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COMMUTATIVE AND SPECTRAL PROPERTIES OF k^{th}-ORDER (SLANT TOEPLITZ + SLANT HANKEL) OPERATORS ON THE POLYDISK

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Abstract. In this paper for $k \geq 2$, we introduce the idea of k^{th}-order (slant Toeplitz + slant Hankel) operators on the polydisk and discuss the commutativity, partial isometry and co-isometry properties. Further, we extend our study to the spectral properties.

Key words: slant Toeplitz Operator, slant Hankel Operator, commutative, spectrum.

1. Introduction and Preliminaries

Since the beginning of nineteenth century, the theory of Toeplitz and Hankel operators is being studied extensively on several spaces like Hardy space, Bergman space, Fock space, etc. These operators have lots of applications in mathematics and mathematical physics and therefore they got a prominent place in the study of operator theory. In 1950, Hartman and Wintner [18] discussed spectral properties of Toeplitz Matrices. In 1974, Fuhrmann [14] derived the necessary and sufficient conditions for the sum of two Hankel operators of closed range to have closed range. In the year 1999, Basor and Ehrhardt [5] studied the sum of Toeplitz and Hankel operator on the Hardy Space and defined it as $\mathcal{M}(\phi) = T_\phi + H_\phi$ (denote $\mathcal{M}(\phi, \phi)$ by $\mathcal{M}(\phi)$) for functions $\phi \in L^\infty(\mathbb{T})$ where T_ϕ and H_ϕ are Toeplitz and Hankel operators, respectively, with symbol ϕ and evaluated its several properties. Later on, they investigated the connections between Fredholmness and invertibility of $\mathcal{M}(\phi)$. These developments were also extended to the study of operator $\mathcal{M}(\phi, \psi) = T_\phi + H_\psi$ for the functions ϕ, $\psi \in L^\infty(\mathbb{T})$. In 1996, the notion of slant Toeplitz operators on $L^2(\mathbb{T})$ was introduced by Ho [15]. S. C. Arora and his research associates [18] introduced the class of slant Hankel operators on $L^2(\mathbb{T})$ and extended his ideas to

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define the compression of slant Hankel operators to $H^2(\mathbb{T})$. In continuation of the works of Basor and Ehrhardt, Didenko and Silbermann [19, 20] discusssed various results on invertibility of Toeplitz plus Hankel operators including its invertibility and inverses. Also, Y. Lu and his research associates [21] discussed the commutativity of k^{th}-order slant Toeplitz. Recently, G. Datt and B. Gupta [6] studied the analogue of slant Hankel operators on the Lebesgue space of n- torus. M. Hazarika and S. Marik[9] analyzed the slant Toeplitz operator on the polydisk. These developments motivated us to study k^{th}-order (slant Toeplitz+ slant Hankel) operators on $L^2(\mathbb{T}^n)$. For adequate literature on Toeplitz, Hankel, slant Toeplitz, slant Hankel operators one can refer [2, 3, 4, 13, 20]. Toeplitz and Hankel operators often appear in differential equations and integral inequalities, which could relate to the properties of operators in function spaces. The papers [1, 7, 10, 11, 12, 17] explore various mathematical and computational methods, including special functions, optimization in fuzzy systems, decision-making under uncertainty, numerical schemes for differential equations and integral inequalities. While they do not directly address Hankel or Toeplitz operators, there are indirect connections. Numerical methods for differential equations and integral equations might involve Toeplitz matrices and functional analysis techniques could intersect with the study of these operators

Let $\mathbb{D}$ be the open unit polydisc in $\mathbb{C}^n$ and $\mathbb{T}^n$, the distinguished boundary of $\mathbb{D}$, denotes the $n-$torus. Although, the function theory on the polydisc is significantly different from the one on the unit disc, the available theory of multiple Fourier series on the n-torus enables one to discuss function spaces (e.g. $L^2(\mathbb{T}^n)$, $L^\infty(\mathbb{T}^n)$) as well as slant Toeplitz operators in the higher dimensional setting. In the whole paper, the space of all Lebesgue integrable functions on $\mathbb{T}^n$ and the collection of all Lebesgue square integrable functions on $\mathbb{T}^n$ are respectively denoted by $L^1(\mathbb{T}^n)$ and $L^2(\mathbb{T}^n)$. The class of all essentially bounded measurable functions on $\mathbb{T}^n$ is expressed by $L^\infty(\mathbb{T}^n)$. Any two functions in these spaces are equal in the sense of equality almost everywhere. The Fourier coefficients of $f \in L^1(\mathbb{T}^n)$ are given by

$$f_{m_1,....,m_n} = \frac{1}{(2\pi)^n} \int_0^{2\pi} ... \int_0^{2\pi} f(e^{i\theta_1}, ..., e^{i\theta_n}) e^{-i(m_1\theta_1+...+m_n\theta_n)} d\theta_1...d\theta_n,$$

for $m_i \in \mathbb{Z}$, $1 \leq i \leq n$. The multiple Fourier series can be seen or treated as Fourier transformation of functions in $L^1(\mathbb{T}^n)$. Therefore , by the multiple Fourier series on $\mathbb{T}^n$, the spaces $L^2(\mathbb{T}^n)$ and $H^2(\mathbb{T}^n)$ can be written as

$$L^2(\mathbb{T}^n) = \Big\{ f : f(z_1, z_2, ..., z_n) = \sum_{(m_1,m_2,...,m_n)\in\mathbb{Z}^n} f_{m_1,m_2,...,m_n} z_1^{m_1} z_2^{m_2} ... z_n^{m_n},$$

$$\sum_{(m_1,m_2,...,m_n)\in\mathbb{Z}^n} |f_{m_1,m_2,...,m_n}|^2 < \infty \Big\}$$

and

$$H^2(\mathbb{T}^n) = \Big\{ f : f(z_1, z_2, ..., z_n) = \sum_{(m_1,m_2,...,m_n)\in\mathbb{Z}_+^n} f_{m_1,m_2,...,m_n} z_1^{m_1} z_2^{m_2} ... z_n^{m_n},$$

$$\sum_{(m_1,m_2,...,m_n)\in\mathbb{Z}_+^n} |f_{m_1,m_2,...,m_n}|^2 < \infty\Big\}$$

where $\mathbb{Z}$ and $\mathbb{Z}_+$ are respectively the set of all integers and the set of all non-negative integers. Clearly, the space $L^2(\mathbb{T}^n)$ is a Hilbert space with the norm given by the inner product

$$\langle f, g\rangle = \frac{1}{(2\pi)^n}\int_0^{2\pi}\int_0^{2\pi}...\int_0^{2\pi} f(e^{i\theta_1},...,e^{i\theta_n})\overline{g(e^{i\theta_1},...,e^{i\theta_n})}d\theta_1 d\theta_2...d\theta_n.$$

