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The Strong 3-Rainbow Index of Graphs Containing some Cycles

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Abstract. A tree of minimum size in an edge-colored connected graph G is a rainbow Steiner tree if no two edges of G are colored the same. For an integer k, the strong k-rainbow index $srx_k(G)$ of G is the smallest number of colors required in an edge-coloring of G so that there exists a rainbow Steiner tree connecting every k-subset S of $V(G)$. We focus on $k = 3$. It is obvious that $srx_3(G) \leq \|G\|$ where $\|G\|$ denotes the size of G. It has been proven that $srx_3(T_n) = \|T_n\|$. This paper investigates the behavior of the $srx_3(T_n)$ under the addition of at least one edge to T_n. We establish sharp upper bounds and exact values of the srx_3 for unicylic and bicyclic graphs. Our results show that $srx_3(G) = \|G\|$ if G is a unicyclic graph with girth 7 or at least 9. In all other cases, where G is either a unicylic graph or bicyclic graph, it holds that $srx_3(G) < \|G\|$.

Key words and Phrases: cycle, rainbow Steiner tree, strong 3-rainbow index, tree.

1. INTRODUCTION

Graph theory provides a powerful framework for modeling and analyzing communication networks, where reliability, security, and efficiency are crucial. Various coloring concepts have been proposed to ensure that networks can support secure and interference-free communication. One such concept is the strong k-rainbow index of a graph, introduced by Awanis and Salman [1], which measures the minimum number of colors needed to color the edges of a connected graph so that every set of k vertices is connected by a rainbow Steiner tree—a tree of minimum size

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whose edges have distinct colors. This parameter is closely tied to combinatorial optimization and connectivity theory, particularly through its relation to Steiner trees.

In the case of the strong 3-rainbow index, the aim is to guarantee minimum rainbow connectivity for every three vertices of a graph. When interpreted in a network model, vertices represent devices such as servers or routes, while edges represent direct communication links between these devices. Assigning distinct colors to the edges of a Steiner tree can be viewed as assigning different frequency channels or encryption keys to ensure interference-free and secure multi-terminal communication. Beyond its practical relevance, determining a strong 3-rainbow index also yields theoretical insights into how cycle constraints and structural properties of a graph influence its rainbow connectivity requirements. By minimizing the number of colors used while maintaining strong connectivity properties, the strong 3-rainbow index contributes to the efficient use of network resources.

Before we discuss the formal definitions of a strong k-rainbow index, the readers are advised to understand the formal definitions of a k-rainbow index first. Let G be a connected graph of order $n \geq 3$ that admits an edge-coloring. The size of G is denoted by $\|G\|$. A tree in G is a *rainbow tree* if all edges of the tree are colored with distinct colors. Let k be an integer with $2 \leq k \leq n$. In this paper, we always consider S as a k-subset of $V(G)$. The *k-rainbow index* $rx_k(G)$ of G is the smallest number of colors required in an edge-coloring of G so that every set S in G is connected by a rainbow tree. The 2-rainbow index of G is also known as the *rainbow connection number* $rc(G)$ of G [2]. Hence, it is easy to see that $rc(G) = rx_2(G) \leq rx_3(G) \leq \ldots \leq rx_n(G)$.

Chakraborty *et al.* in [3] proved a conjecture given by Caro *et al.* [4] which states that computing the rc of a graph is an NP-Hard problem. Hence, it is more difficult to compute the rx_3 of a graph. Some previous researchers studied the upper bounds for rx_3 of graphs (e.g. [5, 6, 7]), the rx_3 of some graphs and some graph operations (e.g., [8, 6, 9, 10, 11]), and the characterization of graphs G with certain values of $rx_3(G)$ (e.g., [9, 12]). We refer to [13, 14] for some detailed surveys on 3-rainbow index.

Later, Awanis and Salman [1] proposed the concept of a strong k-rainbow index. A tree of minimum size in G that connects S is called a *Steiner S-tree* and the minimum size is defined as the *Steiner distance* $d(S)$ of S. The Steiner $\{u, v\}$-tree is also known as the *$u - v$ geodesic* [2]. The *strong k-rainbow index* $srx_k(G)$ of G is the smallest number of colors required in an edge-coloring of G so that every set S in G is connected by a rainbow Steiner S-tree. Such an edge-coloring of G is called a *strong k-rainbow coloring* of G. The strong 2-rainbow index of G is also known as the *strong rainbow connection number* $src(G)$ of G [2]. Awanis and Salman [1] provided sharp lower and upper bounds for the srx_3 of a connected graph G, that is

$$sdiam_k(G) \leq rx_k(G) \leq srx_k(G) \leq \|G\|, \tag{1}$$

where $sdiam_k(G)$ denotes the *k-Steiner diameter* of G and is defined as $sdiam_k(G) = \max\{d(S) : S \text{ is a } k\text{-subset of } G\}$.

In the same paper, Awanis and Salman [1] established the edge-coloring rules for connected graphs containing at least two bridges. Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be these two bridges. Since $G-e_1-e_2$ consists of three components, say G_1, G_2, and G_3, without loss of generality, we may assume that $u_1 \in V(G_1)$, $v_1, u_2 \in V(G_2)$, and $v_2 \in V(G_3)$. Under this condition, any rainbow Steiner tree connecting a set S of three vertices that includes vertices u_1 and v_2 must necessarily contain both bridges e_1 and e_2. This directly leads to Observation 1.1. According to this observation and Eq. (1), Awanis and Salman further established that srx_3 of trees is equal to its size, as stated in Theorem 1.2.

Observation 1.1. [1] *Let G be a strong 3-rainbow colored connected graph of order $n \geq 3$. If e and f are any two bridges of G, then e and f are colored with distinct colors.*

Theorem 1.2. [1] *For a tree T_n of order $n \geq 3$, $srx_3(T_n) = \|T_n\| = n - 1$.*

Many researchers have investigated the srx_3 of graphs resulting from some graph operations, such as some certain graphs and their amalgamation [1], the edge-amalgamation of some graphs [15], the comb product of a tree and a connected graph [16], and the edge-comb product of a path and a connected graph [17]. In addition, we are also interested in exploring the characteristics of graphs G with $srx_3(G) = 2$, as presented in [18].

Since a tree is an acyclic connected graph, adding even a single edge necessarily creates a graph that contains at least one cycle. Cycles, especially those with small girths, are of particular interest because they generate alternative Steiner trees between three vertices, which may affect the existence and structure of rainbow Steiner trees. Therefore, adding one or two edges to a tree increases the graph's connectivity and redundancy. These structural changes may impact the minimum number of colors required to ensure strong 3-rainbow connectivity.

