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The Structure of Cayley Graph of Dihedral Groups of Valency 4

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Abstract. Let G be a group and S be a subset of G in which $e \notin S$ and $S^{-1} \subseteq S$. The Cayley graph of group G with respect to subset S, denoted by $Cay(G,S)$, is an undirected simple graph whose vertices are all elements of G, and two vertices x and y are adjacent if and only if $xy^{-1} \in S$. If $|S| = k$, then $Cay(G,S)$ is called a Cayley graph of valency k. The aim of this paper is to determine the structure of Cayley graph of dihedral groups D_{2n} of order $2n$ when $n = p$ or $2p^2$, where p is an odd prime number. The graph structures are based on circulant graphs with suitable jumps.

Key words and Phrases: Cayley graph, valency, dihedral group, circulant graph, jump

1. Introduction

For any group G and subset S of G such that $e \notin S$, $S^{-1} \subseteq S$, we can associated a graph whose vertex set is the set of element in G and two distinct vertices x, y are adjacent if and only if $xy^{-1} \in S$. This graph firstly was introduced by Arthur Cayley in 1878 and is known as undirected Cayley graph. Later, Max Dehn reintroduced Cayley graphs under the name Gruppenbild (group diagram) in his unpublished lectures on group theory. It leds to the geometric group theory and he used the set of generators for the new geometric representation of group. This will translate groups into geometrical objects that can be considered from the geometric view. For example, it provides a rich source of many symmetric graphs which are known as transitive graphs and they play a serious work in many graph theoretical problems such as hamiltonian path and cycles. One more importance of

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Cayley graphs is that these graphs are known as the oldest algebraic graphs. One of the most important branches of mathematics is algebraic graph theory and it is playing an essential role in other fields in which algebraic methods are applied to some problems in graph theory. There are a series of papers in which many authors defined graphs associated to special algebraic structures and found some relations between graph properties and algebraic properties. For example, commuting and non-commuting graphs of a group (see [1, 2]) or zero divisor graph of a ring(see [3]). We have a similar work on Cayley graphs. Some authors found more graph properties of this graph and they determined the graph structures for a given group G and a subset S of G. To determine the structure of Cayley graph, the subset S plays an important role. For instance, when S is a generating set of group G, the Cayley graph is connected. Moreover, the size of S is also very important. If $|S| = k$, then $Cay(G, S)$ is usually denoted as Cayley graph of valency k. It is clear that if $k = 1$, then S is a singleton set consisting an element of order 2 and so $Cay(G, S)$ of valency 1 must be the union of some edges. For the case $k = 2$, there are two possibilities for subset S. First possibility is that S consists of two elements of order 2 and the second possibility is the case that S contains an element having order not equal to 2 and its inverse. For $k \geq 3$ there will be more possibilities and so to determine the structure of $Cay(G, S)$ in terms of valency k, it is necessary to consider all such possibilities and it is more complicated in general. Furthermore, the structure of group G is also playing important role to find the graph structure of $Cay(G, S)$. Recently, in 2021, S. Alkaseabe and the second author (see [4]) found the graph structure of $Cay(D_{2n}, S)$ of valencies 1, 2 and 3, where D_{2n} is the dihedral group of order $2n$ and $n \geq 3$. In this paper, we aim to give the structure of cayley graphs $Cay(D_{2p}, S)$ and $Cay(D_{2p^2}, S)$ of valency 4, where p is an odd prime number.

The results given in this paper are stated in two different sections according to two Cayley graphs $Cay(D_{2p}, S)$ and $Cay(D_{2p^2}, S)$ of valency 4, respectively. In the rest of this section, we recall some terminologies and notations which are standard and can be found in (see[5, 6, 7, 8, 9, 10, 11, 12, 13, 14]). For a positive integer n, we use the notations Z_n and D_{2n} for the cyclic group of order n and dihedral group of order $2n$, respectively. The presentations of these two groups are as $Z_n =< x \mid x^n = e >= \{e, x, x^2, \ldots, x^{n-1}\}$ and $D_{2n} =< a, b \mid a^n = b^2 = e, bab = a^{-1} >= \{e, a, a^2, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b\}$. Let X be a graph, then the set of vertices and edges are denoted by $V(X)$ and $E(X)$, respectively. For two vertices $x, y \in V(X)$, we denote $x \sim y$ or $x - y$ if there is an edge between x and y. If in a graph there is no edge then is called an empty graph. A graph with no loop and multiple edges is called a simple graph. All graphs in this paper are assumed to be simple and undirected. The complement of X, denoted by $\overline{X}$, is a graph such that $V(X) = V(\overline{X})$ and two vertices are adjacent in $\overline{X}$ if and only if there are not adjacent in X. The degree of vertex $x \in V(X)$ denoted by $deg(x)$, is the number of adjacent vertices of x. A graph X is called connected if for any arbitrary two vertices $x, y \in V(X)$ there exists at least a path between x and y. Otherwise, it is called a disconnected graph. The distance between two vertices $x, y \in V(X)$, denoted by $d_X(x, y)$, is the length of the shortest path between x and y. We denote by K_n, P_n, and C_n the complete graph, the path graph and the cycle graph with n vertices, respectively. The union of two graphs X_1 and X_2, denoted by $X_1 \cup X_2$, is a graph with $V(X_1 \cup X_2) = V(X_1) \cup V(X_2)$, and $E(X_1 \cup X_2) = E(X_1) \cup E(X_2)$. If $X_1 = X_2$, then $X_1 \cup X_1$ will be denoted by $2X_1$ and similarly nX stands for the union of n copies of X. Let G be a group and H be a subset of G. Then the number of right or left cosets of H in G denoted by $[G : H]$.

As we mentioned earlier, for a given group G, the Cayley graph of G with respect to S as a subset of G, denoted by $Cay(G, S)$ is an undirected simple graph with vertex set consists of all elements of G in which we have two conditions $e \notin S$ and $S^{-1} \subseteq S$. Two vertices x and y are adjacent if and only if $xy^{-1} \in S$. The valency of the Cayley graph is defined as the size of subset S. Three important properties of $Cay(G, S)$ are $|S|$- regular, vertex transitive (group of automorphisms of the $Cay(G, S)$ acts on $V(G)$ transitively) and connected whenever subset S of G is a generating set of group G. More background on graph theory and group theory concepts not defined here can be found in [15] and [16].

