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HUB PARAMETERS OF HYPERGRAPH

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Abstract. A hypergraph is an extension of a graph in which one edge can connect any number of vertices. In contrary, an edge connects exactly two vertices in a graph.In this paper we introduce hub-hyperpath, hubset and hubnumber of a hypergraph. Also we defined the hubnumber of a different types of hypergraphs and analyze some of its properties. Then the hub number of a hypergraph is compared with its dual hypergraph. Hubset can be useful for various tasks, such as targeted marketing and social networks. Additionally, finding the hub number of a graph is useful in network security, as it helps in identifying nodes that, if compromised, could have a significant impact on the overall network.

Key words and Phrases: hypergraph, hub-hyperpath, hubset, hubnumber.

1. INTRODUCTION

The hub number is a measure of how well-connected a hypergraph is, and it is used in network analysis and design to determine the robustness and vulnerability of a network. A higher hub number indicates a more robust and less vulnerable network, as there are more hubs to maintain the connectivity of the network even in the event of vertex failures. The hub number of a hypergraph is an important concept in network analysis and is used to measure the robustness of a network. Networks with a high hub number are more robust to vertex failures, as the removal of a small number of vertices does not significantly affect the connectivity of the network. The hub number of a hypergraph can be computed using various algorithms, including brute-force search and heuristic algorithms.

At first, Berge[2] expanded the concept of an ordinary graph to a concept called as a hypergraph, which is the most essential term in graph theory. Then Walsh[3] introduced the concept of a graph's hub number. The very next level of

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the hypergraph theory was established by Bretto[1]. Khalaf, Mathad and Mahde[4] demonstrate graph hub and global hub numbers.

This study depends basically on hub parameters in hypergraph and its applications. Throughout, the hypergraph's hubnumber is denoted as $h_n(\mathbb{H})$. The basic definitions are discussed in Section 2. In Section 3, the definitions of hub-hyperpath, hub set and hubnumber for different types of hypergraphs along with some examples and results were discussed. Section 4 covers comparisons between a hypergraph's hub number and its dual. In Section 5, some realistic applications of a hypergraph's hubnumber are discussed.

2. NOTATIONS

$\mathbb{H}$	-	Hypergraph
$\mathscr{V}$	-	Vertices in $\mathbb{H}$
$\mathscr{E}$	-	Hyperedges in $\mathbb{H}$
$\mathbb{H}_{\mathbb{P}}$	-	Hub-hyperpath in $\mathbb{H}$
$\mathcal{S}$	-	hub set of $\mathbb{H}$
$h_n(\mathbb{H})$	-	Hub number of $\mathbb{H}$
$\mathbb{H}_{P(\mathscr{V})}$	-	Vertices in hub-hyperpath in $\mathbb{H}$
$\mathbb{H}_e$	-	Empty hypergraph
$d\mathbb{H}_r$	-	d-regular hypergraph
$\mathbb{H}_c$	-	Complete hypergraph
$\mathbb{H}^*$	-	Dual of $\mathbb{H}$

3. BASIC DEFINITIONS

In this section, we reviewed over some basic definitions associated to our main concept.

Definition 3.1. [3] A *hubset in a graph* G is a set S of vertices in G, such that any two vertices outside S are connected by a path whose all internal vertices lie in S and the *hubnumber of* G is the minimum cardinality of a hubset in G.

Definition 3.2. [1] An *undirected hypergraph* $\mathbb{H}$ is a pair $\mathbb{H} = (\mathscr{V}, \mathscr{E})$, where

- $\mathscr{V}$ is a set of elements called vertices
- $\mathscr{E}$ is a set of non-empty subsets of $\mathscr{V}$ called hyperedges

Therefore, $\mathscr{E}$ is a subset of $\rho(\mathscr{V})\backslash\{0\}$, where $\rho(\mathscr{V})$ is power set of $\mathscr{V}$.

Definition 3.3. [1] Let $\mathbb{H} = (\mathscr{V}, \mathscr{E})$ be a hypergraph without isolated vertex; a *hyperpath* $\mathbb{H}_{\mathbb{P}}$ in $\mathbb{H}$ from v_i to v_j, is a vertex-hyperedge alternative sequence:
$v_i = v_1 \quad e_1 \quad v_2 \quad e_2 \quad ... \quad v_n \quad e_n \quad v_{n+1} = v_j$ such that

- $v_i = v_1$, v_2,...,$v_n, v_{n+1} = v_j$ are distinct vertices with the possibility that $v_i = v_j$;
- e_1, e_2,...,e_n are distinct hyperedges.