A natural question then arises: What happens to the srx_3 when at least one edge is added to a tree? Specifically, does the srx_3 of the resulting graph remain equal to its size? Motivated by this, the present study investigates the srx_3 of graphs containing some cycles, with a particular focus on unicyclic and bicyclic graphs. First, we establish an upper bound for the srx_3 of these graphs and demonstrate that the bound is sharp. These results are presented in Section 2. Subsequently, we determine the exact values of the srx_3 for unicyclic and bicyclic graphs, which are presented in Sections 3 and 4, respectively.

2. SHARP UPPER BOUND FOR THE STRONG 3-RAINBOW INDEX OF GRAPHS CONTAINING AT MOST TWO CYCLES

Several notations are defined in this paper as follows. For an integer x with $a \leq x \leq b$, let $[a, b]$ denotes a set of all integers x. For an integer t with $1 \leq t \leq 2$, let

- G_t denotes a connected graph of order $n \geq 3$ containing exactly t cycles,

- $C_{g_i} := v_i^1 v_i^2 \dots v_i^{g_i} v_i^1$ for $1 \le i \le t$ denotes a cycle of length $g_i \ge 3$ contained in G_t,
- X denotes a set of all bridges in G_t, and
- $c(U)$ denotes a set of all colors assigned to the edges in $U \subseteq E(G_t)$.

Note that if $t = 2$, then there exists exactly one path connecting the two cycles in G_2. We denote $P := v_1^1 - v_2^1$ as such a path. Since X denotes a set of all bridges in G_t for $t \in [1, 2]$, it follows from Observation 1.1 that

$$|c(X)| \ge \|G_t\| - \sum_{i=1}^{t} g_i. \tag{2}$$

Now, we are ready to provide an upper bound of the $srx_3(G_t)$ for $t \in [1, 2]$. This result is given in the following theorem.

Theorem 2.1. *For $n \ge 3$ and $t \in [1, 2]$, let G_t be a connected graph of order n containing exactly t cycles of length at least 3. Then*

$$srx_3(G_t) \le \|G_t\| - t + 1.$$

Proof. For $t = 1$, it follows from Eq. (1) that $srx_3(G_1) \le \|G_1\|$.

For $t = 2$, we show that $srx_3(G_2) \le \|G_2\| - 1$ by defining a strong 3-rainbow coloring $c : E(G_2) \to [1, \|G_2\| - 1]$, which can be obtained by defining $c(v_1^{\lfloor \frac{g_1}{2} \rfloor + 1} v_1^{\lfloor \frac{g_1}{2} \rfloor + 2}) = c(v_2^{\lfloor \frac{g_2}{2} \rfloor + 1} v_2^{\lfloor \frac{g_2}{2} \rfloor + 2}) = 1$ and assigning colors $2, 3, \dots, \|G_2\| - 1$ to the remaining $\|G_2\| - 2$ edges of G_2. Now, we show that every three vertices of G_2 is connected by a rainbow Steiner tree by considering the following properties.

- For each $i \in [1, 2]$, all edges of C_{g_i} have distinct colors. This ensures that for every three vertices of C_{g_i} for $i \in [1, 2]$, there exists a rainbow Steiner tree connecting them.
- All edges of C_{g_1} and C_{g_2} are colored with distinct colors, except for edges $v_1^{\lfloor \frac{g_1}{2} \rfloor + 1} v_1^{\lfloor \frac{g_1}{2} \rfloor + 2}$ and $v_2^{\lfloor \frac{g_2}{2} \rfloor + 1} v_2^{\lfloor \frac{g_2}{2} \rfloor + 2}$, which are both colored with 1. This implies that for distinct $i, j \in [1, 2]$, $p \in [1, g_i]$, and $q, r \in [1, g_j]$, there exist a rainbow $v_i^1 - v_i^p$ geodesic T_i in C_{g_i} and a rainbow Steiner $\{v_j^1, v_j^q, v_j^r\}$-tree T_j in C_{g_j} such that $c(E(T_i)) \cap c(E(T_j)) = \emptyset$. Therefore, there exists a rainbow Steiner tree connecting one vertex of C_{g_i} and two vertices of C_{g_j}.
- If G_2 contains bridges, then all bridges of G_2 are colored with distinct colors and $c(X) \cap c(E(C_{g_i})) = \emptyset$ for all $i \in [1, 2]$. Consequently, for every three vertices of G_2, where at least one of them is not a vertex of C_{g_i} for $i \in [1, 2]$, there exists a rainbow Steiner tree connecting them.

Thus, the theorem holds. □

The upper bound provided in Theorem 2.1 is sharp. Theorem 2.5 shows that $srx_3(G_t) = \|G_t\| - t + 1$, where G_t is a connected graph containing exactly t odd cycles of length at least 7. Before we proceed to this theorem, we first need several preliminary results as given in Theorem 2.2 and Lemmas 2.3 and 2.4.

Theorem 2.2. [1] *For a cycle C_n of order $n \geq 3$,*

$$srx_3(C_n) = \begin{cases} 2, & n = 3; \\ n-2, & n \in \{4,5,6,8\}; \\ n, & n = 7 \text{ or } n \geq 9. \end{cases}$$

According to theorem above, it is not difficult to define a strong 3-rainbow coloring of C_n for $n = 7$ or $n \geq 9$, since all edges of the cycle can simply be assigned with distinct colors. The challenge lies in defining such an edge-coloring for smaller values of n, where fewer colors must be used while still maintaining the existence of rainbow Steiner trees. Figure 1 below illustrates the strong 3-rainbow colorings of C_n for $n = 3, 4, 5, 6, 8$. Since the graphs studied in this paper contain at most two cycles, and the rainbow Steiner tree connecting every three vertices within the cycle must lies in it, these edge-coloring illustrations are essential to guarantee the existence of rainbow Steiner trees that support the results established in Sections 3 and 4.

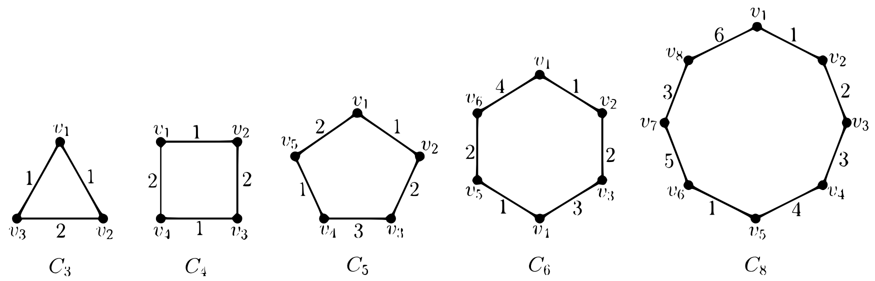

FIGURE 1. Strong 3-rainbow colorings of C_3, C_4, C_5, C_6, and C_8

Lemma 2.3. *For $n \geq 5$ and $g \geq 4$, let G be a strong 3-rainbow colored connected graph of order n containing a cycle C_g. If $e \in E(C_g)$ and f is an arbitrary bridge of G, then e and f are colored with distinct colors.*

Proof. Let $C_g := v_1v_2 \ldots v_gv_1$. Suppose that there exist $e \in E(C_g)$ and a bridge $f \in E(G)$ so that e and f are colored with the same color. Let $e = v_pv_{p+1}$ for $p \in [1, g]$ and $f = xy$, and assume that $d(C_g, x) < d(C_g, y)$. Observe that every rainbow Steiner $\{v_p, v_{p+1}, y\}$-tree must contain edges e and f, which is a contradiction since these two edges have the same color. □

To assist the reader's understanding, an illustration of Lemma 2.3 is provided in Figure 2.