2. Cayley Graphs of Dihedral Groups of Order $2p$

In this section, we investigate and determine the structure of Cayley graph $Cay(G, S)$ on a dihedral group D_{2p} of valency 4, where p is an odd prime number. Before to state our results on the structure of $Cay(D_{2p}, S)$ for $|S| = 4$ and p is an odd prime, we need to define and give some notations on the circulant graph. A circulant graph, denoted by $C_n(j_1, j_2, \ldots, j_m)$, with n vertices, labeled with integers modulo n, and jumps j_1, j_2, $\ldots$, j_m, is an undirected graph in which each vertex i, $0 \le i \le n-1$, is adjacent to all the vertices $i \pm j_k$ mod n, with $1 \le k \le m$. It is obvious that $C_n(1) = C_n$ and

$C_n(1, 2, \ldots, \lfloor n/2 \rfloor) = K_n$. For more details, some circulant graphs $C_4(2)$, $C_5(1, 2)$ and $C_8(1, 2, 3)$ are shown in Figure 1. The circulant graph $C_n(1, k)$ is important for us here. Note that $C_n(1, k)$ can be considered as a graph with n vertices and jump k consisting a cycle of length n, namely"outer cycle"' and cycle (cycles) inside of outer cycle, namely "inner cycle (cycles)". Now, let $G = C_n(1, k)$ be a circulant graph with outer cycle C_n and jump k, then outer and inner cycles will denote for shorten by "O" and "I", respectively. For two vertices u and v, we may display $d^I_G(u, v)$ and $d^O_G(u, v)$ as the distance between u and v through outer and inner cycles, respectively. Let us explain more on these notations with the following example.

Example 2.1. *Let $G = C_7(1, 2)$ and $H = C_7(1, 3)$ be circulant graphs with 7 vertices and jumps 2 and 3, respectively (see Figure 2). Then we have $d^O_G(u, v) = 2$, $d^I_G(u, v) = 1$, $d^O_H(u, v) = 2$ and $d^I_H(u, v) = 3$.*

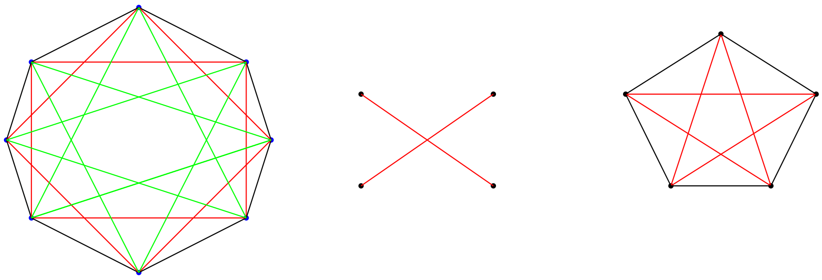

FIGURE 1. $C_8(1,2,3)$, $C_4(2)$, $C_5(1,2)$,

FIGURE 2. $H = C_7(1,3)$, $G = C_7(1,2)$.

The following theorem gives the structure of $Cay(D_{2p}, S)$ when S consists of two elements y, z of order 2 and two elements x and its inverse x^{-1} whenever $o(x) \neq 2$.

Theorem 2.2. *Let* $Cay(D_{2p}, S)$ *be* Caylay graph of dihedral group D_{2p} with $S = \{x, x^{-1}, y, z\}$ such that p is an odd prime number and $x \neq x^{-1}$, $y^2 = z^2 = e$. Let k be the smallest positive integer in which $(zy)^k = x$, then

$$Cay(D_{2p}, S) = C_{2p}(1, 2k).$$

Proof. Suppose that $D_{2p} =< a, b \mid a^p = b^2 = e, bab = a^{-1} >= \{e, a, a^2, ..., a^{p-1}, b, ab, ..., a^{p-1}b\}$. Since $o(a^i b) = 2$ for $i = 0, 1, 2, ..., p-1$ and $o(x) \neq 2$ it whould imply that $x = a^j$ for some $1 \leq j \leq p-1$. Moreover, we have $p = o(x) = o(a^j) = \dfrac{p}{(p,j)}$. We show that the following is a cycle of length p:

$$e \sim x \sim x^2 \sim x^3 \sim x^4 \sim \cdots \sim x^{p-1} \sim x^p = e.$$

Note that $(x^t)(x^{t+1})^{-1} = a^{it}a^{-it-i} = a^{-i} = x^{-1} \in S$, for $t = 0, 1, 2, \ldots, p$. Let $H =< x >= \{e, x, x^2, x^3, \ldots, x^{p-1}\}$ be a cyclic subgroup of D_{2p} of order p, then we have $[D_{2p} : H] = \dfrac{2p}{p} = 2$. Thus, there are two distinct left cosets H and yH such that $y \notin H$. For left coset $yH = \{y, yx, yx^2, yx^3, \ldots, yx^{p-1}\}$, we have a cycle $y \sim yx \sim yx^2 \sim yx^3 \sim yx^4 \sim \cdots \sim yx^{p-1} \sim yx^p = y$, since

$(yx^t)(yx^{t+1})^{-1} = yx^tx^{-1-t}y^{-1} = yx^{-1}y^{-1} = xyy^{-1} = x \in S$, for $t = 0, 1, 2, \ldots, p$. In addition, we can easily see that there is a cycle of length $2p$ as the following:

$$e \sim y \sim x^{p-h} \sim yx^{p-h} \sim x^{p-2h} \sim yx^{p-2h} \sim x^{p-3h} \sim yx^{p-3h} \sim$$
$$\ldots \sim x^{p-(p-1)h} \sim yx^{p-(p-1)h} \sim x^{p-ph} = e,$$

where $x^h = (zy)^{p-1}$ and $x^r = (zy)^1$, $h = p - r$. Because $yx^{h-p} = yx^{-r} = y(zy)^{-1} = yy^{-1}z^{-1} = z \in S$, $x^{p-th}(yx^{p-th})^{-1} = x^{p-th}x^{th-p}y^{-1} = y \in S, (yx^{p-th})$ $(x^{p-(t+1)})^{-1} = yx^{p-th}x^{(t+1)h-p} = y(x)^h = y(zy)^{p-1} = y(zy)^{-1} = yy^{-1}z^{-1} = z \in S$.

