0 in a path connected open set in X, then F is constant.*

Proof. Since any nbd of a point a in the open set contains a convex nbd U so that any two points can be joined by a straight line segment $x = \{ta+(1-t)b, t \in [0,1]\}$. By Mean Value Theorem, as the derivative is 0 we have $F(b) = F(a)$. So F is locally constant. As the domain is path connected, F is constant. □

As in [16] and [17] one can deduce Mean Value theorem for a mapping $F : X \to Y$, where both are TVS and X is locally convex.

7. CONCLUDING REMARKS

In this article a new version of Mean Value theorem is obtained and proved on TVS using gauges. The Implicit Function Theorem is generalised on TVS. The conditions for existence, continuity and differentiability are also provided for a mapping in TVS. In this article, TVS, Hausdorff TVS and Banach Space are linked with the mapping in generalised version of Implicit Function Theorem. It's a fundamental tool in multivariable calculus and has applications in various fields, including physics, economics, and differential geometry. This theorem allows us to express one or more variables in a system of equations as functions of the remaining variables, under certain conditions on the partial derivatives.

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