For further discussion, we always let $A_i = E(C_{g_i}) \setminus \{v_i^{\lfloor \frac{g_i}{2} \rfloor + 1} v_i^{\lfloor \frac{g_i}{2} \rfloor + 2}\}$ if g_i is odd or $A_i = E(C_{g_i}) \setminus \{v_i^{\frac{g_i}{2}} v_i^{\frac{g_i}{2}+1}, v_i^{\frac{g_i}{2}+1} v_i^{\frac{g_i}{2}+2}\}$ if g_i is even, for $i \in [1, 2]$.

Lemma 2.4. *For $n \geq 5$, let G_2 be a bicyclic graph of order n containing two cycles C_{g_1} and C_{g_2} of length at least 3. If c is a strong 3-rainbow coloring of G_2, then*

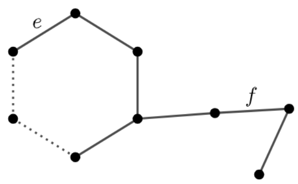

FIGURE 2. An illustration of Lemma 2.3, where edges e and f are colored with distinct colors

(i) $|c(A_i)| \geq g_i - 2$ *for* $g_i \in [3,4]$ *or even* $g_i \geq 10$*,* $|c(A_i)| \geq g_i - 3$ *for* $g_i \in \{5,6,8\}$*, and* $|c(A_i)| \geq g_i - 1$ *for odd* $g_i \geq 7$*;*
(ii) $c(A_i) \cap c(E(C_{g_j})) = \emptyset$ *for distinct* $i, j \in [1,2]$ *and* $g_j \geq 4$*.*

Proof.

(i) For $g_i = 3$, it is clear that $|c(A_i)| \geq 1$. For $g_i \in [4,5]$, since two adjacent edges should have distinct colors, we have $|c(A_i)| \geq 2$. For $g_i = 6$, by considering $\{v_i^1, v_i^3, v_i^6\}$, we obtain that no edge of path $v_i^3 v_i^2 v_i^1 v_i^6$ is colored the same. A similar argument applies to $\{v_i^1, v_i^2, v_i^5\}$. However, edges $v_i^2 v_i^3$ and $v_i^5 v_i^6$ may be colored the same. Thus, $|c(A_i)| \geq 3$. For $g_i = 8$, suppose that $|c(A_i)| \leq 4$. First, by considering $\{v_i^1, v_i^3, v_i^7\}$, we obtain that no edge of path $v_i^7 v_i^8 v_i^1 v_i^2 v_i^3$ is colored the same, which implies we have used all colors in $c(A_i)$. Next, by considering $\{v_i^2, v_i^4, v_i^8\}$ and $\{v_i^2, v_i^6, v_i^8\}$, we obtain that $c(v_i^3 v_i^4) = c(v_i^7 v_i^8)$ and $c(v_i^6 v_i^7) = c(v_i^2 v_i^3)$. However, there is no rainbow Steiner $\{v_i^2, v_i^4, v_i^7\}$-tree, a contradiction. For odd $g_i \geq 7$ or even $g_i \geq 10$, since no edge of C_{g_i} is colored the same by Theorem 2.2, we have $|c(A_i)| \geq g_i - 1$ or $|c(A_i)| \geq g_i - 2$, respectively.

(ii) Let $i, j \in [1,2]$ with $i \neq j$ and $g_j \geq 4$. Let $e = xy$ be an arbitrary edge of C_{g_j}. Observe that every rainbow Steiner $\{x, y, v_i^p\}$-tree for $p \in [\lfloor \frac{g_i}{2} \rfloor + 1, \lfloor \frac{g_i}{2} \rfloor + 2]$ if g_i is odd or $p \in \{\frac{g_i}{2}, \frac{g_i}{2} + 2\}$ if g_i is even must contain edge e and a $v_i^1 - v_i^p$ geodesic, implying that $c(A_i) \cap c(E(C_{g_j})) = \emptyset$.

□

To assist the reader's understanding, an illustration of Lemma 2.4 is provided in Figure 3.

Now, we are ready to prove the sharpness of the upper bound in Theorem 2.1, as given in the following theorem.

Theorem 2.5. *For* $n \geq 3$ *and* $t \in [1,2]$*, let* G_t *be a connected graph of order* n *containing exactly* t *odd cycles of length at least* 7*. Then,* $srx_3(G_t) = \|G_t\| - t + 1$*.*

Proof. For each $i \in [1,t]$, let C_{g_i} be an odd cycle of length $g_i \geq 7$ contained in G_t. It follows from Theorem 2.1 that $srx_3(G_t) \leq \|G_t\| - t + 1$. For the lower bound, suppose that $srx_3(G_t) \leq \|G_t\| - t$. Then there exists a strong 3-rainbow coloring $c : E(G_t) \to [1, \|G_t\| - t]$. Let Y be the set of colors assigned to the edges of C_{g_i}

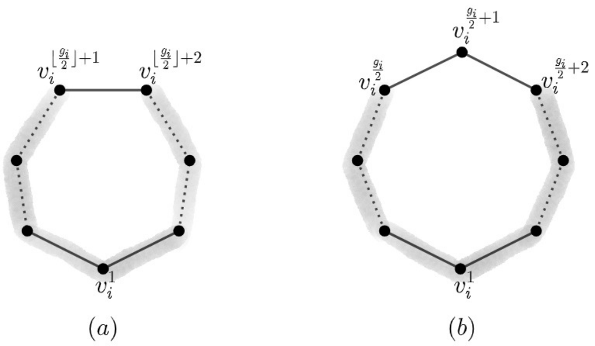

FIGURE 3. An illustration of Lemma 2.4, where the shaded grey area represents the set A_i for the cases when (a) g_i is odd and (b) g_i is even

for all $i \in [1, t]$. According to Lemma 2.3, $c(X) \cap Y = \emptyset$. Thus, it follows from Eq. (2) that $|Y| \leq \sum_{i=1}^{t} g_i - t$. Now, we distinguish two cases.

Case 1. $t = 1$.

It means $c : E(G_1) \to [1, \|G_1\| - 1]$. If $G_1 \cong C_n$, then there are at least two edges of C_n that are colored the same, contradicting Theorem 2.2. If G_1 is a unicyclic graph and not a cycle, then we have $|Y| \leq g_1 - 1$. This implies there are at least two edges of C_{g_1} that are colored the same, contradicting Theorem 2.2.