Put cycle of length $2p$ as outer cycle and two cycles of length p as inner cycles. Now we show that for any two adjacent vertices u and v in each inner cycles $d_G^O(u, v) = 2k$. We know that $D_{2p} = H \cup yH$ and each of cosets H and yH produce an inner cycle of length p. Consider two adjacent vertices e and x in the first inner cycle. Then we have $x^{p-th} = x$ and so it will be happend if $p-th = 1$ or equivalently $p-t(p-r) = p(1-t)+tr = 1$ or $x^{p-th} = x^{tr} = (x^r)^t = (zy)^t$ for some $0 \leq t \leq p-1$. Since $(zy)^k = x$, so by putting $t = k$, we can see that $x^{p-kh} = x$ and therefore, $d_G^O(e, x) = 2k$. Similarly, we may comput distance between any of two other adjacent vertices in any inner cycle and find at distance $2k$ through outer cycle. Hence, $Cay(D_{2p}, S) = C_{2p}(1, 2k)$ and the proof is completed. □

Example 2.3. *Let D_{14} be a dihedral group of order 14. Then we have $D_{14} =< a, b \mid a^7 = b^2 = e, bab = a^{-1} >= \{e, a, a^2, a^3, a^4, a^5, a^6, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b\}$. If $S = \{a^6, a, a^2b, a^4b\}$ where $x = a^6, x^{-1} = a, y = a^2b, z = a^4b$. Then by Theorem 2.2 we can see that $Cay(D_{14}, S) = C_{14}(1, 6)$. (see Figure 3)*

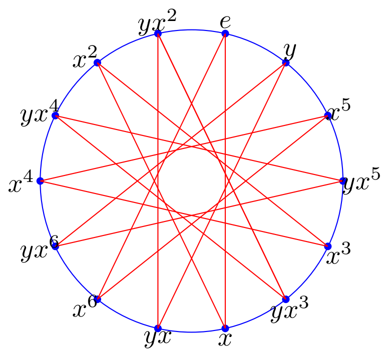

FIGURE 3. $Cay(D_{14}, S) = C_{14}(1, 6)$.

Theorem 2.4. *Let D_{2p} be the dihedral group of order $2p$. If $S \subseteq D_{2p}$ such that $S = \{u, u^{-1}, v, v^{-1}\}$, where $u \neq u^{-1}$, $v \neq v^{-1}$ and m be the smallest positive integer such that $u^m = v$ for some $2 \leq m \leq p-1$. Then we have*

$$Cay(D_{2p}, S) = 2C_p(1, m).$$

Proof. Suppose that $D_{2p} = \{e, x, x^2, x^3, x^4, \dots x^{p-1}, y, yx, yx^2, \dots, yx^{p-1}\}$. Since $u \neq u^{-1}$ and $v \neq v^{-1}$, so u and v are as the form x^i for some $1 \leq i \leq p-1$. Also we can see that $o(yx^j) = 2$ for $j = 1, 2, \dots, p-1$. Let k be the smallest positive integer such that $u = x^k$, and we can let $v = x^{mk}$ for some $2 \leq m \leq p-1$. Now we have an outer cycle of length p as the following:

$$e \sim x^k \sim x^{2k} \sim x^{3k} \sim \cdots \sim x^{(p-1)k} \sim e.$$

Note that $(x^{ik})(x^{(i+1)k})^{-1} = x^{-k} \in S$, for $i = 1, 2, \dots, p-1$. We have also the inner cycle of length p as following:

$$e \sim x^{mk} \sim x^{2mk} \sim x^{3mk} \sim \cdots \sim x^{(p-1)mk} \sim e$$

Because $(x^{imk})(x^{(i+1)mk})^{-1} = x^{-mk} \in S$, for $i = 1, 2, ..., p-1$. We show that these outer and inner cycles produce a circulant graph $C_p(1, m)$ with jump m. For this, it is necessary to prove that every two adjacent vertices in the inner cycle has length m on the outer cycle. It can be seen that

$$d_G^O(e, x^{mk}) = d_G^O(x^{mk}, x^{2mk}) = d_G^O(x^{2mk}, x^{3mk}) = \\ , \dots, d_G^O(x^{(p-1)mk}, e) = m.$$

Since the graph is $4-$regular, we conclude that the rest of the other elements of the graph, whose number is p, is in the form $x^i y$ such that $1 \leq i \leq p$. So there will be the following outer and inner cycles of length p, respectively

$$y \sim x^k y \sim x^{2k} y \sim x^{3k} y \sim \cdots \sim x^{(p-1)k} y \sim y$$

$$y \sim x^{mk} y \sim x^{2mk} y \sim x^{3mk} y \sim \cdots \sim x^{(p-1)mk} y \sim y.$$

We can easily see that

$$d_G^O(y, x^{mk} y) = d_G^O(x^{mk} y, x^{2mk} y) = d_G^O(x^{2mk} y, x^{3mk} y) = \\ , \dots, d_G^O(x^{(p-1)mk} y, y) = m.$$

Hence $Cay(D_{2p}, S)$ is the disjoint union of two circulant graphs $C_p(1, m)$ and so we have

$$Cay(D_{2p}, S) = 2C_p(1, m).$$

□

In the next theorem, we deal with the case that $S = \{x, y, z, w\}$, where $x^2 = y^2 = z^2 = w^2 = e$.

Theorem 2.5. *Let $D_{2p} =< a, b \mid a^p = b^2 = e, bab = a^{-1} >$ be a dihedral group, p is an odd prime number and $S = \{a^i b, a^j b, a^s b, a^t b\} \subseteq D_{2p}$ such that i, j, s and t are distinct and $0 \leqslant i, j, s, t \leqslant p-1$. Let $i + j = s + t$ (mod p) and k be the smallest positive integer in which $k(i-j) + i = s$ or t (mod p). Then we have*

$$Cay(D_{2p}, S) = C_{2p}(1, 2k + 1).$$

Proof. We may consider D_{2p} as the following presentation:

$$D_{2p} =< a^{j-i}, a^i b \mid (a^{j-i})^p = (a^i b)^2 = e, a^i b(a^{j-i})a^i b = (a^{j-i})^{-1} > .$$

We have the following outer cycle of length $2p$

$$e \sim a^i b \sim a^{j-i} \sim a^i b(a^{j-i}) \sim (a^{j-i})^2 \sim a^i b(a^{j-i})^2 \sim (a^{j-i})^3 \sim a^i b(a^{j-i})^3 \sim$$
$$(a^{j-i})^4 \sim a^i b(a^{j-i})^4 \sim \cdots \sim (a^{j-i})^{p-1} \sim a^i b(a^{j-i})^{p-1} \sim e. \quad (1)$$

Notice that $(a^{j-i})^r(a^i b(a^{j-i})^r)^{-1} = (a^{j-i})^r(a^{j-i})^{-r}a^i b = a^i b \in S$ and $a^i b(a^{j-i})^r(a^i b(a^{j-i})^{r+1})^{-1} = a^i b a^{j-i} = a^j b \in S$, where $0 \leq r \leq p-1$. Each of the element of order 2 in D_{2p} is as the form $a^i b(a^{j-i})^r$ such that $0 \leq r \leq p-1$. Since the set S contains 4 elelments of order 2 and $i + j = s + t \ (mod\ p)$, so S can be written as the following:

$$S = \{a^i b, a^i b(a^{j-i})^{-1}, a^i b(a^{j-i})^k, a^i b(a^{j-i})^{-(1+k)}\}.$$