The class $\Big\{e_{m_1,m_2,...m_n}(z_1, z_2, ...z_n) = z_1^{m_1}z_2^{m_2}...z_n^{m_n}, (m_1, m_2, ..., m_n) \in \mathbb{Z}^n\Big\}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$. For $f \in L^2(\mathbb{T})$ and for $\phi \in L^\infty(\mathbb{T}^n)$, the operator $M_\phi^n : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ is the multiplication operator induced by ϕ and is defined as $M_\phi^n(f) = \phi f$ and $J : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ defined as $J(f(z)) = f(\overline{z})$ is the unitary self-adjoint operator. The Toeplitz operator on $L^2(\mathbb{T}^n)$ with symbol ϕ is a bounded linear operator $L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ defined by $\mathbb{T}_\phi^n(f) = PM_\phi(f)$ and the Hankel operator on $L^2(\mathbb{T}^n)$ is defined as $H_\phi^n(f) = PJM_\phi(f)$ for all $f \in H^2(\mathbb{T}^n)$. Also, $\|T_\phi^n\| \le \|\phi\|_\infty$. Throughout the paper we assume that $k \ge 2$. Define W_k^n : $L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ by

$$W_k^n(z_1^{i_1}z_2^{i_2}....z^{i_n}) = \begin{cases} z_1^{\frac{i_1}{k}}z_2^{\frac{i_2}{k}}....z_n^{\frac{i_n}{k}}, & \text{if } k|i_j \ \forall\ 1 \le j \le n \\ 0, & \text{otherwise.}\end{cases}$$

One can refer to find that W_k^n is a bounded linear operator with $\|W_k^n\| = 1$ and the adjoint of W_k^n is given by $(W_k^n)^*(z_1^{i_1}z_2^{i_2}....z^{i_n}) = z_1^{ki_1}z_2^{ki_2}....z^{ki_n}$ for $n \in Z^n$.

In this paper, we introduce the k^{th}- order(slant Toeplitz + slant Hankel) operators on the polydisk discussing the matrix representation and partial isometry and co-isometry in the first section of the paper. In the second section, we are going to discuss the commutativity of k^{th} order(slant Toeplitz + slant Hankel) operators on the polydisk using harmonic symbols and we show that these operators commutes if and only if their harmonic symbols are scalar multiple of each other. In the last section of the paper we are discussing the point spectra and spectrum of the k^{th}- order(slant Toeplitz + slant Hankel) operator.

2. Matrix representation of slant (Toeplitz + Hankel) operator

In this section, a k^{th}-order (slant Toeplitz + slant Hankel) operator on $L^2(\mathbb{T}^n)$ is defined and develope a matrical way to represent the operator. The basic properties of partial Isometry and Co-isometry are also studied.

Definition 2.1. [9] *For $\phi \in L^\infty(\mathbb{T}^n)$ and an integer $k \ge 2$, a k^{th}- order slant Toeplitz operator $A_\phi^{k,n}$ with symbol ϕ, on the Lebesgue space $L^2(\mathbb{T}^n)$, is defined to be*

$$A_\phi^{k,n} = W_k^n M_\phi,$$

where M_ϕ is the multiplication operator on $L^2(\mathbb{T}^n)$ induced by ϕ.

Definition 2.2. [6] For an integer $k \geq 2$, a k^{th}- order slant Hankel operator of level n on $L^2(\mathbb{T}^n)$ is defined as $S_{\psi}^{k,n} = W_k^n J M_{\phi}$, where $\phi \in L^{\infty}(\mathbb{T}^n)$.

Definition 2.3. *Let ϕ and $\psi \in L^{\infty}(\mathbb{T}^n)$. For any $f \in L^2(\mathbb{T}^n)$ and $k \geq 2$, a k^{th}-order (slant Toeplitz + slant Hankel) operators on $L^2(\mathbb{T}^n)$, $T_{M_{(\phi,\psi)}}^{k,n}$ is defined as*

$$\begin{aligned} T_{M_{(\phi,\psi)}}^{k,n} = A_{\phi}^{k,n} + S_{\psi}^{k,n} &= W_k^n M_{\phi} + W_k J M_{\psi} \\ &= W_k^n (M_{\phi} + J M_{\psi}). \end{aligned}$$

When ϕ and ψ are equal $T_{M(\phi,\psi)}^{k,n}$ reduces to

$$T_{M_{\phi}}^{k,n} = A_{\phi}^{k,n} + S_{\phi}^{k,n} = W_k^n (I+J) M_{\phi} \text{ and}$$

$$(T_{M_{\phi}}^{k,n})^* = M_{\overline{\phi}} (I+J)(W_k^n)^*.$$

Let $a_{m,m'}$ denotes $(m,m')^{th}$ entry of the matrix representation of $T_{M_{(\phi,\psi)}}^{k,n}$ with respect to the orthonormal basis $\{e_u(z_1, z_2, \ldots, z_n) = z_1^{u_1} z_2^{u_2} \ldots z_n^{u_n}\}_{u \in \mathbb{Z}^n}$ of $L^2(\mathbb{T}^n)$ where $m = (m_1, m_2, \ldots, m_n), m' = (m_1', m_2', \ldots, m_n')$ and $u = (u_1, u_2, \ldots, u_n)$ respectively. Let $\epsilon_j = (x_1, x_2, ..., x_n)$ where $x_i = \delta_{ij}$ for $j \in [1, n]$ and

$$\delta_{ij} = \begin{cases} 1 & \text{if i = j} \\ 0 & \text{if otherwise.} \end{cases}$$

Let

$$\phi(z_1^{i_1} z_2^{i_2} \ldots . z^{i_n}) = \sum_{m \in \mathbb{Z}^n} \phi_m e_m (z_1^{i_1} z_2^{i_2} \ldots . z^{i_n})$$

and

$$\psi(z_1^{i_1} z_2^{i_2} \ldots . z^{i_n}) = \sum_{m \in \mathbb{Z}^m} \psi_m e_m (z_1^{i_1} z_2^{i_2} \ldots . z^{i_n})$$

be functions in $L^{\infty}(\mathbb{T}^n)$. Then for $m = (m_1, m_2, \ldots, m_n), m' = (m_1', m_2', \ldots, m_n') \in \mathbb{Z}^n$.

$$\begin{aligned} a_{m',m} &= \left\langle T_{M(\phi,\psi)}^{k,n} e_m(z_1 z_2 \ldots . z_n), e_{m'}(z_1 z_2 \ldots . z_n) \right\rangle \\ &= \left\langle A_{\phi}^{k,n} e_m(z_1 z_2 \ldots . z_n), e_{m'}(z_1 z_2 \ldots . z_n) \right\rangle + \left\langle S_{\psi}^{k,n} e_m(z_1 z_2 \ldots . z_n), e_{m'}(z_1 z_2 \ldots . z_n) \right\rangle \\ &= \left\langle A_{\phi}^{k,n} e_{m+k\epsilon_j}(z_1 z_2 \ldots . z_n), e_{m'+\epsilon_j}(z_1 z_2 \ldots . z_n) \right\rangle \\ &\quad + \left\langle S_{\psi}^{k,n} e_{m-k\epsilon_j}(z_1 z_2 \ldots . z_n), e_{m'+\epsilon_j}(z_1 z_2 \ldots . z_n) \right\rangle \\ &= \phi_{km'-m} + \psi_{-km'-m}. \end{aligned}$$

Let us consider the sequence of scalars $\{a_m\}_{m \in \mathbb{Z}^n}$ and $\{u_{m'}\}_{m' \in \mathbb{Z}^n}$. A matrix of the type