Case 2. $t = 2$.

It means $c : E(G_2) \to [1, \|G_2\| - 2]$. Note that $|Y| \leq g_1 + g_2 - 2$. However, it follows from Theorem 2.2 and Lemma 2.4 that $|c(A_1) \cup c(E(C_{g_2}))| \geq g_1 + g_2 - 1$, which is impossible. □

Following the result above, an immediate question arises: What is the $srx_3(G_t)$ for G_t that does not satisfy the premise of Theorem 2.5? The answers to this question are given in Sections 3 and 4.

3. THE STRONG 3-RAINBOW INDEX OF UNICYCLIC GRAPHS

Let G_1 be a unicyclic graph of order $n \geq 3$ and girth $g_1 \geq 3$. Chartrand *et al.* [6] have determined the rx_3 of unicyclic graphs as follows.

Theorem 3.1. [6] *For $n, g_1 \geq 3$, let G_1 be a unicyclic graph of order n and girth g_1. Then,*

$$rx_3(G_1) = \begin{cases} n-1, & \text{if } g_1 = 3; \\ n-2, & \text{otherwise.} \end{cases}$$

Motivated by the result above, we are interested in studying the srx_3 of unicyclic graphs. Awanis and Salman in [1] have determined the srx_3 of cycles (see Theorem 2.2). Hence, in this section, we determine the srx_3 of unicyclic graphs that is not a cycle as given in the following theorem.

Theorem 3.2. *For $n \geq 4$ and $g_1 \geq 3$, let G_1 be a unicyclic graph of order n and girth g_1 that is not a cycle. Then,*

$$srx_3(G_1) = \begin{cases} n-1, & \text{if } g_1 = 3; \\ n-2, & \text{if } g_1 \in \{4,5,6,8\}; \\ n, & \text{otherwise.} \end{cases}$$

Proof. Note that $\|G_1\| = n$. It follows from Eq. (1) and Theorem 3.1 that $srx_3(G_1) \geq n-1$ for $g_1 = 3$ and $srx_3(G_1) \geq n-2$ for $g_1 \in \{4,5,6,8\}$. Meanwhile for $g_1 = 7$ or $g_1 \geq 9$, it follows from Eq. (2), Theorem 2.2, and Lemma 2.3 that $srx_3(G_1) \geq n$.

Next, we prove the upper bound. For each $i \in [1, n-g_1]$, let e_i be the i-th bridge of G_1. As $srx_3(C_3) = 2$, $srx_3(C_{g_1}) = g_1 - 2$ for $g_1 \in \{4,5,6,8\}$, and $srx_3(C_{g_1}) = g_1$ for $g_1 = 7$ or $g_1 \geq 9$ by Theorem 2.2, there exists a strong 3-rainbow coloring c' of C_{g_1}. Thus, we define a strong 3-rainbow coloring of G_1 as follows.

$$c(e) = \begin{cases} c'(e), & \text{if } e \in E(C_{g_1}); \\ srx_3(C_{g_1}) + i, & \text{if } e = e_i \text{ for each } i \in [1, n-g_1]. \end{cases}$$

Now, we show that every three vertices of G_1 is connected by a rainbow Steiner tree by considering the following properties.

- All edges of C_{g_1} are colored according to the edge-coloring rule c'. This guarantees that for every three vertices of C_{g_1}, there exists a rainbow Steiner tree connecting them.
- All bridges of G_1 are colored with distinct colors and $c(X) \cap c(E(C_{g_1})) = \emptyset$. Consequently, for every three vertices of G_1, where at least one of them is not a vertex of C_{g_1}, there exists a rainbow Steiner tree connecting them.

□

As an example, let G_1 be an unicylic graph of order $n = 12$ and girth $g_1 = 6$. According to Theorem 2.2, we have $srx_3(C_6) = 4$. Let c' be a strong 3-rainbow coloring of C_6 that follows the edge-coloring pattern illustrated in Figure 1. Based on the edge-coloring c defined in the proof of theorem above, we assign the first 4 colors to the edges of C_6 using the edge-coloring c', and then assign colors $5, 6, \ldots, 10$ to the bridges of G_1. As a result, we obtain a strong 3-rainbow coloring of G_1 as illustrated in Figure 4(a). By using a similar procedure, we also obtain a strong 3-rainbow coloring of G_1 with order $n = 13$ and girth $g_1 = 7$ as illustrated in Figure Figure 4(b).

We now conclude this section by outlining the main results obtained as given in the following corollary.

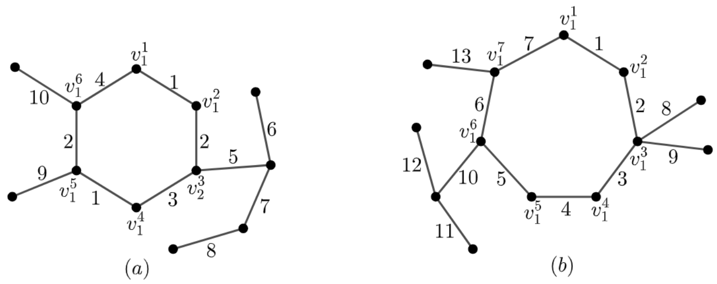

FIGURE 4. Strong 3-rainbow colorings of G_1, where (a) $n = 12$ and $g_1 = 6$, and (b) $n = 13$ and $g_1 = 7$

Corollary 3.3. *For $n, g_1 \geq 3$, let G_1 be a unicyclic graph of order n and girth g_1. Then,*

$$srx_3(G_1) = \begin{cases} n-1, & \text{if } g_1 = 3; \\ n-2, & \text{if } g_1 \in \{4,5,6,8\}; \\ n, & \text{otherwise.} \end{cases}$$

Recall that adding an edge to a tree creates a unicyclic graph. Following the corollary above, we obtain that the srx_3 of unicyclic graphs is equal to its size if it has a girth of 7 or at least 9. Otherwise, its srx_3 is less than its size.

4. THE STRONG 3-RAINBOW INDEX OF BICYCLIC GRAPHS

For $n \geq 5$ and $g_1, g_2 \geq 3$, let G_2 be a bicyclic graph of order n containing two cycles C_{g_1} and C_{g_2}. Recall that we denote $C_{g_i} := v_i^1 v_i^2 \dots v_i^{g_i} v_i^1$ for each $i \in [1,2]$ and $P := v_1^1 - v_2^1$ as the only path connecting C_{g_1} and C_{g_2}. Let $e = xy$ be an arbitrary bridge of G_2. Thus, it is clear that for each $i \in [1,2]$, there exists exactly one vertex $v_i^p \in V(C_{g_i})$ for $p \in [1, g_i]$ such that $d(v_i^p, x) < d(v_i^q, x)$ for all $q \in [1, g_i]$ with $q \neq p$.