Now, since

$$a^i b(a^{j-i})^k(a^{j-i})^{(r-1)2k+1}((a^{j-i})^{r(2k+1)})^{-1} = a^i b(a^{j-i})^k(a^{j-i})^{-(2k+1)}$$
$$= a^i b(a^{j-i})^{-(1+k)} \in S$$

and $(a^{j-i})^{r(2k+1)}(a^i b(a^{j-i})^k(a^{j-i})^{r(2k+1)})^{-1} = (a^{j-i})^{-k}a^i b = a^i b(a^{j-i})^k \in S$, we have the following inner cycle of length $2p$

$$e \sim a^i b(a^{j-i})^k \sim (a^{j-i})^{(2k+1)} \sim a^i b(a^{j-i})^k(a^{j-i})^{2k+1} \sim (a^{j-i})^{2(2k+1)} \sim$$
$$a^i b(a^{j-i})^k(a^{j-i})^{2(2k+1)} \sim (a^{j-i})^{3(2k+1)} \sim a^i b(a^{j-i})^k(a^{j-i})^{3(2k+1)} \sim \cdots \sim$$
$$(a^{j-i})^{(p-1)(2k+1)} \sim a^i b(a^{j-i})^k(a^{j-i})^{(p-1)(2k+1)} \sim e. \quad (2)$$

Suppose that $x = a^i b$ and $y = a^{j-i}$. Then we can rewrite outer and inner cycles of length $2p$ of (1) and (2) with these notations as the following:
outer cycle:

$$e \sim x \sim y \sim xy \sim y^2 \sim xy^2 \sim \cdots \sim y^{p-1} \sim xy^{p-1} \sim e$$

inner cycle:

$$e \sim xy^k \sim y^{2k+1} \sim xy^{3k+1} \sim y^{4k+2} \sim xy^{5k+2} \sim y^{6k+3} \sim$$
$$xy^{7k+3} \sim \cdots \sim y^{(2p-2)k+(p-1)} \sim xy^{(2p-1)k+(p-1)} \sim e.$$

If we consider the inner cycle, then we can see that:

$$d_G^O(e, xy^k) = 2k+1, \quad d_G^O(xy^k, y^{2k+1}) = 2k+1, \quad d_G^O(y^{2k+1}, xy^{3k+1}) = 2k+1,$$
$$d_G^O(xy^{3k+1}, y^{4k+2}) = 2k+1,\ d_G^O(y^{4k+2}, xy^{5k+2}) = 2k+1,\ d_G^O(xy^{5k+2}, y^{6k+3}) = 2k+1,$$
$$d_G^O(y^{6k+3}, y^{7k+3}) = 2k+1, \ldots, d_G^O(y^{(2p-2)k+(p-1)}, xy^{(2p-1)k+(p-1)}) = 2k+1,$$
$$d_G^O(xy^{(2p-1)k+(p-1)}, e) = 2k+1.$$

So, it implies that we have a circulant $C_{2p}(1, 2k+1)$ as required and the proof is completed. $\square$

Remark 2.6. *The condition $i+j = s+t \pmod p$ is necessary in Theorem 2.4, to have circulant graph. Because, we can see the following example which can not be displayed as circulant graph.*

Example 2.7. *Let D_{14} be the dihedral group of order 14. Suppose $S = \{b, a^2b, a^3b, a^4b\}$. We can see that the condition of Theorem 2.5 is not satisfied and, it is not a circulant graph (see Figure 4).*

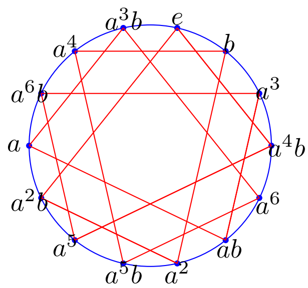

FIGURE 4. $Cay(D_{14}, S)$

3. CAYLEY GRAPH OF DIHEDRAL GROUPS OF ORDER $2p^2$

In this section, we are going to find the structure graph of valency 4, for dihedral group of order $2p^2$. Let D_{2p^2} be dihedral group of order $2p^2$, where p is prime. If $p = 2$, then we have dihedral group D_8 of order 8. In the following Theorem we can give the structure of $Cay(D_8, S)$, whenever $|S| = 4$.

Theorem 3.1. *Let D_8 be a dihedral group of order 8 with $S = \{x, y, z, w\}$such that $x^2 = y^2 = z^2 = w^2 = e$, then we have*

$$Cay(D_8, S) = C_8(1, 3).$$

Proof. It is clear that in this case S can be only presented as $S = \{b, ab, a^2b, a^3b\}$. We have the following outer and inner cycle of length 8, respectively.
outer cycle:

$$e \sim b \sim a^3 \sim ab \sim a^2 \sim a^2b \sim a \sim a^3b \sim e$$

inner cycle:

$$e \sim ab \sim a \sim b \sim a^2 \sim a^3b \sim a^3 \sim a^2b \sim e.$$

Moreover, we have

$$d_G^O(e, ab) = 3 \quad , d_G^O(ab, a) = 3, \quad d_G^O(a, b) = 3, \quad d_G^O(b, a^2) = 3,$$

$$d_G^O(a^2, a^3b) = 3, \quad d_G^O(a^3b, a^3) = 3, \quad d_G^O(a^3, a^2b) = 3, \quad d_G^O(a^2b, e) = 3.$$

Hence we will have a circulant graph $C_8(1, 3)$. $\square$

Now, if $S = \{x, x^{-1}, y, z\} \subset D_8$ such that $x \neq x^{-1}, y^2 = z^2 = e$, then each element of S is the form $S = \{a, a^3, y, z\}$. We may consider the following cases:

$$(i) zy = a \quad (ii) zy = a^2 \quad (iii) zy = a^3.$$

In the following theorem, we will discuss about the graph structure of $Cay(D_8, S)$ in the above cases.

