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A SHORT NOTE ON BANDS OF GROUPS

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Abstract. In this paper, we give necessary and sufficient conditions on a semigroup S to be a semilattice of groups, a normal band of groups and a rectangular band of groups.

Key words and Phrases: Semigroup, band, semilattice, band of semigroup.

Abstrak. Pada paper ini, kami menyatakan syarat perlu dan cukup dari suatu semigrup S untuk menjadi semilatis dari grup, pita normal dari grup, dan pita persegipanjang dari grup.

Kata kunci: Semigrup, pita, semilatis, pita dari semigrup.

1. Introduction and Preliminaries

Before we present the basic definitions we give a short history of the subject. In [4], Clifford introduced bands of semigroups and determined their structure. In [3], Ciric and S. Bogdanovic studied sturdy bands of semigroups. Then, this concept is studied by many authors, for example see [6, 11]. In [7, 8, 9, 10], Lajos studied semilattices of groups. In [1], Bogdanovic presented a characterization of semilattices of groups using the notion of weakly commutative semigroup. The purpose of this paper is as stated in the abstract.

A semigroup S is a *group*, if for every $a, b \in S$, $a \in bS \cap Sb$. A semigroup S is a *band*, if for every $a \in S$, $a^2 = a$. A commutative band is called a *semilattice*.

Let S be a semigroup. If there exists a band $\{S_\alpha \mid \alpha \in \mathcal{C}\}$ of mutually disjoint subsemigroups S_α such that

(1) $S = \bigcup_{\alpha \in \mathcal{C}} S_\alpha$,

(2) for every $\alpha, \beta \in \mathcal{C}$, $S_\alpha S_\beta \subseteq S_{\alpha\beta}$,

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then we say S is a *band of semigroups of type* $\mathcal{C}$.

A congruence ρ of a semigroup S is a *semilattice congruence* of S if the factor S/ρ is a semilattice. If there exists a congruence relation ρ on a semigroup S such that S/ρ is a semilattice and every ρ-class is a group, then we say S is a *semilattice of groups*.

2. Main Results

Let S be a semigroup. Then, S^1 is "S with an identity adjoined if necessary"; if S is not already a monoid, a new element is adjoined and defined to be an identity. For an element a of S, the relevant ideals are: (1) The *principal left ideal generated by* a: $S^1a = \{sa \mid s \in S^1\}$, this is the same as $\{sa \mid s \in S\} \cup \{a\}$; (2) The *principal right ideal generated by* a: $aS^1 = \{as \mid s \in S^1\}$, this is the same as $\{as \mid s \in S\} \cup \{a\}$.

Let $a, b \in S$. We use the following well known notations:

$$a|_r b \;\Leftrightarrow\; b \in aS^1 \;\text{ and }\; a|_l b \;\Leftrightarrow\; b \in S^1a,$$
$$a|_t b \;\Leftrightarrow\; a|_r b,\; a|_l b.$$

For elements $a, b \in S$, Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are defined by

$$a\mathcal{L}b \;\Leftrightarrow\; a|_l b,\; b|_l a,$$
$$a\mathcal{R}b \;\Leftrightarrow\; a|_r b,\; b|_r a,$$
$$a\mathcal{H}b \;\Leftrightarrow\; a|_t b,\; b|_t a.$$

Indeed, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

Lemma 2.1. *$\mathcal{R}$ is a left congruence relation and $\mathcal{L}$ is a right congruence relation on S.*

PROOF. It is well-known in algebraic semigroup theory [4].

An element x of a semigroup S is said to be *left* (*right*) *regular* if $x = yx^2$ ($x = x^2y$) for some $y \in S$, or equivalently, $x\mathcal{L}x^2$ ($x\mathcal{R}x^2$). The second condition in the following theorem is equivalent to a semigroup being left regular and right regular.

Theorem 2.2. *A semigroup S is a semilattice of groups if and only if*

$$(\forall a, b \in S)\; ba|_t ab,\; a^2|_t a. \tag{1}$$

PROOF. Suppose that a semigroup S is a semilattice of groups and $S = \bigcup_{\alpha \in \mathcal{C}} S_\alpha$. If $a \in S_\alpha$ and $b \in S_\beta$, then $ab, ba \in S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is a group, $ba \in abS \cap Sab$. Since $a, a^2 \in S_\alpha$, we conclude that $a^2|_t a$.