In this section, we provide the exact value of the srx_3 of bicyclic graphs. We first need the following results.

Observation 4.1. *For $n \geq 5$, let G_2 be a strong 3-rainbow colored bicyclic graph of order n containing two cycles. Let one of its cycles be $C_3 := v_1v_2v_3v_1$. Let $v_p, v_q, v_r \in V(C_3)$ for distinct $p, q, r \in [1,3]$ and $e = xy$ be an arbitrary bridge of G_2 such that $d(v_p, x) < d(v_q, x) \leq d(v_r, x)$. Then, edges v_pv_q and v_pv_r should have distinct colors from e.*

Proof. Let $\{v_1, v_2, v_3\} = \{v_p, v_q, v_r\}$. Assume that $d(v_p, x) < d(v_p, y)$. Thus, by considering $\{v_p, v_q, y\}$ and $\{v_p, v_r, y\}$, it is clear that edges v_pv_q and v_pv_r should have distinct colors from bridge e. □

The following observation is an immediate consequence of Observation 4.1.

Observation 4.2. *For $n \geq 5$, let G_2 be a strong 3-rainbow colored bicyclic graph of order n containing two cycles. Let one of its cycles be $C_3 := v_1v_2v_3v_1$. Then, at most one color of the bridges of G_2 can be used on C_3.*

Proof. Let $\{v_1, v_2, v_3\} = \{v_p, v_q, v_r\}$. Let c be a strong 3-rainbow coloring of G_2. Suppose that there are two colors of the bridges of G_2, say 1 and 2, which are used on C_3. Let $e = xy$ and $e' = x'y'$ be two distinct bridges of G_2 with $c(e) = 1$ and $c(e') = 2$. Recall that there exist $v_p, v_q \in V(C_3)$ such that $d(v_p, x) < d(u, x)$ for $u \in V(C_3) \setminus \{v_p\}$ and $d(v_q, x') < d(w, x')$ for $w \in V(C_3) \setminus \{v_q\}$. Assume that $d(v_p, x) < d(v_p, y)$ and $d(v_q, x') < d(v_q, y')$. Thus, it follows by Observation 4.1 that edges v_qv_r and v_pv_r may be colored with 1 and 2, respectively. If $p = q$, then there exists an edge of C_3 with colors 1 and 2, which is impossible. Thus, $p \neq q$. However, observe that every rainbow Steiner $\{v_r, y, y'\}$-tree must contain bridges e and e' and two edges of C_3, say f and f', where at least one of f and f' is colored with 1 or 2, a contradiction. □

Let c be a strong 3-rainbow coloring of G_2. For $g_i \in \{5, 6, 8\}$ with $i \in [1, 2]$, it follows from Theorem 2.2 that we need at least $g_i - 2$ distinct colors to color the edges of C_{g_i}. Hence, we assign these colors to the edges of C_{g_i} by adopting the edge-coloring pattern illustrated in Figure 1, as given in (A1). This ensures the existence of a rainbow Steiner tree connecting every three vertices of C_{g_i}.

(A1) For $g_i = 5$, define $c(v_i^1v_i^2) = c(v_i^4v_i^5)$ and $c(v_i^2v_i^3) = c(v_i^1v_i^5)$. For $g_i = 6$, define $c(v_i^2v_i^3) = c(v_i^5v_i^6)$ and $c(v_i^3v_i^4) = c(v_i^1v_i^6)$. For $g_i = 8$, define $c(v_i^2v_i^3) = c(v_i^6v_i^7)$ and $c(v_i^4v_i^5) = c(v_i^1v_i^8)$. Furthermore, assign $g_i - 4$ distinct colors that are not used to the preceding edges to the remaining $g_i - 4$ edges of C_{g_i}.

As an example, let $C_{g_i} \cong C_5$ be one of the cycles contained in G_2. By using the edge-coloring rules given in (A1), we assign three distinct colors, say a, b, and c, so that $c(v_i^1v_i^2) = c(v_i^4v_i^5) = a$, $c(v_i^2v_i^3) = c(v_i^1v_i^5) = b$, and $c(v_i^3v_i^4) = c$. Hence, we have an edge-coloring of C_5 as illustrated in Figure 5. By using a similar argument, we also obtain edge-colorings of C_6 and C_8 as illustrated in Figure 5.

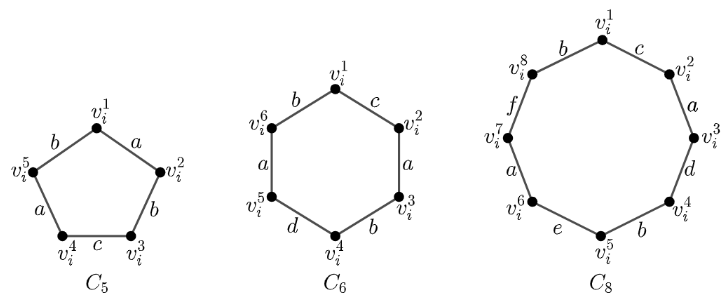

FIGURE 5. Illustration of edge-coloring rules given in (A1), where a, b, c, d, e, and f are distinct colors

We now determine the srx_3 of bicyclic graphs as given in the following theorem.

Theorem 4.3. *For $n \geq 5$ and $g_2 \geq g_1 \geq 3$, let G_2 be a bicyclic graph of order n containing two cycles of lengths g_1 and g_2. Then*

$$srx_3(G_2) = \begin{cases} \|G_2\| - 3, & \text{if } g_1 = 3 \text{ and } g_2 \in [3,4], \text{ or} \\ & g_1 \in \{5,6,8\} \text{ and } g_2 = 7 \text{ or } g_2 \geq 9; \\ \|G_2\| - 4, & \text{if } g_1 = 3 \text{ and } g_2 \in \{5,6,8\}, \text{ or} \\ & g_1 = 4 \text{ and } g_2 \in \{4,5,6,8\}; \\ \|G_2\| - 2, & \text{if } g_1 \in [3,4] \text{ and } g_2 = 7 \text{ or } g_2 \geq 9; \\ \|G_2\| - 5, & \text{if } g_1, g_2 \in \{5,6,8\}; \\ \|G_2\| - 1, & \text{otherwise.} \end{cases}$$

Proof. To prove the exact value of $srx_3(G_2)$, it is necessary to establish both the lower and upper bounds. The lower bound is obtained by considering two main cases based on the lengths of the cycles in G_2: (i) both cycles have length at least 4, and (ii) at least one cycle has length 3. In each case, the bound is derived by applying key lemmas and theorems linking cycle structure to rainbow connectivity. For the upper bound, we define a strong 3-rainbow coloring in a case-by-case manner, mirroring the cycle-length distinctions, with the following rules: (i) cycles are colored to guarantee internal rainbow Steiner trees, and (ii) bridges are assigned distinct colors not used in the cycles to prevent overlap in any Steiner tree spanning both cycle and tree parts. For every three vertices, the coloring guarantees a rainbow Steiner tree entirely within a cycle or through uniquely colored bridges. In each case, the number of colors used matches the lower bound, thereby establishing sharpness. Having set these proof sketches, we are now ready to present the detailed proofs.