Theorem 3.2. *$Cay(D_8, S)$ in cases (i) and (iii) is the circulant graph $C_8(1, 2)$.*

Proof. We have the following outer cycle:

$$e \sim y \sim zy \sim y(zy) \sim (zy)^2 \sim y(zy)^2 \sim (zy)^3 \sim y(zy)^3 \sim e.$$

In case (i), since $zy = a$ we have an outer cycle $e \sim y \sim a \sim ya \sim a^2 \sim ya^2 \sim a^3 \sim ya^3 \sim e$, and two inner cycles $e \sim a \sim a^2 \sim a^3 \sim e$ and $y \sim ya \sim ya^2 \sim ya^3 \sim y$. We can easily check that $Cay(D_8, S) = C_8(1, 2)$. For case (iii) we can follow the same method as above and again get circulant graph $C_8(1, 2)$. □

Theorem 3.3. *$Cay(D_8, S)$ in case (ii) is the circulant graph $C_8(1, 3)$.*

Proof. In case (ii), if $S = \{a, a^3, b, a^2b\}$, we have an outer cycle $e \sim a \sim ab \sim a^3 \sim a^2 \sim a^2b \sim a^3b \sim b \sim e$ and an inner cycle $e \sim a^3 \sim a^3b \sim a \sim a^2 \sim b \sim ab \sim a^2b \sim e$. We have jump of length 3, because

$$d_G^O(e, a^3) = 3, \quad d_G^O(a^3, a^3b) = 3, \quad d_G^O(a^3b, a) = 3, \quad d_G^O(a, a^2) = 3,$$
$$d_G^O(a^2, b) = 3, \quad d_G^O(b, ab) = 3, \quad d_G^O(ab, a^2b) = 3, \quad d_G^O(a^2b, e) = 3.$$

Hence we have $Cay(D_8, S) = C_8(1, 3)$. If $S = \{a, a^3, ab, a^3b\}$, we have an outer cycle $e \sim a \sim a^2b \sim a^3 \sim a^2 \sim a^3b \sim b \sim ab \sim e$ and an inner cycle:

$$e \sim a^3 \sim b \sim a \sim a^2 \sim ab \sim a^2b \sim a^3b \sim e$$

which produce a circulant graph with jump of length 3. Because

$$d_G^O(e, a^3) = 3, \quad d_G^O(a^3, b) = 3, \quad d_G^O(b, a) = 3, \quad d_G^O(a, a^2) = 3,$$
$$d_G^O(a^2, ab) = 3, \quad d_G^O(ab, a^2b) = 3, \quad d_G^O(a^2b, a^3b) = 3, \quad d_G^O(a^3b, e) = 3.$$

Thus we conclude that $Cay(D_8, S) = C_8(1, 3)$, in case (ii). □

Now, assume That p is an odd prime number. We have $D_{2p^2} =< a, b \mid a^{p^2} = b^2 = e, bab = a^{-1} >= \{e, a, a^2, \ldots, a^{p^2-1}, b, ab, a^2b, \ldots, a^{p^2-1}b\}$. Let S be a subset of D_{2p^2} of size 4. Then we have the following three possibilities:

(i) $S = \{x, x^{-1}, y, z\}, \quad x \neq x^{-1}, \quad y^2 = z^2 = e$
(ii) $S = \{x, y, z, w\}, \quad x^2 = y^2 = z^2 = w^2 = e$
(iii) $S = \{x, x^{-1}, y, y^{-1}\}, \quad x \neq x^{-1}, y \neq y^{-1}$.

It is clear that if $x, y, z \in D_{2p^2}$ such that $x \neq x^{-1}$ and $y^2 = z^2 = e$, then x can be only presented as a^i and y, z can be presented as a^jb and a^sb, respectively such that $1 \leqslant i \leqslant p^2 - 1$ and $0 \leqslant j \neq s \leqslant p^2 - 1$. If $x = a^i, y = a^jb$ and $z = a^sb$, then we can see that $zy = a^sba^jb = a^{s-j}$ and therefor zy can be presented as x^r for some $1 \leqslant r \leqslant p^2 - 1$. Now, the following cases, will arise.

Case 1: $o(x) = p^2, o(zy) = p$.

In this case, we should have $r = tp$, where $1 \leqslant t \leqslant p-1$.

Case 2: $o(x) = p^2, o(zy) = p^2$.
Case 3: $o(x) = p, o(zy) = p$.
Case 4: $o(x) = p, o(zy) = p^2$.

In cases 2, 3 and 4, r can not be a multiplication of p. Because, if for example $r = tp, o(x) = o(zy) = p$, then we have $p = o(zy) = o(x^r) = \dfrac{o(x)}{(r, o(x))} = \dfrac{p}{(tp, p)} = 1$ which is a contradiction. So, $r \neq p, 2p, \ldots, (p-1)p$. Now we can start investigation of the structure of $Cay(D_{2p^2}, S)$, when $S = \{x, x^{-1}, y, z\}$, $x \neq x^{-1}, y^2 = z^2 = e$ according to the above four cases. In the following theorem, we consider case 1.

Theorem 3.4. *$Cay(D_{2p^2}, S)$ in case 1 is a graph consisting p disjoint cycles of length $2p$ and a Hamiltonian cycle in which every two adjacent vertices are belong to distinct cycles of length $2p$.*

Proof. Suppose that $D_{2p^2} =< a, b \mid a^{p^2} = b^2 = e, bab = a^{-1} >= \{e, a, a^2, \ldots, a^{p^2-1}, b, ab, a^2b, \ldots, a^{p^2-1}b\}$ and $S = \{x, x^{-1}, y, z\} \subset D_{2p^2}$. Since $x \neq x^{-1}$ we have $o(x) = m > 2$. Let $o(x) = p^2$. In this case, since $zy = x^p$ we have the following outer cycle of length $2p$.

$$e \sim y \sim zy \sim y(zy) \sim (zy)^2 \sim y(zy)^2 \sim \cdots \sim (zy)^{p-1} \sim y(zy)^{p-1} \sim e.$$

So there is a dihedral subgroup K such that

$$K =< y, x^p >= \{e, y, x^p, yx^p, x^{2p}, yx^{2p}, \ldots, x^{(p-1)p}, yx^{(p-1)p}\}.$$

We have $[D_{2p^2} : K] = \dfrac{2p^2}{2p} = p$. Thus there are p distinct right cosets

$$K, \quad Kx, \quad Kx^2, \ldots, \quad Kx^{p-1}.$$

For each right coset Kx^t, we have a cycle

$$x^t \sim yx^t \sim x^{p+t} \sim yx^{p+t} \sim \ldots \sim x^{(p-1)p+t} \sim yx^{(p-1)p+t}$$

for $t = 1, 2, \ldots, p-1$. Therefore, we have p cycles of length $2p$. Since $x \sim x^2, x^2 \sim x^3, x^3 \sim x^4, \ldots, x^{p-2} \sim x^{p-1}, x^{p-1} \sim x^p$. So, $K \cup Kx \cup Kx^2 \cup Kx^3 \cup \ldots \cup Kx^{p-1} = D_{2p^2}$ produces a graph consisting p disjoint cycles of length $2p$ and a Hamiltonian cycle in which every two adjacent vertices are belong to distinct cycles of length $2p$, as required. $\square$