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \ldots & \alpha^1 & \beta^1 & \gamma^1 & \ldots \\ \ldots & \mu^1 & \nu^1 & \xi^1 & \ldots \\ \ldots & \rho^1 & \tau^1 & \chi^1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$\begin{aligned}
\alpha^1 &= \phi_{(-(k+1),m'_2,\dots,m'_n)} + \psi_{(k+1,m_2,\dots,m_n)},\\
\beta^1 &= \phi_{(2k,m'_2,\dots,m'_n)} + \psi_{(0,m_2,\dots,m_n)},\\
\gamma^1 &= \phi_{(-(2k+1),m'_2,\dots,m'_n)} + \psi_{(-1,m_2,\dots,m_n)},\\
\mu^1 &= \phi_{(-1,m'_2,\dots,m'_n)} + \psi_{(-1,m_2,\dots,m_n)},\\
\nu^1 &= \phi_{(-k,m'_2,\dots,m'_n)} + \psi_{(-k,m_2,\dots,m_n)},\\
\xi^1 &= \phi_{(-(k+1),m'_2,\dots,m'_n)} + \psi_{(-(k+1),m_2,\dots,m_n)},\\
\rho^1 &= \phi_{(-1,m'_2,\dots,m'_n)} + \psi_{(-(k+1),m_2,\dots,m_n)},\\
\tau^1 &= \phi_{(0,m'_2,\dots,m'_n)} + \psi_{(-2k,m_2,\dots,m_n)},\\
\chi^1 &= \phi_{(-1,m'_2,\dots,m'_n)} + \psi_{(-(2k+1),m_2,\dots,m_n)}
\end{aligned}$$

is said to be the (slant Toeplitz + slant Hankel) matrix of level 1. A block matrix of the form

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & \alpha^2 & \beta^2 & \gamma^2 & \dots \\ \dots & \mu^2 & \nu^2 & \xi^2 & \dots \\ \dots & \rho^2 & \tau^2 & \chi^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$\begin{aligned}
\alpha^2 &= \phi_{(-(k+1),m'_3,\dots,m'_n)} + \psi_{(k+1,m_2,\dots,m_n)},\\
\beta^2 &= \phi_{(2k,m'_3,\dots,m'_n)} + \psi_{(0,m_3,\dots,m_n)},\\
\gamma^2 &= \phi_{(-(2k+1),m'_3,\dots,m'_n)} + \psi_{(-1,m_3,\dots,m_n)},\\
\mu^2 &= \phi_{(-1,m'_3,\dots,m'_n)} + \psi_{(-1,m_3,\dots,m_n)},\\
\nu^2 &= \phi_{(-k,m'_3,\dots,m'_n)} + \psi_{(-k,m_3,\dots,m_n)},\\
\xi^2 &= \phi_{(-(k+1),m'_3,\dots,m'_n)} + \psi_{(-(k+1),m_3,\dots,m_n)},\\
\rho^2 &= \phi_{(-1,m'_3,\dots,m'_n)} + \psi_{(-(k+1),m_3,\dots,m_n)},\\
\tau^2 &= \phi_{(0,m'_3,\dots,m'_n)} + \psi_{(-2k,m_3,\dots,m_n)},\\
\chi^2 &= \phi_{(-1,m'_3,\dots,m'_n)} + \psi_{(-(2k+1),m_3,\dots,m_n)}
\end{aligned}$$

is said to be the (slant Toeplitz + slant Hankel) matrix of level 2. Continuing in this way, we get the (slant Toeplitz + slant Hankel) matrix of level n as

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & \alpha^{n-1} & \beta^{n-1} & \gamma^{n-1} & \dots \\ \dots & \mu^{n-1} & \nu^{n-1} & \xi^{n-1} & \dots \\ \dots & \rho^{n-1} & \tau^{n-1} & \chi^{n-1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$\alpha^{n-1} = \phi_{(-(k+1))} + \psi_{(k+1)}, \beta^{n-1} = \phi_{(2k)} + \psi_{(0)},$$
$$\gamma^{n-1} = \phi_{(-(2k+1))} + \psi_{(-1)}, \mu^{n-1} = \phi_{(-1)} + \psi_{(-1)},$$
$$\nu^{n-1} = \phi_{(-k)} + \psi_{(-k)}, \xi^{n-1} = \phi_{(-(k+1))} + \psi_{(-(k+1))},$$
$$\rho^{n-1} = \phi_{(-1)} + \psi_{(-(k+1))}, \tau^{n-1} = \phi_{(0)} + \psi_{(-2k)},$$
$$\chi^{n-1} = \phi_{(-1)} + \psi_{(-(2k+1))}.$$

Lemma 2.4. *If* $\phi(z) = \sum_{(p_1,p_2,\ldots,p_n)\in z^n} a_{(p_1,p_2,\ldots,p_n)}(z_1^{p_1} z_2^{p_2} \ldots . z_n^{p_n}),$

$$\psi(z) = \sum_{(q_1,q_2,\ldots,q_n)\in z^n} b_{(q_1,q_2,\ldots,q_n)}(z_1^{q_1} z_2^{q_2} \ldots . z_n^{q_n})$$

be two functions in $L^\infty(\mathbb{T}^n)$ *and* W_k *is a bounded linear operator on* $L^2(\mathbb{T}^n)$ *then*
$W_k^n\ T^{k,n}_{M_{(\phi,\psi)}}(W_k^n)^* e_{m_1,m_2,\ldots,m_n}(z_1 z_2 \ldots . z_n)$
$= \Big(\sum_{(p_1,p_2,\ldots,p_n)\in z^n} a_{(k^2p_1,k^2p_2,\ldots,k^2p_n)}(z_1^{p_1} z_2^{p_2} \ldots . z_n^{p_n}) +$
$\sum_{(q_1,q_2,\ldots,q_n)\in z^n} b_{(k^2q_1,k^2q_2,\ldots,k^2q_n)}(z_1^{q_1} z_2^{q_2} \ldots . z_n^{q_n})\Big)(z_1^{\frac{m_1}{k}} z_2^{\frac{m_2}{k}} \ldots . z_n^{\frac{m_n}{k}}).$

Proof. Let $\phi(z) = \sum_{(p_1,p_2,\ldots,p_n)\in z^n} a_{(p_1,p_2,\ldots,p_n)}(z_1^{p_1} z_2^{p_2} \ldots . z_n^{p_n}),$

$$\psi(z) = \sum_{(q_1,q_2,\ldots,q_n)\in z^n} b_{(q_1,q_2,\ldots,q_n)}(z_1^{q_1} z_2^{q_2} \ldots . z_n^{q_n})$$

be two functions in $L^\infty(\mathbb{T}^n)$ and W_k is a bounded linear operator on $L^2(\mathbb{T}^n)$. Then,

$$\begin{aligned}
&W_k^n\ T^{k,n}_{M_{(\phi,\psi)}}(W_k^n)^* e_m(z_1 z_2 \ldots . z_n)\\
&= (W_k^n)W_k^n(M_\phi + JM_\psi)(W_k^n)^* e_m(z_1 z_2 \ldots . z_n)\\
&= (W_k^n)(W_k^n)\Big(\sum_{(p_1,p_2,\ldots,p_n)\in z^n} a_{(p_1,p_2,\ldots,p_n)}(z_1^{p_1} z_2^{p_2} \ldots . z_n^{p_n}) +\\
&J \sum_{(q_1,q_2,\ldots,q_n)\in z^n} b_{(q_1,q_2,\ldots,q_n)}(z_1^{q_1} z_2^{q_2} \ldots . z_n^{q_n})\Big)(z_1^{km_1} z_2^{km_2} \ldots z_n^{km_n})\\
&= W_k^n \Bigg[\sum_{(p_1,p_2,\ldots,p_n)\in z^n} a_{(p_1,p_2,\ldots,p_n)}\Big(z_1^{\frac{p_1+km_1}{k}} z_2^{\frac{p_2+km_2}{k}} \ldots z_n^{\frac{p_n+km_m}{k}}\Big) +\\
&\sum_{(q_1,q_2,\ldots,q_n)\in z^n} b_{(q_1,q_2,\ldots,q_n)}\Big(z_1^{\frac{-q_1+km_1}{k}} z_2^{\frac{-q_2+km_2}{k}} \ldots z_n^{\frac{-q_n+km_n}{k}}\Big)\Bigg]
\end{aligned}$$