Note that $\|G_2\| = n + 1$. First, we prove the lower bound. We distinguish two cases as follows.

Case 1. The two cycles contained in G_2 have length at least 4.

Let c be a strong 3-rainbow coloring of G_2. Note that for distinct $i, j \in [1,2]$, we have

$$srx_3(G_2) \geq |c(X)| + |c(A_i)| + |c(E(C_{g_j}))| \tag{3}$$

by Lemmas 2.3 and 2.4(ii). Without loss of generality, let $i = 1$ and $j = 2$. Thus, it follows from Eq. (2), Theorem 2.2, and Lemma 2.4(i) that $srx_3(G_2) \geq \|G_2\| - 3$ for $g_1 \in \{5,6,8\}$ and $g_2 = 7$ or $g_2 \geq 9$, $srx_3(G_2) \geq \|G_2\| - 4$ for $g_1 = 4$ and $g_2 \in \{4,5,6,8\}$, $srx_3(G_2) \geq \|G_2\| - 2$ for $g_1 = 4$ and $g_2 = 7$ or $g_2 \geq 9$, and $srx_3(G_2) \geq \|G_2\| - 5$ for $g_1, g_2 \in \{5,6,8\}$.

We now consider the case where both g_1 and g_2 are equal to 7 or at least 9. If g_1 and g_2 are both odd, or if g_1 and g_2 have distinct parity, then by using a similar argument, we have $srx_3(G_2) \geq \|G_2\| - 1$. Thus, the remaining case to consider is when both g_1 and g_2 are even. According to Eq. (3), we need at least $\|G_2\| - 2$ distinct colors to color all edges of X, A_1 and C_{g_2}. Now, consider edges

$v_1^{\frac{g_1}{2}} v_1^{\frac{g_1}{2}+1}$ and $v_1^{\frac{g_1}{2}+1} v_1^{\frac{g_1}{2}+2}$. By using Theorem 2.2 and considering $\{v_1^{\frac{g_1}{2}}, v_1^{\frac{g_1}{2}+2}, v_2^p\}$ for $p \in \{\frac{g_2}{2}, \frac{g_2}{2}+2\}$, we have $\{c(v_1^{\frac{g_1}{2}} v_1^{\frac{g_1}{2}+1}), c(v_1^{\frac{g_1}{2}+1} v_1^{\frac{g_1}{2}+2})\} \nsubseteq c(A_1) \cup c(A_2)$. This forces $\{c(v_1^{\frac{g_1}{2}} v_1^{\frac{g_1}{2}+1}), c(v_1^{\frac{g_1}{2}+1} v_1^{\frac{g_1}{2}+2})\} = \{c(v_2^{\frac{g_2}{2}} v_2^{\frac{g_2}{2}+1}), c(v_2^{\frac{g_2}{2}+1} v_2^{\frac{g_2}{2}+2})\}$. However, observe that every rainbow Steiner $\{v_1^{\frac{g_1}{2}}, v_1^{\frac{g_1}{2}+2}, v_2^{\frac{g_2}{2}+1}\}$-tree must contain edges $v_1^{\frac{g_1}{2}} v_1^{\frac{g_1}{2}+1}$, $v_1^{\frac{g_1}{2}+1} v_1^{\frac{g_1}{2}+2}$, and either $v_2^{\frac{g_2}{2}} v_2^{\frac{g_2}{2}+1}$ or $v_2^{\frac{g_2}{2}+1} v_2^{\frac{g_2}{2}+2}$, a contradiction.

Case 2. At least one of the cycles contained in G_2 has length 3.

Without loss of generality, let $g_1 = 3$ and $g_2 \geq 3$. To simplify the discussion, let us denote $srx_3(G_2) = Z$. Suppose that $srx_3(G_2) \leq Z - 1$. Then there exists a strong 3-rainbow coloring $c : E(G_2) \to [1, Z-1]$. Since $|c(X)| \geq \|G_2\| - g_2 - 3$ by Eq. (2), we have at most $Z - \|G_2\| + g_2 + 2$ remaining colors. Let the set of these remaining colors be denoted by $Y = [1, Z - \|G_2\| + g_2 + 2]$. Further, we distinguish three subcases depending on the length g_2 as follows.

Subcase 2.1. $g_2 \in [3, 4]$.

Since $Z = \|G_2\| - 3$, we have $Y = [1, g_2 - 1]$. For $g_2 = 3$, $Y = [1, 2]$. If G_2 does not contain bridges, then $|c(X)| = 0$ and $v_1^1 = v_2^1$. However, by considering $\{v_1^2, v_1^3, v_2^2\}$, we need at least three distinct colors to color edge $v_2^1 v_2^2$ and two edges in a rainbow Steiner $\{v_1^1, v_1^2, v_1^3\}$-tree, which is impossible. Thus, G_2 contains bridges. Note that $c(A_i) \subseteq c(X) \cup Y$ for each $i \in [1, 2]$. However, according to Observation 4.2, edges $v_i^1 v_i^2$ or $v_i^1 v_i^3$ should be colored with colors from Y for each $i \in [1, 2]$. Without loss of generality, let $\{c(v_1^1 v_1^2), c(v_2^1 v_2^2)\} \subseteq Y$. Since $c(v_1^1 v_1^p) \neq c(v_2^1 v_2^q)$ for $p, q \in [2, 3]$, let $c(v_1^1 v_1^2) = 1$ and $c(v_2^1 v_2^2) = 2$. Now, consider edge $v_1^1 v_1^3$. Note that $c(v_1^1 v_1^3) \in c(X) \cup \{1\}$. If $c(v_1^1 v_1^3) \in c(X)$, then let $e = xy$ be the bridge of G_2 where $c(e) = c(v_1^1 v_1^3)$, and assume that $d(C_{g_1}, x) < d(C_{g_1}, y)$. Now, consider edge $v_1^2 v_1^3$. According to Observation 4.2, $c(v_1^2 v_1^3) \notin c(X)$. This forces $c(v_1^2 v_1^3) \in Y$. However, there is no rainbow Steiner $\{v_1^3, v_2^2, y\}$-tree, a contradiction. If $c(v_1^1 v_1^3) = 1$, then $c(v_1^2 v_1^3) \notin Y$. This forces $c(v_1^2 v_1^3) \in c(X)$. Let $e = xy$ be the bridge of G_2 where $c(e) = c(v_1^2 v_1^3)$, and assume that $d(C_{g_1}, x) < d(C_{g_1}, y)$. However, there is no rainbow Steiner $\{v_1^2, v_1^3, y\}$-tree, a contradiction.

