A SHORT NOTE ON BANDS OF GROUPS

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Abstract. In this paper, we give necessary and sufficient conditions on a semigroup $S$ to be a semilattice of groups, a normal band of groups and a rectangular band of groups.

Key words and Phrases: Semigroup, band, semilattice, band of semigroup.

Abstrak. Pada paper ini, kami menyatakan syarat perlu dan cukup dari suatu semigrup $S$ untuk menjadi semilatish dari grup, pita normal dari grup, dan pita persegi panjang dari grup.

Kata kunci: Semigrup, pita, semilatish, pita dari semigrup.

1. Introduction and Preliminaries

Before we present the basic definitions we give a short history of the subject. In [4], Clifford introduced bands of semigroups and determined their structure. In [3], Ciric and S. Bogdanovic studied sturdy bands of semigroups. Then, this concept is studied by many authors, for example see [6, 11]. In [7, 8, 9, 10], Lajos studied semilattices of groups. In [1], Bogdanovic presented a characterization of semilattices of groups using the notion of weakly commutative semigroup. The purpose of this paper is as stated in the abstract.

A semigroup $S$ is a group, if for every $a, b \in S$, $a \in bS \cap Sb$. A semigroup $S$ is a band, if for every $a \in S$, $a^2 = a$. A commutative band is called a semilattice.

Let $S$ be a semigroup. If there exists a band $\{S_\alpha \mid \alpha \in \mathcal{C}\}$ of mutually disjoint subsemigroups $S_\alpha$ such that

\begin{enumerate}
\item $S = \bigcup_{\alpha \in \mathcal{C}} S_\alpha,$
\item for every $\alpha, \beta \in \mathcal{C}$, $S_\alpha S_\beta \subseteq S_{\alpha \beta},$
\end{enumerate}
then we say $S$ is a band of semigroups of type $C$.

A congruence $\rho$ of a semigroup $S$ is a semilattice congruence of $S$ if the factor $S/\rho$ is a semilattice. If there exists a congruence relation $\rho$ on a semigroup $S$ such that $S/\rho$ is a semilattice and every $\rho$-class is a group, then we say $S$ is a semilattice of groups.

2. Main Results

Let $S$ be a semigroup. Then, $S^1$ is “$S$ with an identity adjoined if necessary”; if $S$ is not already a monoid, a new element is adjoined and defined to be an identity. For an element $a$ of $S$, the relevant ideals are: (1) The principal left ideal generated by $a$: $aS^1 = \{sa \mid s \in S^1\}$, this is the same as $\{sa \mid s \in S\} \cup \{a\}$; (2) The principal right ideal generated by $a$: $S^1a = \{as \mid s \in S^1\}$, this is the same as $\{as \mid s \in S\} \cup \{a\}$.

Let $a, b \in S$. We use the following well known notations:

\[ aL b \iff b \in aS^1 \text{ and } aL b \iff b \in S^1a, \]
\[ aL b \iff aL b, aL b. \]

For elements $a, b \in S$, Green’s relations $L$, $R$ and $H$ are defined by

\[ aL b \iff aL b, bL a, \]
\[ aR b \iff aR b, bL a, \]
\[ aH b \iff aH b, bL a. \]

Indeed, $H = L \cap R$.

**Lemma 2.1.** $R$ is a left congruence relation and $L$ is a right congruence relation on $S$.

**Proof.** It is well-known in algebraic semigroup theory [4].

An element $x$ of a semigroup $S$ is said to be left (right) regular if $x = yx^2$ ($x = x^2y$) for some $y \in S$, or equivalently, $xLx^2$ ($xRx^2$). The second condition in the following theorem is equivalent to a semigroup being left regular and right regular.

**Theorem 2.2.** A semigroup $S$ is a semilattice of groups if and only if

\[ (\forall a, b \in S) \ b[a|a, a^2|a. \]

**Proof.** Suppose that a semigroup $S$ is a semilattice of groups and $S = \bigcup_{\alpha} S_{\alpha}$. If $a \in S_{\alpha}$ and $b \in S_{\beta}$, then $ab, ba \in S_{\alpha \beta}$. Since $S_{\alpha \beta}$ is a group, $ba \in abS \cap S_{\alpha \beta}$. Since $a$, $a^2 \in S_{\alpha}$, we conclude that $a^2|a$.