Theorem 3.5. *$Cay(D_{2p^2}, S)$ in cases $o(x) = p^2, o(zy) = p^2$ and $o(x) = p, o(zy) = p^2$ is a circulant graph $C_{2p^2}(1, 2k)$, where k is the smallest positive integer such that $x = (zy)^k$.*

Proof. Suppose that $D_{2p^2} =< a, b \mid a^{p^2} = b^2 = e, bab = a^{-1} >= \{e, a, a^2, \ldots, a^{p^2-1}, b, ab, a^2b, \ldots, a^{p^2-1}b\}$ and $S = \{x, x^{-1}, y, z\} \subset D_{2p^2}$. Then in case 2 we can follow the same method as in Theorem 2.2 and again we get

$$Cay(D_{2p^2}, S) = C_{2p^2}(1, 2k).$$

Now, we consider the case $o(x) = p, o(zy) = p^2$. Since $o(zy) = p^2$ and $o(x) = p$, we have the following outer cycle of length $2p^2$

$$e \sim y \sim zy \sim y(zy) \sim (zy)^2 \sim y(zy)^2 \sim (zy)^3 \sim y(zy)^3 \sim \ldots \sim y(zy)^{p^2-2}$$
$$\sim (zy)^{p^2-1} \sim y(zy)^{p^2-1} \sim (zy)^{p^2} = e,$$

because $[y(zy)^t][(zy)^{t+1}]^{-1} = y(zy)^t(zy)^{-t-1} = y(zy)^{-1} = yy^{-1}z^{-1} = z \in S$ and $(zy)^t[y(zy)^t]^{-1} = (zy)^t(zy)^{-t}y^{-1} = y^{-1} = y \in S$, for $t = 0, 1, \ldots, p^2 - 1$. Assume that $H =< x >$ is a cyclic subgroup of order p, then $H = \{e, x, x^2, \ldots, x^{p-1}\}$ consists a cycle of length p as the following

$$e \sim x \sim x^2 \sim x^3 \sim \cdots \sim x^{p-1} \sim e.$$

We have $[D_{2p^2} : H] = \dfrac{2p^2}{p} = 2p$. Thus there are $2p$ distinct right cosets $Hg_1, Hg_2, \ldots, Hg_{2p}$ such that $g_1 = e \in H$ and $g_2, g_3, \ldots, g_{2p} \notin H$. For each right coset $Hg_j = \{g_j, xg_j, x^2g_j, \ldots, x^{p-1}g_j\}$, we have an inner cycle $g_j \sim xg_j \sim x^2g_j \sim \cdots \sim x^{p-1}g_j \sim x^pg_j = g_j$, for $j = 1, 2, \ldots, 2p$. Therefore, there are $2p$ inner cycles of length p and since $(zy)^k = x$, we have

$$d_G^O(e, x) = 2k, \quad d_G^O(x, x^2) = 2k, \quad d_G^O(x^2, x^3) = 2k,$$
$$d_G^O(x^3, x^4) = 2k, \ldots, \quad d_G^O(x^{p-1}, e) = 2k$$

and similarly

$$d_G^O(g_j, xg_j) = 2k, \quad d_G^O(xg_j, x^2g_j) = 2k, \quad d_G^O(x^2g_j, x^3g_j) = 2k,$$
$$d_G^O(x^3g_j, x^4g_j) = 2k, \ldots, d_G^O(x^{p-1}g_j, g_j) = 2k.$$

Hence we have

$$Cay(D_{2p^2}, S) = C_{2p^2}(1, 2k).$$

□

Theorem 3.6. *$Cay(D_{2p^2}, S)$ in case $o(x) = p, o(zy) = p$ is the union of p circulant graph $C_{2p^2}(1, 2k)$, where k is the smallest positive integer such that $x = (zy)^k$.*

Proof. Assume that $H =< x >$ is a cyclic subgroup of order p. Then $H = \{e, x, x^2, \ldots, x^{p-1}\}$ consist of cycle graph of length p as the following

$$e \sim x \sim x^2 \sim x^3 \sim \cdots \sim x^{p-1} \sim e.$$

We have $[D_{2p^2} : H] = \dfrac{2p^2}{p} = 2p$. Thus, there are $2p$ distinct right cosets and so we have $2p$ inner cycles of length p as the following:

$$g_j \sim g_j(zy)^k \sim g_j(zy)^{2k} \sim \cdots \sim g_j(zy)^{p-1} \sim g_j. \quad j = 1, 2, \ldots, p.$$

Since $o(zy) = p$, we have a dihedral subgroup of order $2p$ as the following

$$D_{2p} =< y, zy >= \{e, y, zy, y(zy), (zy)^2, y(zy)^2, \ldots, (zy)^{p-1}, y(zy)^{p-1}\}.$$

Thus we have the following outer cycle of length $2p$

$$e \sim y \sim zy \sim y(zy) \sim (zy)^2 \sim y(zy)^2 \sim \cdots \sim (zy)^{p-1} \sim y(zy)^{p-1} \sim e,$$

because

$$[y(zy)^t][(zy)^{t+1}]^{-1} = y(zy)^k(zy)^{-k-1} = y(zy)^{-1} = yy^{-1}z^{-1} = z \in S$$

and

$$(zy)^k[y(zy)^k]^{-1} = (zy)^k(zy)^{-k}y^{-1} = y^{-1} = y \in S.$$

On other hand, $[D_{2p^2} : D_{2p}] = \dfrac{2p^2}{2p} = p$, which implies that $D_{2p^2} = D_{2p} \cup g_2 D_{2p} \cup g_3 D_{2p} \cup \ldots \cup g_p D_{2p}$, where each $g_j D_{2p}$ consist of a cycle of length $2p$ as the following for each $j = 1, 2, \ldots, p$

$$g_j \sim g_j y \sim g_j(zy) \sim g_j y(zy) \sim g_j(zy)^2 \sim g_j y(zy)^2 \sim$$
$$\ldots \sim g_j y(zy)^{p-1} \sim g_j(zy)^p = g_j.$$

We also have $\dfrac{[D_{2p^2} : H]}{[D_{2p^2} : D_{2p}]} = \dfrac{2p}{p} = 2$ which implies that there are two inner cycles in every outer cycle. We have a jump of length $2k$, because

$$d^O_G(g_j, g_j(zy)^k) = 2k, \quad d^O_G(g_j(zy)^k, g_j(zy)^{2k}) = 2k,$$
$$d^O_G(g_j(zy)^{2k}, g_j(zy)^{3k}) = 2k, \ldots, d^O_G(g_j(zy)^{p-1}, g_j) = 2k.$$

Hence, we have

$$Cay(D_{2p^2}, S) = p\ C_{2p}(1, 2k).$$

□

If $S = \{x, y, z, w\}$ such that $x^2 = y^2 = z^2 = w^2 = e$, then each of element in S has the form $a^i b$, for some $0 \leq i \leq p^2 - 1$. Suppose that $x = a^i b$, $y = a^j b$, $z = a^s b$, $w = a^t b$. Let $i + j = s + t$ $(mod\ p)$ and k be the smallest positive integer in which $k(i - j) + i = s$ or t $(mod\ p)$. Then we may consider the following possibilities:

(i) $o(yx) \neq p,\ o(wz) \neq p$
(ii) $o(yx) \neq p,\ o(wz) = p$
(iii) $o(yx) = o(wz) = p$.