Meanwhile for $g_2 = 4$, $Y = [1, 3]$. Since $c(E(C_4)) \cap c(X) = \emptyset$ by Lemma 2.3, we have $c(E(C_4)) \subseteq Y$. Since $c(A_2) \geq 2$ by Lemma 2.4(i), without loss of generality, let $c(v_2^1 v_2^2) = 1$ and $c(v_2^1 v_2^4) = 2$. Now, consider edges $v_1^1 v_1^2$ and $v_1^1 v_1^3$. Since $\{c(v_1^1 v_1^2), c(v_1^1 v_1^3)\} \nsubseteq [1, 2]$ by Lemma 2.4(ii), we have $\{c(v_1^1 v_1^2), c(v_1^1 v_1^3)\} \subseteq c(X) \cup \{3\}$. According to Observation 4.2, edges $v_1^1 v_1^2$ or $v_1^1 v_1^3$ should be colored with 3. Thus, without loss of generality, we have either $c(v_1^1 v_1^2) = 3$ and $c(v_1^1 v_1^3) \in c(X)$, or $c(v_1^1 v_1^2) = c(v_1^1 v_1^3) = 3$. By using a similar argument as case $g_2 = 3$, we will obtain a contradiction.

Subcase 2.2. $g_2 \in \{5, 6, 8\}$.

Since $Z = \|G_2\| - 4$, we have $Y = [1, g_2 - 2]$. Note that $c(E(C_{g_2})) \geq g_2 - 2$ and $c(E(C_{g_2})) \cap c(X) = \emptyset$ by Theorem 2.2 and Lemma 2.3, respectively.

Thus, $c(E(C_{g_2})) = Y$. Now, consider edges $v_1^1v_1^2$ and $v_1^1v_1^3$. By Lemma 2.4(ii), $\{c(v_1^1v_1^2), c(v_1^1v_1^3)\} \not\subseteq Y$. This forces $\{c(v_1^1v_1^2), c(v_1^1v_1^3)\} \subseteq c(X)$, contradicting Observation 4.2.

Subcase 2.3. $g_2 = 7$ or $g_2 \geq 9$.

Since $Z = \|G_2\| - 2$, we have $Y = [1, g_2]$. By using a similar argument as Subcase 2.2, we obtain that $\{c(v_1^1v_1^2), c(v_1^1v_1^3)\} \subseteq c(X)$, contradicting Observation 4.2.

The following figure presents an illustration of the contradiction arising in Subcases 2.2 and 2.3.

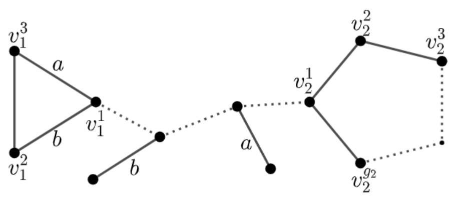

FIGURE 6. An illustration of the contradiction for Subcases 2.2 and 2.3.

Now, we proceed to prove the upper bound. For g_1, g_2 are equal to 7 or at least 9, it follows from Theorem 2.1 that $srx_3(G_2) \leq \|G_2\| - 1$.

For $g_1 = 3$ and $g_2 \geq 3$, we define a strong 3-rainbow coloring c of G_2 as follows.

(1) Define $c(v_1^1v_1^2) = c(v_1^1v_1^3) = 1$ and $c(v_1^2v_1^3) = 2$.
(2) For $g_2 = 3$, do step (a). For $g_2 = 4$, do step (b). For $g_2 \in \{5, 6, 8\}$, do step (c). For $g_2 = 7$ or $g_2 \geq 9$, do step (d).
   (a) Define $c(v_2^1v_2^2) = c(v_2^1v_2^3) = 3$ and $c(v_2^2v_2^3) = 2$. Furthermore, if $\|G_2\| > 6$, then assign colors $4, 5, \ldots, \|G_2\| - 3$ to the $\|G_2\| - 6$ bridges of G_2.
   (b) Define $c(v_2^1v_2^2) = c(v_2^3v_2^4) = 3$ and $c(v_2^2v_2^3) = c(v_2^1v_2^4) = 4$. Furthermore, if $\|G_2\| > 7$, then assign colors $5, 6, \ldots, \|G_2\| - 3$ to the $\|G_2\| - 7$ bridges of G_2.
   (c) Assign colors $2, 3, \ldots, g_2 - 1$ to the edges of C_{g_2} by using the edge-coloring rules given in (A1) such that $c(v_2^{\lfloor \frac{g_2}{2} \rfloor + 1}v_2^{\lfloor \frac{g_2}{2} \rfloor + 2}) = 2$. Furthermore, if $\|G_2\| > g_2 + 3$, then assign colors $g_2, g_2 + 1, \ldots, \|G_2\| - 4$ to the $\|G_2\| - g_2 - 3$ bridges of G_2.
   (d) Assign colors $2, 3, \ldots, \|G_2\| - 2$ to the remaining $\|G_2\| - 3$ edges of G_2 such that $c(v_2^{\lfloor \frac{g_2}{2} \rfloor + 1}v_2^{\lfloor \frac{g_2}{2} \rfloor + 2}) = 2$.

For $g_1 = 4$ and $g_2 \geq 4$, we define a strong 3-rainbow coloring c of G_2 as follows.

(1) Define $c(v_1^1v_1^2) = c(v_1^3v_1^4) = 1$ and $c(v_1^2v_1^3) = c(v_1^1v_1^4) = 2$.

(2) For $g_2 = 4$, do step (a). For $g_2 \in \{5, 6, 8\}$, do step (b). For $g_2 = 7$ or $g_2 \geq 9$, do step (c).
   (a) Define $c(v_2^1 v_2^2) = c(v_2^3 v_2^4) = 3$ and $c(v_2^2 v_2^3) = c(v_2^1 v_2^4) = 4$. Furthermore, if $\|G_2\| > 8$, then assign colors $5, 6, \ldots, \|G_2\| - 4$ to the $\|G_2\| - 8$ bridges of G_2.
   (b) Assign colors $3, 4, \ldots, g_2$ to the edges of C_{g_2} by using the edge-coloring rules given in (A1). Furthermore, if $\|G_2\| > g_2 + 4$, then assign colors $g_2 + 1, g_2 + 2, \ldots, \|G_2\| - 4$ to the $\|G_2\| - g_2 - 4$ bridges of G_2.
   (c) Assign colors $3, 4, \ldots, \|G_2\| - 2$ to the remaining $\|G_2\| - 4$ edges of G_2.

For $g_1 \in \{5, 6, 8\}$, and $g_2 \geq g_1 \geq 5$, we define a strong 3-rainbow coloring c of G_2 as follows.