$$= \Bigg(\sum_{(p_1,p_2,\ldots,p_n)\in z^n} a_{(k^2p_1,k^2p_2,\ldots,k^2p_n)}(z_1^{p_1} z_2^{p_2} \ldots. z_n^{p_n}) +$$
$$\sum_{(q_1,q_2,\ldots,q_n)\in z^n} b_{(k^2q_1,k^2q_2,\ldots,k^2q_n)}(z_1^{q_1} z_2^{q_2} \ldots. z_n^{q_n}) \Bigg) (z_1^{\frac{m_1}{k}} z_2^{\frac{m_2}{k}} \ldots. z_n^{\frac{m_n}{k}})$$

□

2.1. **Partial isometry and Co-isometry.**

Lemma 2.5. *A k^{th}- order (slant Toeplitz + slant Hankel) operators on $L^2(\mathbb{T}^n)$, $T^{k,n}_{M_\phi}$ is partial isometry if and only if $\phi|\phi|^2(I+J)^2$.*

Proof. An operator T is partial isometry if and only if $T = TT^*T$. So,

$$\begin{aligned} T^{k,n}_{M_\phi} &= (T^{k,n}_{M_\phi})(T^{k,n}_{M_\phi})^*(T^{k,n}_{M_\phi}) \\ &= W^n_k(I+J)M_\phi M_{\overline{\phi}}(I+J)(W^n_k)^*(W^n_k)(I+J)M_\phi \\ &= W^n_k(I+J)M_{|\phi|^2}(I+J)(I+J)M_\phi \\ &= W^n_k(I+J)M_{\phi|\phi|^2(I+J)^2}. \end{aligned}$$

Hence, $T^{k,n}_{M_\phi}$ is partially isometry iff $\phi = \phi.|\phi|^2(I+J)^2$. □

Lemma 2.6. *A k^{th}- order (slant Toeplitz + slant Hankel) operators on $L^2(\mathbb{T}^n)$, $T^{k,n}_{M_\phi}$ is co-isometry iff $|\phi|^2 W^n_k(I+J)^2 = 1$.*

Proof. An operator T is co-isometry iff $TT^* = I$. So,

$$\begin{aligned} &(T^{k,n}_{M(\phi)})(T^{k,n}_{M(\psi)})^* \\ &= W^n_{k(I+J)}M_\phi M_{\overline{\phi}}(I+J)(W^n_k)^* \\ &= W^n_k(I+J)\phi\overline{\phi}(I+J)(W^n_k)^* \\ &= |\phi|^2 W^n_k(I+J)^2(W^n_k)(W^n_k)^* \\ &= |\phi|^2 W^n_k(I+J)^2 \\ &= M_|\phi|^2 W^n_k(I+J)^2. \end{aligned}$$

Therefore, $T^{k,n}_{M(\phi)}$ is isometry iff $|\phi|^2 W^n_k(I+J)^2 = 1$. □

Theorem 2.7. *$T^{k,n}_{M_\phi}$ is co-isometry if and only if $(T^{k,n}_{M_{(\phi)}})^*$ preserves norm, i.e., if and only if* $\displaystyle\sum_{r_j=0,1\le j\le n}^{k-1} |\phi(\frac{\theta_1+2r_1\pi}{k},..,\frac{\theta_n+2r_n\pi}{k})|^2|I+j|^2 = k^n.$

Proof. For any $f \in L^2(\mathbb{T}^n)$,

$$
\begin{aligned}
&||(T^{k,n}_{M_{(\phi)}})^*(f)||^2 \\
&= ||M_{\overline{\phi}}(I+J)(w^n_k)^*(f)||^2 \\
&= ||M_{\overline{\phi}}(I+J)f(k\theta_1, k\theta_2, \dots, k\theta_n)||^2 \\
&= \Big\langle \overline{\phi}(\theta^1, \theta^2, \dots, \theta^n)(I+J)f(k\theta_1, k\theta_2, \dots, k\theta_n), \overline{\phi}(\theta^1, \theta^2, \dots, \theta^n)(I+J) \\
&\qquad \times f(k\theta_1, k\theta_2, \dots, k\theta_n)\Big\rangle \\
&= \frac{1}{(2\pi)^n} \underbrace{\int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi}}_{n\ times} |\phi(\theta^1, \theta^2, \dots, \theta^n)|^2 (I+J)f(k\theta_1, k\theta_2, \dots, k\theta_n) \\
&\qquad \times \overline{(I+J)f(k\theta_1, k\theta_2, \dots, k\theta_n)} d\theta_1 d\theta_2 \cdots d\theta_n \\
&= \frac{1}{(2\pi)^n} \underbrace{\int_0^{2\pi}\int_0^{2\pi}...\int_0^{2\pi}}_{n\ times} |(\phi(\theta^1, \theta^2, ..., \theta^n)|^2 |(I+J) \\
&\times f(k\theta_1, k\theta_2, ..., k\theta_n)|^2 d\theta_1 d\theta_2 ... d\theta_n \\
&= \frac{1}{(2\pi)^n} \underbrace{\int_0^{2\pi}\int_0^{2\pi}...\int_0^{2\pi}}_{n\ times} |\phi(\frac{\theta_1}{k}, \frac{\theta_2}{k}, ...\frac{\theta_n}{k})|^2 |(I+J) \\
&\times f(\theta_1, \theta_2, ...\theta_n)|^2 d\theta_1 d\theta_2 ... d\theta_n \\
&= \frac{1}{(2\pi)^n} \underbrace{\int_0^{2\pi}\int_0^{2\pi}...\int_0^{2\pi}}_{n\ times} \sum_{r_j, 1\le j\le n}^{k-1} \Big|\phi(\frac{\theta_1 + 2r_1\pi}{k}, \frac{\theta_2 + 2r_2\pi}{k}, ...\frac{\theta_n + 2r_n\pi}{k})\Big|^2 \\
&\qquad \times \Big|(I+J)f(\theta_1, \theta_2, ..., \theta_n)\Big|^2 d\theta_1 d\theta_2 ... d\theta_n \\
&= \frac{1}{(2\pi)^n} \underbrace{\int_0^{2\pi}\int_0^{2\pi}...\int_0^{2\pi}}_{n\ times} \frac{1}{k^n} \sum_{r_j, 1\le j\le n}^{k-1} \Big|\phi(\frac{\theta_1 + 2r_1\pi}{k}, \frac{\theta_2 + 2r_2\pi}{k}, ...\frac{\theta_n + 2r_n\pi}{k})\Big|^2 \\
&\qquad \times \Big|(I+J)f(\theta_1, \theta_2, ..., \theta_n)\Big|^2 d\theta_1 d\theta_2 ... d\theta_n \\
&= ||M_{\psi(I+J)}(f)||^2,
\end{aligned}
$$

where

$$\psi(\theta_1, \theta_2, ..., \theta_n) = \sqrt{\frac{1}{k^n} \times \sum_{r_j, 1\le j\le n} |\phi(\frac{\theta_1 + 2r_1\pi}{k}, \frac{\theta_2 + 2r_2\pi}{k}, ...\frac{\theta_n + 2r_n\pi}{k})|^2}.$$