Conversely, we define the relation η on S as follows:

$$a \;\eta\; b \Leftrightarrow a|_t b,\; b|_t a. \tag{2}$$

Obviously, $\eta \subseteq \mathcal{H}$, where $\mathcal{H}$ is the Green relation. Now, suppose that $a\mathcal{H}b$. Then, $a \in bS \cap Sb$ and $b \in aS \cap Sa$. Hence, $a \;\eta\; b$, and so $\mathcal{H} = \eta$. Suppose that $a\mathcal{H}b$ and

$c \in S$. Then, $ac \in bSc$. Thus, there exists $t \in S$ such that $ac = btc$. By (1), we have

$$ac = btc \in btc^2S \subseteq bc^2tS \subseteq bcS.$$

Similarly, $bc \in acS$. Hence, $ac\mathcal{R}bc$ and so $\mathcal{R}$ is a right congruence relation. By Lemma 2.1, we conclude that $\mathcal{R}$ is a congruence relation. Since $a \in Sb$, there exists $m \in S$ such that $a = mb$. By (1), we obtain

$$ca = cmb \in Smcb \subseteq Scb.$$

So, $\mathcal{L}$ is a left congruence relation. By Lemma 2.1, we conclude that $\mathcal{L}$ is a congruence relation. Therefore, $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$ is a congruence relation. For every $a \in S$, we have $a^2 \in aS \cap Sa$. Then, by (1), $a \in a^2S \cap Sa^2$ which implies that $a\mathcal{H}a^2$. Also, by (1), we obtain $ab\mathcal{H}ba$. Therefore, $\mathcal{H}$ is a congruence semilattice. Now, let $S = \bigcup_{\alpha \in \mathcal{C}} S_\alpha$, where $\mathcal{C}$ is a semilattice and S_α is $\mathcal{H}$-class, for every $\alpha \in \mathcal{C}$. We prove that S_α is a group, for every $\alpha \in \mathcal{C}$. Suppose that $a\mathcal{H}b$. Then, for some $\alpha \in \mathcal{C}$, $a, b \in S_\alpha$ and $a\mathcal{H}b^2$. Hence, there exists $x \in S$ such that $a = b^2x$. If $a, b \in S_\alpha$ and $x \in S_\beta$, then $\alpha\beta = \alpha$. From (1), we conclude that there exists $y \in S$ such that $a = a^2y$. If $y \in S_\gamma$, then $\alpha\gamma = \alpha$. So, we have

$$a = a^2y = aay = b^2xay = bbxay \in bS_{\alpha\beta\alpha\gamma} = bS_\alpha.$$

Similarly, we can prove that $a \in S_\alpha b$ and $b \in S_\alpha a \cap aS_\alpha$. Thus, $a|_tb$ and $b|_ta$ in S_α. Therefore, S is a semilattice of groups S_α.

Definition 2.3. *A band $\mathcal{B}$ is called normal if for every $a, b, c \in \mathcal{B}$, $cabc = cbac$.*

Theorem 2.4. *A semigroup S is a normal band of groups if and only if*

$$(\forall a, b, c, d \in S)\ abcd|_tacbd,\ a|_ta^2. \tag{3}$$

PROOF. Suppose that a semigroup S is a normal band of groups and $S = \bigcup_{\alpha \in \mathcal{C}} S_\alpha$. If $a \in S_\alpha$, $b \in S_\beta$, $c \in S_\gamma$ and $d \in S_\delta$, then $abcd \in S_{\alpha\beta\gamma\delta}$. Since $\mathcal{C}$ is a normal band, $acbdbacd, abdcacbd \in S_{\alpha\beta\gamma\delta}$. So, we have

$$(\forall a, b, c, d \in S)\ abcd \in acbdbacdS \subseteq acbdS \text{ and } abcd \in Sabdcacbd \subseteq Sacbd.$$

Conversely, we consider the relation η. Similar to the proof of Theorem 2.2, we obtain $\mathcal{H} = \eta$. In order to prove $\mathcal{H}$ is a congruence relation, it is enough to show that $\mathcal{R}$ is a right congruence relation and $\mathcal{L}$ is a left congruence relation. Suppose that $a\mathcal{R}b$. Then, there exists $s \in S$ such that

$$ac = bsc \in bsc^2S \subseteq bcscS \subseteq bcS.$$

Similarly, $bc \in acS$. Suppose that $a\mathcal{L}b$. Then, there exists $m \in S$ such that

$$ca = cmb \in Sc^2mb \subseteq Scmcb \subseteq Scb.$$

Similarly, $cb \in Sca$. Let $a, b, c \in S$. By (3), $abca\mathcal{H}acba$ and $a\mathcal{H}a^2$. Therefore, $\mathcal{H}$ is a congruence normal band.