Theorem 3.7. *$Cay(D_{2p^2}, S)$ in cases $o(yx) \neq p,\ o(wz) \neq p$ and $o(yx) \neq p,\ o(wz) = p$ is circulant graph $C_{2p^2}(1, 2k + 1)$, where k is the smallest positive integer such that $k(i - j) + i = s$ or t $(mod\ p)$ and $i + j = s + t$ $(mod\ p)$.*

Proof. The case $o(yx) \neq p,\ o(wz) \neq p$ is similar to Theorem 2.5 and we omit the proof. In case $o(yx) \neq p,\ o(wz) = p$ since $o(yx) \neq p$, we have $o(a^{j-i}) = p^2$. Suppose that $x = a^i b$ and $y = a^{j-i}$. Then by Theorem 2.5, we have outer cycle of length $2p^2$:

$$e \sim x \sim y \sim xy \sim y^2 \sim xy^2 \sim \cdots \sim y^{p^2-1} \sim xy^{p^2-1} \sim e.$$

We can rewrite the set S with the above notations as the following:

$$S = \{x, xy^{p^2-1}, xy^k, xy^{(2p^2-1)k+(p^2-1)}\},$$

such that k is the smallest positive integer such that $xy^k \in S$. Now, since $o(a^{t-s}) = p$, we have inner cycle of length $2p$ as the following:

$$e \sim xy^k \sim y^{2k+1} \sim xy^{3k+1} \sim y^{4k+2} \sim xy^{5k+2} \sim y^{6k+3} \sim xy^{7k+3} \sim$$

$$\sim \cdots \sim y^{(2p-2)k+(p-1)} \sim xy^{(2p-1)k+(p-1)} \sim e.$$

We define subgroup D_{2p} of D_{2p^2} as follows:

$$D_{2p} =< y^{2k+1}, xy^k | (y^{2k+1})^p = (xy^k)^2 = e, xy^k(y^{2k+1})xy^k = (y^{2k+1})^{-1} >=$$

$$\{e, y^{2k+1}, y^{4k+2}, y^{6k+3}, \ldots, y^{(2p-2)k+(p-1)}, xy^k, xy^{3k+1}, xy^{5k+2}, \ldots, xy^{(2p-1)k+(p-1)}\}.$$

We have $[D_{2p^2} : D_{2p}] = \dfrac{2p^2}{2p} = p$. Thus there are p distinct right cosets which imply that there are p inner cycles of length $2p$ as follows:
for each $j = 1, 2, \ldots, 2p$,

$$g_j \sim g_jxy^k \sim g_jy^{2k+1} \sim g_jxy^{3k+1} \sim g_jy^{4k+2} \sim g_jxy^{5k+2} \sim$$
$$\sim \cdots \sim g_jy^{(2p-2)k+(p-1)} \sim g_jxy^{(2p-1)k+(p-1)} \sim g_j.$$

Moreover, for each $j = 1, 2, \ldots, 2p$, we have

$$d_G^O(g_j, g_jxy^k) = 2k+1, \quad d_G^O(g_jxy^k, g_jy^{2k+1}) = 2k+1,$$

$$d_G^O(g_jy^{2k+1}, g_jxy^{3k+1}) = 2k+1, \ldots, \quad d_G^O(g_jxy^{(2p-1)k+(p-1)}, g_j) = 2k+1.$$

Hence, we conclude that $Cay(D_{2p^2}, S) = C_{2p^2}(1, 2k+1)$. □

Theorem 3.8. *$Cay(D_{2p^2}, S)$ in case (iii) is the union of p circulant graph $C_{2p}(1, 2k+1)$, where k is the smallest positive integer such that $k(i-j)+i = s$ or t (mod p) and $i+j = s+t$ (mod p).*

Proof. Since we have $o(a^{j-i}) = p$, similar to Theorem 2.5, we suppose that $x = a^ib$ and $y = a^{j-i}$ Thus we have an outer cycle as the following:

$$e \sim x \sim y \sim xy \sim y^2 \sim xy^2 \sim \cdots \sim y^{p-1} \sim xy^{p-1} \sim e.$$

Since $o(a^{t-s}) = p$, we have inner cycle

$$e \sim xy^k \sim y^{2k+1} \sim xy^{3k+1} \sim y^{4k+2} \sim xy^{5k+2} \sim y^{6k+3} \sim xy^{7k+3} \sim$$
$$\sim \cdots \sim y^{(2p-2)k+(p-1)} \sim xy^{(2p-1)k+(p-1)} \sim e.$$

Note that $[D_{2p^2} : D_{2p}] = \dfrac{2p^2}{2p} = p$. Hence by Theorem 3.7, we have p outer cycles and p inner cycles as following, respectively.

$$g_j \sim g_jx \sim g_jy \sim g_jxy \sim g_jy^2 \sim g_jxy^2 \sim \ldots \sim g_jy^{(p-1)} \sim g_jxy^{(p-1)} \sim g_j,$$

for $j = 1, 2, \ldots, p$,
and

$$g_j \sim g_jxy^k \sim g_jy^{2k+1} \sim g_jxy^{3k+1} \sim g_jy^{4k+2} \sim g_jxy^{5k+2} \sim$$
$$\sim \cdots \sim g_jy^{(2p-2)k+(p-1)} \sim g_jxy^{(2p-1)k+(p-1)} \sim g_j,$$

for $j = 1, 2, \ldots, p$,

such that every outer cycle consists of one inner cycle. If we consider all inner cycles we have for $j = 1, 2, \ldots, p$,

$$d_G^O(g_j, g_jxy^k) = 2k+1,\ d_G^O(g_jxy^k, g_jy^{2k+1}) = 2k+1,$$
$$d_G^O(g_jy^{2k+1}, g_jxy^{3k+1}) = 2k+1, \ldots, d_G^O(g_jxy^{(2p-1)k+(p-1)}, g_j) = 2k+1.$$

Hence, we have $Cay(D_{2p^2}, S) = p\ C_{2p}(1, 2k+1)$. □

Now, assume that $S = \{u, u^{-1}, v, v^{-1}\}$ such that $u \neq u^{-1}, v \neq v^{-1}$ then we may consider the following possibillities:

(i) $o(u) \neq p,\ o(v) \neq p$
(ii) $o(u) \neq p,\ o(v) = p$
(iii) $o(u) = o(v) = p$.