$||M_{\psi(I+J)}(f)|| = ||f||$ if and only if $\psi(I+J)$ is unimodular
i.e., $|\psi(I+J)| = 1$

$$\implies \left|\sqrt{\frac{1}{k^n} \times \sum_{r_j, 1\leq j\leq n}^{k-1} \left|\phi(\frac{\theta_1 + 2r_1\pi}{k}, \frac{\theta_2 + 2r_2\pi}{k}, ...\frac{\theta_n + 2r_n\pi}{k})\right|^2 (I+J)}\right|^2 = 1^2.$$

$$\implies \left|\frac{1}{k^n} \times \sum_{r_j, 1\leq j\leq n}^{k-1} \left|\phi(\frac{\theta_1 + 2r_1\pi}{k}, \frac{\theta_2 + 2r_2\pi}{k}, ...\frac{\theta_n + 2r_n\pi}{k})\right|^2 (I+J)^2\right| = 1.$$

$$\implies \sum_{r_j, 1\leq j\leq n}^{k-1} \left|\phi(\frac{\theta_1 + 2r_1\pi}{k}, \frac{\theta_2 + 2r_2\pi}{k}, \dots \frac{\theta_n + 2r_n\pi}{k})\right|^2 \left|I+J\right|^2 = k^n.$$

□

3. Commutativity

In these section we are going to discuss the commutativity of k^{th} order (slant Toeplitz + slant Hankel) operators on the polydisk using harmonic symbols and we show that these operators commutes if and only if their harmonic symbols are scalar multiple of each other.

Theorem 3.1. *If $\phi(z) = \sum_{s\in z^n} b_s e_s(z_1 z_2 \dots z_n)$, $\psi(z) = \sum_{r\in z^n} a_r e_r(z_1 z_2 \dots z_n)$ be two functions in $L^\infty(\mathbb{T}^n)$ where $s = (s_1, s_2, \dots, s_n)$ and $r = (r_1, r_2, \dots, r_n)$ then $T^{k,n}_{M_{(\phi)}}$ commutes with $T^{k,n}_{M_{(\psi)}}$ if and only if $r = s$.*

Proof. Let $\phi(z) = \sum_{s\in z^n} b_s e_s(z_1 z_2 \dots z_n)$, $\psi(z) = \sum_{r\in z^n} a_r e_r(z_1 z_2 \dots z_n)$ be two functions in $L^\infty(\mathbb{T}^n)$ where $s = (s_1, s_2, \dots, s_n)$ and $r = (r_1, r_2, \dots, r_n)$. Let $e_m(z_1 z_2 \dots z_n)$ be a orthonormal basis of $L^2(\mathbb{T}^n)$. Then,

$$
\begin{aligned}
&(T^{k,n}_{M_{(\phi)}})^* (T^{k,n}_{M_{(\psi)}})^* e_m(z_1 z_2 \dots z_n) \\
=&M_{\overline{\phi}}(I+J)(W^n_k)^* M_{\overline{\psi}}(I+J)(z_1^{km_1} z_2^{km_2} \dots z_n^{km_n}) \\
=&M_{\overline{\phi}}(I+J)(W^n_k)^* \Bigg[\sum_{(r_1, r_2, \dots, r_n)\in\mathbb{Z}^n} \overline{a}_{(r_1, r_2, \dots, r_n)} (z_1^{-r_1+km_1} z_2^{-r_2+km_2} \dots z_n^{-r_n+km_n}) \\
&+ \sum_{(r_1, r_2, \dots, r_n)\in\mathbb{Z}^n} \overline{a}_{(r_1, r_2, \dots, r_n)} (z_1^{-r_1-km_1} z_2^{-r_2-km_2} \dots z_n^{-r_n-km_n}) \Bigg] \\
=&M_{\overline{\phi}} \Bigg[\sum_{(r_1, r_2, \dots, r_n)\in\mathbb{Z}^n} \overline{a}_{(r_1, r_2, \dots, r_n)} (z_1^{-kr_1+k^2m_1} z_2^{-kr_2+k^2m_2} \dots z_n^{-kr_n+k^2m_n}) \\
&+ \sum_{(r_1, r_2, \dots, r_n)\in\mathbb{Z}^n} \overline{a}_{(r_1, r_2, \dots, r_n)} (z_1^{-kr_1-k^2m_1} z_2^{-kr_2-k^2m_2} \dots z_n^{-kr_n-k^2m_n})
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{(r_1,r_2,\dots,r_n)\in\mathbb{Z}^n} \overline{a}_{(r_1,r_2,\dots,r_n)}(z_1^{kr_1-k^2m_1} z_2^{kr_2-k^2m_2} \dots z_n^{kr_n-k^2m_n}) \\
&+ \sum_{(r_1,r_2,\dots,r_n)\in\mathbb{Z}^n} \overline{a}_{(r_1,r_2,\dots,r_n)}(z_1^{kr_1+k^2m_1} z_2^{kr_2+k^2m_2} \dots z_n^{kr_n+k^2m_n})\Big] \\
= &\sum_{r\in\mathbb{Z}^n}\sum_{s\in\mathbb{Z}^n} \overline{a}_r\overline{b}_s(z_1^{-s_1-kr_1+k^2m_1} z_2^{-s_2-kr_2+k^2m_2} \dots z_n^{-s_n-kr_n+k^2m_n}) \\
&+ \sum_{r\in\mathbb{Z}^n}\sum_{s\in\mathbb{Z}^n} \overline{a}_r\overline{b}_s(z_1^{-s_1-kr_1-k^2m_1} z_2^{-s_2-kr_2-k^2m_2} \dots z_n^{-s_n-kr_n-k^2m_n}) \\
&+ \sum_{r\in\mathbb{Z}^n}\sum_{s\in\mathbb{Z}^n} \overline{a}_r\overline{b}_s(z_1^{-s_1+kr_1-k^2m_1} z_2^{-s_2+kr_2-k^2m_2} \dots z_n^{-s_n+kr_n-k^2m_n}) \\
&+ \sum_{r\in\mathbb{Z}^n}\sum_{s\in\mathbb{Z}^n} \overline{a}_r\overline{b}_s(z_1^{-s_1+kr_1+k^2m_1} z_2^{-s_2+kr_2+k^2m_2} \dots z_n^{-s_n+kr_n+k^2m_n}) \qquad (1)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&(T^{k,n}_{M(\psi)})^*(T^{k,n}_{M(\phi)})^* e_m(z_1z_2\dots z_n) \\
= &\sum_{r\in\mathbb{Z}^n}\sum_{s\in\mathbb{Z}^n} \overline{a}_r\overline{b}_s(z_1^{-r_1-ks_1+k^2m_1} z_2^{-r_2-ks_2+k^2m_2} \dots z_n^{-r_n-ks_n+k^2m_n}) \\
&+ \sum_{r\in\mathbb{Z}^n}\sum_{s\in\mathbb{Z}^n} \overline{a}_r\overline{b}_s(z_1^{-r_1-ks_1-k^2m_1} z_2^{-r_2-ks_2-k^2m_2} \dots z_n^{-r_n-ks_n-k^2m_n}) \\
&+ \sum_{r\in\mathbb{Z}^n}\sum_{s\in\mathbb{Z}^n} \overline{a}_r\overline{b}_s(z_1^{-r_1+ks_1-k^2m_1} z_2^{-r_2+ks_2-k^2m_2} \dots z_n^{-r_n+ks_n-k^2m_n}) \\
&+ \sum_{r\in\mathbb{Z}^n}\sum_{s\in\mathbb{Z}^n} \overline{a}_r\overline{b}_s(z_1^{-r_1+ks_1+k^2m_1} z_2^{-r_2+ks_2+k^2m_2} \dots z_n^{-r_n+ks_n+k^2m_n}) \qquad (2)
\end{aligned}
$$