Now, suppose that $a\mathcal{H}b$. Then, $a\mathcal{H}b^2$ and so $a\mathcal{L}b^2$. Hence, there exists $x \in S$

such that $a = xb^2$. If $\alpha, \beta \in \mathcal{C}$, $a, b \in S_\alpha$ and $x \in S_\beta$, then $\alpha = \beta\alpha$. By (3), for every $a \in S$ there exists $y \in S$ such that $a = ya^2$. If $y \in S_\gamma$, then $\alpha = \gamma\alpha$. Thus, we have

$$a = ya^2 = yaa = yaxb^2 = yaxbb \in S_{\gamma\alpha\beta\alpha}b = S_\alpha b.$$

Similarly, we can prove that $b \in S_\alpha a$. Since $a\mathcal{R}b$, we conclude that $a \in bS_\alpha$ and $b \in aS_\alpha$. Therefore, S_α is a group and S is a normal band of groups.

Definition 2.5. *A semigroup S is called a rectangular band if for every $a, b \in S$, $aba = a$.*

Theorem 2.6. *A semigroup S is a rectangular band of groups if and only if*

$$(\forall a, b \in S)\ a|_t aba. \tag{4}$$

Proof. Suppose that a semigroup S is a rectangular band of groups and $S = \bigcup\limits_{\alpha \in \mathcal{C}} S_\alpha$. Then, for every $a, b \in S$, $aba \in S$. Therefore,

$$a \in abaS \text{ and } a \in Saba.$$

Conversely, suppose that (4) holds. If $a\mathcal{H}b$, then for every $c \in S$ we have $ac \in bSc \subseteq bcbSc \subseteq bcS$. Similarly, $bc \in acS$ and so $ac\mathcal{R}bc$. On the other hand, $ca \in cSb \subseteq cSbcb \subseteq Scb$ and $cb \in Sca$. Thus, $ca\mathcal{L}cb$. Therefore, $\mathcal{R}$ is a right congruence relation and $\mathcal{L}$ is a left congruence relation, and so $\mathcal{H}$ is a congruence relation. Since for every $a, b \in S$, $a \in Saba$ and $a \in abaS$, S is a congruence rectangular band.

Now, suppose that $a\mathcal{H}b$. Then, $a\mathcal{H}b^2$ and there exists $\alpha \in \mathcal{C}$ such that $a, b \in S_\alpha$. So, there exist $m, n \in S$ such that $a = mb^2$ and $b = na$. If $\beta, \gamma \in \mathcal{C}$, $m \in S_\gamma$ and $n \in S_\beta$, then $\alpha = \gamma\alpha$ and $\alpha = \beta\alpha$. So, we have

$$a = mb^2 = mnab \in S_{\gamma\beta\alpha}b = S_\alpha b.$$

Similarly, we can prove that $a \in bS_\alpha$ and $b \in aS_\alpha \cap S_\alpha a$. Therefore, S_α is a group.

Corollary 2.7. *S is a left zero band of groups if and only if for every $a, b \in S$, $a|_t ab$.*

3. Concluding Remarks

In this article, we studied some aspects of band of semigroups and groups. Let H be a non-empty set and let $\mathcal{P}^*(H)$ be the family of all non-empty subsets of H. A *hyperoperation* on H is a map $\star : H \times H \to \mathcal{P}^*(H)$ and the couple $(H, \star)$ is called a *hypergroupoid*. If A and B are non-empty subsets of H, then we denote $A \star B = \bigcup_{a \in A,\, b \in B} a \star b$. A hypergroupoid $(H, \star)$ is called a *semihypergroup* if for all x, y, z of H, we have $(x \star y) \star z = x \star (y \star z)$ [5]. In future, we shall study the band of semihypergroups.

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