Conversely, we define the relation $\eta$ on $S$ as follows:

\[ a \eta b \iff a|b, b|a. \]

Obviously, $\eta \subseteq H$, where $H$ is the Green relation. Now, suppose that $aHb$. Then, $a \in bS \cap Sb$ and $b \in aS \cap Sa$. Hence, $a \eta b$, and so $H = \eta$. Suppose that $aHb$ and
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c ∈ S. Then, ac ∈ bSc. Thus, there exists t ∈ S such that ac = btc. By (1), we have
\[ ac = btc ∈ btec^2S ⊆ bc^2tS ⊆ bcS. \]

Similarly, bc ∈ acS. Hence, acRbc and so \( R \) is a right congruence relation. By
Lemma 2.1, we conclude that \( R \) is a congruence relation. Since \( a ∈ Shb \), there exists
\( m ∈ S \) such that \( a = mb \). By (1), we obtain
\[ ca = cmb ∈ Smcb ⊆ Scb. \]

So, \( L \) is a left congruence relation. By Lemma 2.1, we conclude that \( L \) is a con-
gruence relation. Therefore, \( H = R ∩ L \) is a congruence relation. For every \( a ∈ S, \)
we have \( a^2 ∈ aS ∩ Sa \). Then, by (1), \( a ∈ a^2S ∩ Sa^2 \) which implies that \( aHa^2 \).

Also, by (1), we obtain \( abHba \). Therefore, \( H \) is a congruence semilattice. Now, let
\( S = \bigcup S_α, \) where \( C \) is a semilattice and \( S_α \) is \( H \)-class, for every \( α ∈ C \). We prove
that \( S_α \) is a group, for every \( α ∈ C \). Suppose that \( aHa^2 \). Then, for some \( α ∈ C, \)
a, b ∈ \( S_α \) and \( aHa^2 \). Hence, there exists \( x ∈ S \) such that \( a = b^2x \). If \( a, b ∈ S_α \)
and \( x ∈ S_β \), then \( αβ = α \). From (1), we conclude that there exists \( y ∈ S \) such that
\( a = a^2y \). If \( y ∈ S_γ \), then \( αγ = α \). So, we have
\[ a = a^2y = aay = b^2xay = bbyxay ∈ bS_αβαγ = bS_α. \]

Similarly, we can prove that \( a ∈ S_αb \) and \( b ∈ S_αa ∩ aS_α \). Thus, \( a|b \) and \( b|a \) in \( S_α \).
Therefore, \( S \) is a semilattice of groups \( S_α \).

**Definition 2.3.** A band \( B \) is called normal if for every \( a, b, c ∈ B \), \( cabc = cbac \).

**Theorem 2.4.** A semigroup \( S \) is a normal band of groups if and only if
\[ (∀a, b, c, d ∈ S) \quad abcd|abcd, \quad a|ab^2. \]

**Proof.** Suppose that a semigroup \( S \) is a normal band of groups and \( S = \bigcup S_α \).

If \( a ∈ S_α \), \( b ∈ S_β \), \( c ∈ S_γ \) and \( d ∈ S_δ \), then \( abcd ∈ S_αβγδ \). Since \( C \) is a normal band,
\( a(abcd) \), \( abcd ∈ S_αβγδ \). So, we have
\[ (∀a, b, c, d ∈ S) \quad abcd ∈ abcdS ⊆ acbdS \quad and \quad abcd ∈ Sabcdacbd ⊆ Sacbd. \]

Conversely, we consider the relation \( η \). Similar to the proof of Theorem 2.2, we obtain \( H = η \). In order to prove \( H \) is a congruence relation, it is enough to show that \( R \) is a right congruence relation and \( L \) is a left congruence relation. Suppose
that \( aRb \). Then, there exists \( s ∈ S \) such that
\[ ac = bsc ∈ bsc^2S ⊆ bsScS ⊆ bcS. \]

Similarly, \( bc ∈ acS. \) Suppose that \( aLb \). Then, there exists \( m ∈ S \) such that
\[ ca = cmb ∈ Sc^2mb ⊆ Scmcb ⊆ Scb. \]

Similarly, \( cb ∈ Sca. \) Let \( a, b, c ∈ S \). By (3), \( abcaHcba \) and \( aHα^2 \). Therefore, \( H \) is a congruence normal band.