From (1) and (2),
$(T^{k,n}_{M(\phi)})^*(T^{k,n}_{M(\psi)})^* e_m(z_1z_2\dots z_n) = (T^{k,n}_{M(\psi)})^*(T^{k,n}_{M(\phi)})^* e_m(z_1z_2\dots z_n)$ if and only if $s = r$ where $s, r \in \mathbb{Z}^n$ i.e $s_1 = r_1, s_2 = r_2, \dots, s_n = r_n$.
Hence, $T^{k,n}_{M(\phi)}$ and $T^{k,n}_{M(\psi)}$ commutes if and only if $s = r$, where $s, r \in \mathbb{Z}^n$. □

Corollary 3.2. *If* $\phi(z) = \sum_{p\in z^n} a_p e_p(z_1z_2\dots z_n)$, $\psi(z) = \sum_{q\in z^n} a_q e_q(z_1z_2\dots z_n)$, $\zeta(z) = \sum_{r\in z^n} a_r e_r(z_1z_2\dots z_n)$ *and* $\eta(z) = \sum_{s\in z^n} a_s e_s(z_1z_2\dots z_n)$ *be functions in* $L^\infty(\mathbb{T}^n)$ *where* $p = (p_1, p_2, \dots, p_n)$, $q = (q_1, q_2, \dots, q_n)$, $r = (r_1, r_2, \dots, r_n)$ *and* $s = (s_1, s_2, \dots, s_n)$ *then* $T^{k,n}_{M(\phi,\psi)}$ *commutes with* $T^{k,n}_{M(\zeta,\eta)}$ *if and only if* $p = r = \frac{1-k}{k+1}q$ *and* $q = s$.

Theorem 3.3. *If* $\phi(z) = \overline{g(z)} + f(z), \psi(z) = \overline{p(z)} + q(z), \xi(z) = \overline{r(z)} + s(z)$ *and* $\eta(z) = \overline{u(z)} + v(z) \in L^\infty(\mathbb{T}^n)$ *where*

$$f(z) = \sum_{\substack{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{M} a_{(m_1,m_2,\ldots,m_n)} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n},$$

$$g(z) = \sum_{\substack{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{N} a_{-(m_1,m_2,\ldots,m_n)} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}$$

$$p(z) = \sum_{\substack{(p_1,p_2,\ldots,p_n)\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{P} b_{(p_1,p_2,\ldots,p_n)} z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n}$$

$$q(z) = \sum_{\substack{(p_1,p_2,\ldots,p_n)\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{Q} b_{-(p_1,p_2,\ldots,p_n)} z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n}$$

$$r(z) = \sum_{\substack{(r_1,r_2,\ldots,r_n)\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{R} c_{(r_1,r_2,\ldots,r_n)} z_1^{r_1} z_2^{r_2} \ldots z_n^{r_n},$$

$$s(z) = \sum_{\substack{-(r_1,r_2,\ldots,r_n)\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{S} c_{(r_1,r_2,\ldots,r_n)} z_1^{r_1} z_2^{r_2} \ldots z_n^{r_n}$$

$$u(z) = \sum_{\substack{(u_1,u_2,\ldots,u_n)\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{U} d_{(u_1,u_2,\ldots,u_n)} z_1^{u_1} z_2^{u_2} \ldots z_n^{u_n},$$

$$v(z) = \sum_{\substack{(u_1,u_2,\ldots,u_n)\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{V} d_{-(u_1,u_2,\ldots,u_n)} z_1^{u_1} z_2^{u_2} \ldots z_n^{u_n}$$

and $M, N, P, Q, R, S, U, V \in \mathbb{Z}^n$ *with each entries in the* $n-$*tuple is greater than zero than* $T^{(k,n)}_{M_{(\phi,\psi)}}$ *commutes with* $T^{(k,n)}_{M_{(\xi,\eta)}}$ *if and only if* $u = p,\ r = m,\ a = c, b = d, u = \dfrac{k+1}{k-1}m$ *and* $r(k+1) = p(1-k)$.

Proof. Let $\phi(z) = \overline{g(z)} + f(z), \psi(z) = \overline{p(z)} + q(z), \xi(z) = \overline{r(z)} + s(z)$ and $\eta(z) = \overline{u(z)} + v(z) \in L^\infty(\mathbb{T}^n)$ be the functions as defined in the theorem. Then,

$$\Big\{T^{(k,n)}_{M_{(\phi,\psi)}}\Big\}^*\Big\{T^{(k,n)}_{M_{(\xi,\eta)}}\Big\}^* e_{(\theta_1,\theta_2,\ldots,\theta_n)}(z_1,z_2,\ldots,z_n)$$

$$= \Big\{W_k^n M_\phi + JM_\psi\Big\}^*\Big\{W_k^n M_\xi + JM_\eta\Big\}^*(z_1^{\theta_1}, z_2^{\theta_2},\ldots,z_n^{\theta_n})$$

$$= (M_{\overline{\phi}} + M_{\overline{\psi}}J)(W_k^n)^*\Big\{(\overline{r(s)} + s(z)) + (\overline{u(s)} + v(z))J\Big\}(z_1^{k\theta_1}, z_2^{k\theta_2},\ldots,z_n^{k\theta_n})$$

$$= (M_{\overline{\phi}} + M_{\overline{\psi}}J)(W_k^n)^*\Big\{ \sum_{\substack{(r_1,r_2,\ldots,r_n)\in\mathbb{Z}^n\\ r_i\geq 0, 1\leq i\leq n}}^{R} \overline{c}_{(r_1,r_2,\ldots,r_n)}\overline{z}_1^{r_1-k\theta_1}\overline{z}_2^{r_2-k\theta_2}\ldots\overline{z}_n^{r_n-k\theta_n}$$

$$+ \sum_{\substack{(r_1,r_2,\ldots,r_n)\in\mathbb{Z}^n\\ r_i\geq 0, 1\leq i\leq n}}^{S} c_{-(r_1,r_2,\ldots,r_n)} z_1^{r_1+k\theta_1} z_2^{r_2+k\theta_2}\ldots z_n^{r_n+k\theta_n}$$