Theorem 3.9. *$Cay(D_{2p^2}, S)$ in cases $o(u) \neq p,\ o(v) \neq p$ and $o(u) \neq p,\ o(v) = p$ is the union of two circulant graph $C_{p^2}(1, m)$, where m is the smallest positive integer such that $u^m = v$ for some $2 \leq m \leq p^2 - 1$.*

Proof. Case $o(u) \neq p,\ o(v) \neq p$ is similar to Theorem 2.4 and we omit the proof. Suppose that $D_{2p^2} = \{e, x, x^2, \ldots, x^{p^2-1}, y, yx, yx^2, \ldots, yx^{p^2-1}\}$, and k is the smallest positive integer such that $u = x^k$. In case (ii) since $o(u) \neq p$ we should have $o(u) = p^2$. Let $H =< u >=< x^k >$, we have $[D_{2p^2} : H] = \dfrac{2p^2}{p^2} = 2$. Thus we have two outer cycle of length p^2 as following:

$$e \sim x^k \sim x^{2k} \sim x^{3k} \sim \cdots \sim x^{(p^2-1)k} \sim e,$$
$$y \sim yx^k \sim yx^{2k} \sim yx^{3k} \sim \cdots \sim yx^{(p^2-1)k} \sim y.$$

Since $o(v) = p$ we may suppose that $L =< v >=< x^{mk} >$ then we have $[D_{2p^2} : L] = \dfrac{2p^2}{p} = 2p$ and $[H : L] = \dfrac{p^2}{p} = p$. Hence, we have $2p$ inner cycles of length p as the following: $g_j \sim g_jx^{mk} \sim g_jx^{2mk} \sim \cdots \sim g_jx^{(p-1)mk} \sim g_j$, for $j = 1, 2, \ldots, p$. and $yg_j \sim yg_jx^{mk} \sim yg_jx^{2mk} \sim \cdots \sim yg_jx^{(p-1)mk} \sim yg_j$, for $j = 1, 2, \ldots, p$. every outer cycle conside p inner cycles, with jump m, because

$$d_G^O(g_j, g_jx^{mk}) = m, d_G^O(g_jx^{mk}, g_jx^{2mk}) = m, d_G^O(g_jx^{2mk}, g_jx^{3mk}) = m,$$
$$d_G^O(g_jx^{3mk}, g_jx^{4mk}) = m, \ldots, d_G^O(g_jx^{(p-1)mk}, g_j) = m$$

for $j = 1, 2, \ldots, p$
and in another outer cycle we also have

$$d_G^O(yg_j, yg_jx^{mk}) = m, d_G^O(yg_jx^{mk}, yg_jx^{2mk}) = m, d_G^O(yg_jx^{2mk}, yg_jx^{3mk}) = m,$$
$$d_G^O(yg_jx^{3mk}, yg_jx^{4mk}) = m, \ldots, d_G^O(yg_jx^{(p-1)mk}, yg_j) = m.$$

Hence we have $Cay(D_{2p^2}, S) = 2\ C_{p^2}(1, m)$. □

Theorem 3.10. *$Cay(D_{2p^2}, S)$ in case $o(u) = o(v) = p$. is the union of $2p$ circulant graph $C_p(1, m)$, where m be the smallest positive integer number such that $u^m = v$ for some $2 \leq m \leq p^2 - 1$.*

Proof. Suppose that $D_{2p^2} = \{e, x, x^2, \ldots, x^{p^2-1}, y, yx, yx^2, \ldots, yx^{p^2-1}\}$, and k be the smallest positive integer such that $u = x^k$. Since $o(u) = p$, we can let $v = x^{mk}$ such that $2 \leq m \leq p-1$. Similar to Theorem 2.4 we have outer cycle of length p as following:

$$e \sim x^k \sim x^{2k} \sim x^{3k} \sim \cdots \sim x^{(p-1)k} \sim e.$$

Since $(x^{ik})(x^{(i+1)k})^{-1} = x^{-k} \in S$ for $1 \leq i \leq p-1$. Suppose that $H =< x^k >$ then we have $[D_{2p^2} : H] = \dfrac{2p^2}{p} = 2p$. Thus there are $2p$ distinct right cosets $Hg_1 = H, Hg_2, \ldots, Hg_{2p}$, such that $g_1 = e \in H$ and $g_1, g_2, \ldots, g_{2p} \notin H$. It tends out for each right coset $Hg_j = \{g_j, x^k g_j, x^{2k} g_j, \ldots, x^{(p-1)k} g_j\}$, we have outer cycle $g_j \sim x^k g_j \sim x^{2k} g_j, \ldots, \sim x^{(p-1)k} g_j \sim g_j$ for $j = 1, \ldots, 2p$. Therefor we have $2p$ outer cycle of length p. Let $L =< x^{mk} >$. Since $[D_{2p^2} : L] = \dfrac{2p^2}{p} = 2p$, the number of inner cycles is $2p$ that are located in every outer cycle. Actually for every outer cycle we have an inner cycle of length p as following:

$$g_j \sim x^{mk} g_j \sim x^{2mk} g_j \sim \cdots \sim x^{(p-1)mk} g_j \sim g_j, \ forj = 1, 2, \ldots, 2p.$$

We conclude that every outer cycle has jump of length m, because we have for $j = 1, 2, \ldots, 2p$.

$$d_G^O(g_j, x^{mk} g_j) = m, d_G^O(x^{mk} g_j, x^{2mk} g_j) = m, d_G^O(x^{2mk} g_j, x^{3mk} g_j) = m,$$

$$d_G^O(x^{3mk} g_j, x^{4mk} g_j) = m, \ldots, d_G^O(x^{(p-1)mk} g_j, g_j) = m.$$

Since the degree of each vertex is 4, we conclude that each vertex is connected to two adjacent vertices and to two vertices with a distance of m. Hence we have $Cay(D_{2p^2}, S) = 2p\ C_p(1, m)$ as required. □

We end the paper by giving an open problem which can be improved the paper.

Open Problem. What is the structure of $Cay(D_{2n}, S)$ of valency 4 for every $n \geq 3$ (not necessary prime power)? For what values of n the graph is circulant?

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