(1) Assign colors $1, 2, \ldots, g_1 - 2$ to the edges of C_{g_1} by using the edge-coloring rules given in (A1) such that $c(v_1^{\lfloor \frac{g_1}{2} \rfloor + 1} v_1^{\lfloor \frac{g_1}{2} \rfloor + 2}) = g_1 - 2$.
(2) For $g_2 \in \{5, 6, 8\}$, do step (a). For $g_2 = 7$ or $g_2 \geq 9$, do step (b).
   (a) Assign colors $g_1 - 2, g_1 - 1, \ldots, g_1 + g_2 - 5$ to the edges of C_{g_2} by using the coloring rules given in (A1) such that $c(v_2^{\lfloor \frac{g_2}{2} \rfloor + 1} v_2^{\lfloor \frac{g_2}{2} \rfloor + 2}) = g_1 - 2$. Furthermore, if $\|G_2\| > g_1 + g_2$, then assign colors $g_1 + g_2 - 4, g_1 + g_2 - 3, \ldots, \|G_2\| - 5$ to the $\|G_2\| - g_1 - g_2$ bridges of G_2.
   (b) Assign colors $g_1 - 2, g_1 - 1, \ldots, \|G_2\| - 3$ to the remaining $\|G_2\| - g_1$ edges of G_2 such that $c(v_2^{\lfloor \frac{g_2}{2} \rfloor + 1} v_2^{\lfloor \frac{g_2}{2} \rfloor + 2}) = g_1 - 2$.

The next step is to show that there exists a rainbow Steiner tree connecting every 3-subset S of $V(G_2)$. Since the edge-colorings c assign distinct colors to all bridges of G_2 and ensure that $c(X) \cap c(E(C_{g_i})) = \emptyset$ for all $i \in [1, 2]$, it suffices to consider the subsets $S = \{u, v, w\}$ under the following two cases.

(i) $u, v, w \in V(C_{g_i})$ for some $i \in [1, 2]$. For $g_i \in [3, 4]$, the existence of a rainbow Steiner S-tree is immediate. Moreover, for $g_i \in \{5, 6, 8\}$, the edge-colorings c assign $g_i - 2$ distinct colors to the edges of C_{g_i} according to the edge-coloring rules given in (A1), while for $g_i = 7$ or $g_i \geq 9$, all edges of C_{g_i} are colored with distinct colors. Thus, in all these cases, the existence of a rainbow Steiner S-tree is also guaranteed.
(ii) $u \in V(C_{g_1})$ and $v, w \in V(C_{g_2})$. Observe that there exists a rainbow $v_1^1 - u$ geodesic T_1 in C_{g_1}, a rainbow Steiner $\{v_2^1, v, w\}$-tree T_2 in C_{g_2}, and a rainbow $v_1^1 - v_2^1$ geodesic T_3, such that $c(E(T_a)) \cap c(E(T_b)) = \emptyset$ for distinct $a, b \in [1, 3]$. Thus, the tree $T = T_1 \cup T_2 \cup T_3$ is a rainbow Steiner S-tree.

□

As an example, let G_2 be a bicyclic graph of order $n = 18$ with $g_1 = 3$ and $g_2 = 5$. According to the edge-coloring c defined in the proof of theorem above, we first assign colors 1 and 2 to the edges of C_3 so that the edges $v_1^1 v_1^2$ and $v_1^1 v_1^3$ receive the color 1, and then assign colors 2, 3 and 4 to the edges of C_5 so that the edge $v_2^3 v_2^4$ receives color 2. The remaining edges, which are the bridges of G_2, are then colored with colors $5, 6, \ldots, 15$. As a result, we obtain a strong 3-rainbow coloring of G_2 as illustrated in Figure 7(a). By using a similar procedure, we also obtain a strong 3-rainbow coloring of G_2 with $n = 24$, $g_1 = 6$ and $g_2 = 7$, as illustrated in Figure 7(b).

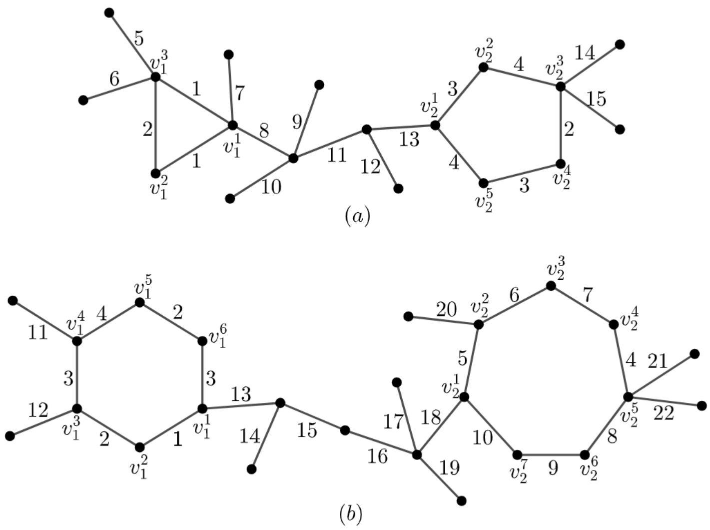

FIGURE 7. Strong 3-rainbow colorings of G_2 where (a) $n = 18$, $g_1 = 3$ and $g_2 = 5$, and (b) $n = 24$, $g_1 = 6$ and $g_2 = 7$

According to Theorem 4.3, it can be concluded that the srx_3 of bicyclic graphs, which is a graph obtained by adding two edges to a tree, is always less than its size.

5. CONCLUSION

It is known that $srx_3(T_n) = \|T_n\|$ (see Theorem 1.2). Therefore, this paper investigated how the addition of one or two edges to a tree T_n affected the srx_3 of the resulting graphs. We first provided sharp upper bounds for $srx_3(G)$ where G is a unicyclic or bicyclic graph, and then determined the exact values of $srx_3(G)$ for such graphs. Our results showed that $srx_3(G) = \|G\|$ if G is a unicyclic graph with girth 7 or at least 9; in all other cases, $srx_3(G) < \|G\|$. More specifically, Corollary 3.3 and Theorem 4.3 showed that for $t \in [1, 2]$, $srx_3(G) = \|G\| - t + 1$ if G is a connected graph containing t odd cycles of lengths at least 7. These results raise a natural question regarding the sharpness of this bound for connected graphs with more cycles, as formulated belows.

Problem 5.1. *For $t \geq 3$, is the upper bound $\|G\| - t + 1$ always sharp for a connected graph G containing t odd cycles of lengths at least 7?*

Additionally, it is important to note that adding two edges to a tree T_n may produce a graph containing exactly three cycles, one of which is commonly known as a theta graph. This observation naturally motivates further investigation of the following problem.

Problem 5.2. *What is the exact values of $srx_3(G)$ where G is a theta graph of order $n \geq 4$?*

Moreover, exploring the behavior of srx_3 in general cyclic graphs presents a promising direction for future research. We hope that the results presented in this paper provide a foundation for the broader characterization of the srx_3 for cyclic graphs.

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