$$+ \sum_{\substack{(u_1,u_2,\ldots,u_n)\in\mathbb{Z}^n\\ u_i\geq 0, 1\leq i\leq n}}^{U} \overline{d}_{(u_1,u_2,\ldots,u_n)}\overline{z}_1^{u_1+k\theta_1}\overline{z}_2^{u_2+k\theta_2}\ldots\overline{z}_n^{u_n+k\theta_n}$$

$$+ \sum_{\substack{(u_1,u_2,\ldots,u_n)\in\mathbb{Z}^n\\ u_i\geq 0, 1\leq i\leq n}}^{V} d_{-(u_1,u_2,\ldots,u_n)} z_1^{u_1-k\theta_1} z_2^{u_2-k\theta_2}\ldots z_n^{u_n-k\theta_n})\Big\}$$

$$= (\overline{\phi(z)} + \overline{\psi(z)}J)\Big\{ \sum_{\substack{(r_1,r_2,\ldots,r_n)\in\mathbb{Z}^n\\ r_i\geq 0, 1\leq i\leq n}}^{R} c_{(r_1,r_2,\ldots,r_n)}\overline{z}_1^{k(r_1-k\theta_1)}\overline{z}_2^{k(r_2-k\theta_2)}\ldots\overline{z}_n^{k(r_n-k\theta_n)}$$

$$+ \sum_{\substack{(r_1,r_2,\ldots,r_n)\in\mathbb{Z}^n\\ r_i\geq 0, 1\leq i\leq n}}^{S} c_{-(r_1,r_2,\ldots,r_n)} z_1^{k(r_1+k\theta_1)} z_2^{k(r_2+k\theta_2)}\ldots z_n^{k(r_n+k\theta_n)}$$

$$+ \sum_{\substack{(u_1,u_2,\ldots,u_n)\in\mathbb{Z}^n\\ u_i\geq 0, 1\leq i\leq n}}^{U} \overline{d}_{(u_1,u_2,\ldots,u_n)}\overline{z}_1^{k(u_1+k\theta_1)}\overline{z}_2^{k(u_2+k\theta_2)}\ldots\overline{z}_n^{k(u_n+k\theta_n)}$$

$$+ \sum_{\substack{(u_1,u_2,\ldots,u_n)\in\mathbb{Z}^n\\ u_i\geq 0, 1\leq i\leq n}}^{V} \overline{d}_{-(u_1,u_2,\ldots,u_n)} z_1^{k(u_1-k\theta_1)} z_2^{k(u_2-k\theta_2)}\ldots z_n^{k(u_n-k\theta_n)}\Big\}$$

$[Let's\ use\ the\ short\ form\ from\ here\ onwards\ to\ visualized\ better]$

$$= \Big[\Big\{\sum_{\substack{m\in\mathbb{Z}^n\\ m_i\geq 0, 1\leq i\leq n}}^{M} \overline{a}_m\overline{z}^m + \sum_{\substack{m\in\mathbb{Z}^n\\ m_i\geq 0, 1\leq i\leq n}}^{N} a_{-m}z^m\Big\} + \Big\{\sum_{\substack{p\in\mathbb{Z}^n\\ p_i\geq 0, 1\leq i\leq n}}^{P} \overline{b}_p\overline{z}^p$$

$$+ \sum_{\substack{p\in\mathbb{Z}^n\\ p_i\geq 0, 1\leq i\leq n}}^{Q} b_{-p}z^p\Big\}J\Big] \times \Big[\sum_{\substack{r\in\mathbb{Z}^n\\ r_i\geq 0, 1\leq i\leq n}}^{R} \overline{c}_r\overline{z}^{k(r-k\theta)} + \sum_{\substack{r\in\mathbb{Z}^n\\ r_i\geq 0, 1\leq i\leq n}}^{S} c_{-r}z^{k(r+k\theta)}$$

$$+ \sum_{\substack{u\in\mathbb{Z}^n\\ u_i\geq 0, 1\leq i\leq n}}^{U} \overline{d}_u\overline{z}^{k(u+k\theta)} + \sum_{\substack{u\in\mathbb{Z}^n\\ u_i\geq 0, 1\leq i\leq n}}^{V} d_{-u}z^{k(u-k\theta)}\Big]$$

$$= \Bigg\{ \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{M} \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{R} \overline{a}_m \, \overline{c}_r \overline{z}^{kr+m} + \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{N} \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{R} a_{-m}\overline{c}_r \overline{z}^{kr-m}$$

$$+ \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{P} \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{V} \overline{b}_p d_{-u} \overline{z}^{ku+p} + \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{Q} \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{V} b_{-p} d_{-u} \overline{z}^{ku-p} \Bigg\} z^{k\theta}$$

$$+ \Bigg\{ \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{M} \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{V} \overline{a}_m d_{-u} z^{ku-m} + \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{N} \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{V} a_{-m} d_{-u} z^{ku+m}$$

$$+ \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{P} \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{R} \overline{b}_p \overline{c}_r z^{kr-p} + \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{Q} \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{R} b_{-p} \overline{c}_r z^{kr+p} \Bigg\} z^{-k\theta}$$

$$+ \Bigg\{ \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{M} \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{S} \overline{a}_m c_{-r} z^{kr-m} + \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{N} \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{S} a_{-m} c_{-r} z^{kr+m}$$

$$+ \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{P} \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{U} \overline{b}_p \overline{d}_u z^{ku-p} + \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{Q} \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{U} b_{-p} d_{-u} z^{ku+p} \Bigg\} z^{k\theta}$$

$$+ \Bigg\{ \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{M} \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{U} \overline{a}_m \overline{d}_u \overline{z}^{ku+m} + \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{N} \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{U} a_{-m} \overline{d}_u \overline{z}^{ku-m}$$

$$+ \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{P} \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{S} \overline{b}_p c_{-r} \overline{z}^{kr+p} + \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{Q} \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{S} b_{-p} c_{-r} \overline{z}^{kr-p} \Bigg\} \overline{z}^{k\theta} \quad (3)$$

Similarly,

$$\left\{T^{(k,n)}_{M_{(\xi,\eta)}}\right\}^* \left\{T^{(k,n)}_{M_{(\phi,\psi)}}\right\}^* e_{(\theta_1,\theta_2,\ldots,\theta_n)}(z_1, z_2, \ldots, z_n)$$

$$= \Bigg\{ \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{M} \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{R} \overline{a}_r \, \overline{c}_m \overline{z}^{km+r} + \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{M} \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{S} a_{-r}\overline{c}_m \overline{z}^{km-r}$$

$$+ \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{Q} \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{U} \overline{b}_u d_{-p} \overline{z}^{kp+u} + \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{Q} \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{V} b_{-u} d_{-p} \overline{z}^{kp-u} \Bigg\} z^{k\theta}$$

$$+ \Bigg\{ \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{R} \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{Q} \overline{a}_r d_{-p} z^{kp-r} + \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{S} \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{Q} a_{-r} d_{-p} z^{kp+r}$$

$$
\begin{aligned}
&+ \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{U} \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{M} \overline{b}_u\overline{c}_m z^{km-u} + \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{V} \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{M} b_{-u}\overline{c}_m z^{km+u} \Bigg\} z^{-k\theta} \\
&+ \Bigg\{ \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{R} \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{N} \overline{a}_r c_{-m} z^{km-r} + \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{S} \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{N} a_{-r} c_{-m} z^{km+r} \\
&+ \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{P} \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{U} \overline{b}_u\overline{d}_p z^{kp-u} + \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{P} \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{V} b_{-u}\overline{d}_p z^{kp+u} \Bigg\} z^{k\theta} \\
&+ \Bigg\{ \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{R} \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{P} \overline{a}_r\overline{d}_p \overline{z}^{kp+r} + \sum_{\substack{r\in\mathbb{Z}^n \\ r_i\geq 0, 1\leq i\leq n}}^{S} \sum_{\substack{p\in\mathbb{Z}^n \\ p_i\geq 0, 1\leq i\leq n}}^{P} a_{-r}\overline{d}_p \overline{z}^{kp-r} \\
&+ \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{U} \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{N} \overline{b}_u c_{-m} \overline{z}^{km+u} + \sum_{\substack{u\in\mathbb{Z}^n \\ u_i\geq 0, 1\leq i\leq n}}^{V} \sum_{\substack{m\in\mathbb{Z}^n \\ m_i\geq 0, 1\leq i\leq n}}^{N} b_{-u} c_{-m} \overline{z}^{km+u} \Bigg\} \overline{z}^{k\theta}
\end{aligned} \tag{4}
$$