Now, suppose that \( aHb \). Then, \( aHb^2 \) and so \( aLb^2 \). Hence, there exists \( x ∈ S \)
such that \( a = xb^2 \). If \( \alpha, \beta \in \mathbb{C}, a, b \in S_\alpha \) and \( x \in S_\beta \), then \( \alpha = \beta \alpha \). By (3), for every \( a \in S \) there exists \( y \in S \) such that \( a = ya^2 \). If \( y \in S_\gamma \), then \( \alpha = \gamma \alpha \). Thus, we have

\[
a = ya^2 = yaa = yaxb^2 = yaxbb \in S_{\gamma \alpha \beta \alpha} b = S_\alpha b.
\]

Similarly, we can prove that \( b \in S_\alpha a \). Since \( aRb \), we conclude that \( a \in bS_\alpha \) and \( b \in aS_\alpha \). Therefore, \( S_\alpha \) is a group and \( S \) is a normal band of groups.

**Definition 2.5.** A semigroup \( S \) is called a rectangular band if for every \( a, b \in S \),

\[
aba = a.
\]

**Theorem 2.6.** A semigroup \( S \) is a rectangular band of groups if and only if

\[
(\forall a, b \in S) \ a|_1 aba.
\]

**Proof.** Suppose that a semigroup \( S \) is a rectangular band of groups and \( S = \bigcup_{\alpha \in C} S_\alpha \). Then, for every \( a, b \in S \), \( aba \in S \). Therefore,

\[
a \in aba S \text{ and } a \in Saba.
\]

Conversely, suppose that (4) holds. If \( a \mathcal{H} b \), then for every \( c \in S \) we have \( ac \in bSc \subseteq bcbSc \subseteq bcS \). Similarly, \( ec \in acS \) and so \( acRbe \). On the other hand, \( ca \in S \subseteq cSbcb \subseteq Sca \) and \( cb \in Sca \). Thus, \( ca \mathcal{L} b \). Therefore, \( \mathcal{R} \) is a right congruence relation and \( \mathcal{L} \) is a left congruence relation, and so \( \mathcal{H} \) is a congruence relation. Since for every \( a, b \in S \), \( a \in Saba \) and \( a \in abaS \), \( S \) is a congruence rectangular band.

Now, suppose that \( a \mathcal{H} b \). Then, \( a \mathcal{H} b^2 \) and there exists \( \alpha \in \mathcal{C} \) such that \( a, b \in S_\alpha \). So, there exist \( m, n \in S \) such that \( a = mb^2 \) and \( b = na \). If \( \beta, \gamma \in \mathcal{C}, m \in S_\gamma \) and \( n \in S_\beta \), then \( \alpha = \gamma \alpha \) and \( \alpha = \beta \alpha \). So, we have

\[
a = mb^2 = mnab \in S_{\gamma \beta \alpha} b = S_\alpha b.
\]

Similarly, we can prove that \( a \in bS_\alpha \) and \( b \in aS_\alpha \cap S_\alpha a \). Therefore, \( S_\alpha \) is a group.

**Corollary 2.7.** \( S \) is a left zero band of groups if and only if for every \( a, b \in S \),

\[
a|_1 ab.
\]

### 3. Concluding Remarks

In this article, we studied some aspects of band of semigroups and groups. Let \( H \) be a non-empty set and let \( \mathcal{P}^\ast(H) \) be the family of all non-empty subsets of \( H \). A hyperoperation on \( H \) is a map \( \star : H \times H \rightarrow \mathcal{P}^\ast(H) \) and the couple \( (H, \star) \) is called a hypergroupoid. If \( A \) and \( B \) are non-empty subsets of \( H \), then we denote \( A \star B = \bigcup_{a \in A, b \in B} a \star b \). A hypergroupoid \( (H, \star) \) is called a semihypergroup if for all \( x, y, z \) of \( H \), we have \( (x \star y) \star z = x \star (y \star z) \) [5]. In future, we shall study the band of semihypergroups.
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REFERENCES