Comparing the Equation (3) and Equation (4) equal, it is evident that they are equal if and only if $u = p$, $r = m$, $a = c, b = d, u = \dfrac{k+1}{k-1}m$ and $r(k+1) = p(1-k)$. □

4. Spectral Properties

In these section, we are discussing the point spectra and spectrum of k^th order order(slant Toeplitz + slant Hankel) operators on the polydisk.

Theorem 4.1. *If $\phi \in L^\infty(\mathbb{T}^n)$ and invertible, then the point spectra of*

$$T^{k,n}_{M_\phi} = T^{k,n}_{M_\phi(z_1^k, z_2^k, z_3^k, \ldots, z_n^k)}$$

Proof. Let λ be an element $\sigma_p(T^{k,n}_{M_\phi})$. Then, $T^{k,n}_{M_\phi}(f) = \lambda f$ for some non-zero function f belonging to $L^2(\mathbb{T}^n)$. Since $\phi \in L^\infty(\mathbb{T}^n)$ and f $\neq 0 \in L^2(\mathbb{T}^n)$ so, $M_\phi(f) \neq 0$. Now,

$$
\begin{aligned}
& M_\phi T^{k,n}_{M_\phi}(f) = M_\phi(\lambda f) = \lambda M_\phi f \\
\Longrightarrow\ & M_\phi W^n_k(I+J)M_\phi(f) = \lambda M_\phi f \\
\Longrightarrow\ & M_\phi W^n_k(I+J)(\phi f) = \lambda(\phi f) \\
\Longrightarrow\ & \phi W^n_k(I+J)(\phi f) = \lambda(\phi f) \\
\Longrightarrow\ & \phi(z_1, z_2, z_3, \ldots, z_n) W^n_k(I+J)(\phi f) = \lambda(\phi f) \\
\Longrightarrow\ & W^n_k \phi(z_1^k, z_2^k, z_3^k, \ldots, z_n^k)(I+J)(\phi f) = \lambda(\phi f)
\end{aligned}
$$

$$
\begin{aligned}
&\Longrightarrow W_k^n M_\phi(z_1^k, z_2^k, z_3^k, \ldots, z_n^k)(I+J)(\phi f) = \lambda(\phi f)\\
&\Longrightarrow W_k^n (I+J) M_\phi(z_1^k, z_2^k, z_3^k, \ldots, z_n^k)(\phi f) = \lambda(\phi f)\\
&\Longrightarrow T^{k,n}_{M_{\phi(z_1^k, z_2^k, z_3^k, \ldots, z_n^k)}}(\phi . f) = \lambda(\phi f)\\
&\Longrightarrow \lambda \in \sigma_p(T^{k,n}_{M_\phi(z_1^k, z_2^k, z_3^k, \ldots, z_n^k)})\\
&\Longrightarrow \sigma_p(T^{k,n}_{M_\phi}) \subseteq \sigma_p(T^{k,n}_{M_\phi(z_1^k, z_2^k, z_3^k, \ldots, z_n^k)}). \qquad (5)
\end{aligned}
$$

Conversely, let $\lambda \in \sigma_p(T^{k,n}_{M_\phi(z_1^k, z_2^k, z_3^k, \ldots, z_n^k)})$. Then, there exist a non- zero function g in $L^2(\mathbb{T}^n)$ such that

$$T^{k,n}_{M_\phi(z_1^k, z_2^k, z_3^k, \ldots, z_n^k)})(g) = \lambda g.$$

Let $G = \overline{\phi}.g$. Then, $G \neq 0$.

$$
\begin{aligned}
&T^{k,n}_{M_\phi(z_1^k, z_2^k, z_3^k, \ldots, z_n^k)})(g) = \lambda g\\
&\Longrightarrow W_k^n M_\phi(z_1^k, z_2^k, z_3^k, \ldots, z_n^k)(I+J)(\phi G) = \lambda . g.\\
&\Longrightarrow \phi W_k^n (I+J)\phi(G) = \lambda . g.\\
&\Longrightarrow \frac{1}{\overline{\phi}} W_k^n (I+J) M_\phi(G) = \lambda . g.\\
&\Longrightarrow W_k^n (I+J) M_\phi(G) = \lambda(\overline{\phi} g).\\
&\Longrightarrow T^{k,n}_{M_\phi}(G) = \lambda(G).\\
&\Longrightarrow \lambda \in \sigma_p(T^{k,n}_{M_\phi}).\\
&\Longrightarrow \sigma_p(T^{k,n}_{M_\phi(z_1^k, z_2^k, z_3^k, \ldots, z_n^k)}) \subseteq \sigma_p(T^{k,n}_{M_\phi}). \qquad (6)
\end{aligned}
$$

From (5) and (6), we see that the point spectra of $\sigma_p(T^{k,n}_{M_\phi(z_1^k, z_2^k, z_3^k, \ldots, z_n^k)})$ is $\sigma_p(T^{k,n}_{M_\phi})$. □

Corollary 4.2. *Let $\phi \in L^\infty(\mathbb{T}^n)$. Then the spectrum of $T^{k,n}_{M_\phi}$ is same as of $T^{k,n}_{M_\phi(z_1^k, z_2^k, z_3^k, \ldots, z_n^k)}$.*

Conclusion and future aspects. In this study, we introduced and explored the properties of k^{th}-order (slant Toeplitz + slant Hankel) operators on the polydisk. We thoroughly examined their commutativity, partial isometry, co-isometry and spectral properties. The commutativity of these operators was found to depend on whether their harmonic symbols are scalar multiples of each other. Additionally, we provided a detailed analysis of the point spectrum and the overall spectrum for these operators. These findings extend existing knowledge on Toeplitz and Hankel operators, specifically in higher dimensions such as the polydisk, enriching the understanding of their algebraic and spectral behaviours.

Future research can delve deeper into the potential applications of k^{th}-order (slant Toeplitz + slant Hankel) operators in mathematical physics and functional analysis. Extending the theory to other function spaces, like Bergman or Sobolev spaces, could provide further insights. Additionally, exploring the role of these operators in signal processing, control theory and quantum mechanics could unlock new practical applications. There is also scope for investigating the invertibility and Fredholmness of these operators in more complex multidimensional settings, potentially opening new research avenues in operator theory and beyond.

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