Definition 3.4. [2] An empty hypergraph $\mathbb{H}_e$ is a hypergraph $\mathbb{H} = (\mathscr{V}, \mathscr{E})$ with $\mathscr{V} \neq \{\}$ and $\mathscr{E} = \phi$.

Definition 3.5. [2] A d-regular hypergraph $d\mathbb{H}_r$ is a hypergraph $\mathbb{H} = (\mathscr{V}, \mathscr{E})$ with degree(d) of each vertices are equal.

Definition 3.6. [1] A *complete hypergraph*$\mathbb{H}_c$ is a hypergraph with $\mathbb{H} = (\mathscr{V}, \mathscr{E} = (\rho(\mathscr{V})\backslash\phi))$

4. HUB PARAMETERS IN HYPERGRAPH

The importance of finding a hub number lies in its usefulness in understanding the structure and robustness of a network. Knowing the hub number of a hypergraph can help in identifying critical nodes in the network that are important for maintaining connectivity. This section establishes definitions for a hub hyperpath, hubset, and hubnumber of a hypergraph. Some of its characteristics are also discussed.

Definition 4.1. Let $\mathbb{H} = (\mathscr{V}, \mathscr{E})$ be a hypergraph and $\mathcal{S} \subseteq \mathscr{V}$ with two end vertices $v_i, v_j \in \mathscr{V}$. Then the *hub-hyperpath*$\mathbb{H}_{\mathbb{P}}$ in $\mathbb{H}$ from v_i to v_j is an alternative sequence of vertex and hyperedge say, $v_1 \quad e_1 \quad v_2 \quad e_2 \quad ... \quad v_n \quad e_n \quad v_{n+1}$, where $v_i = v_1$ & $v_j = v_{n+1}$ such that

- $v_i = v_1$, v_2,...,$v_n, v_{n+1} = v_j$ are distinct vertices
- e_1, e_2,...,e_n are distinct hyperedges with $v_i \in e_r$, $v_j \in e_s$; and $e_r \cap e_s \neq \phi$ with $r \neq s$
- the hub-hyperpath between v_i & v_j is a path with $\mathbb{H}_{P(\mathscr{V})}\backslash\{v_i, v_j\} \in \mathcal{S}$

Note: The degenerate case arises when,

(i) v_i and v_j belongs to same hyperedge and
(ii) two end vertices are same (i.e)$v_i = v_j$

In the above two cases hub-hyperpath does not exist.

Definition 4.2. A hypergraph $\mathbb{H} = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V}$ is a set of all vertices and $\mathscr{E}$ is set of non-empty subsets of $\mathscr{V}$ is called hyperedges. Then a *hubset in a hypergraph* is said to be a set $\mathcal{S} \subseteq \mathscr{V}$, there exist a hub-hyperpath $\mathbb{H}_P$ which connects the two end vertices $v_i, v_j \in \mathscr{V}$ such that $v_i, v_j \notin H_S$ and $\mathbb{H}_{P(\mathscr{V})}\backslash\{v_i, v_j\} \in \mathcal{S}$.

Definition 4.3. The *hubnumber of a hypergraph* $\mathbb{H}$, is denoted by $h(\mathbb{H})$ is the minimum cardinality of a hubset in $\mathbb{H}$.

Example 4.4. Consider a hypergraph$\mathbb{H}$ with vertex set $\mathscr{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ & edge set $\mathscr{E} = \{\{e_1|\{v_1, v_2, v_3\}\}, \{e_2|\{v_2, v_3\}\}, \{e_3|\{v_3, v_4, v_5\}\}, \{e_4|\{v_5, v_6\}\}\}$

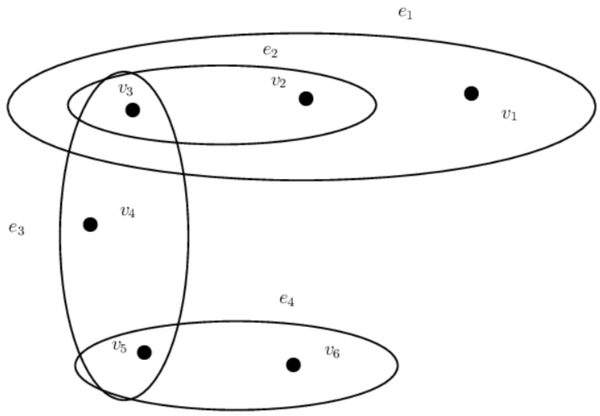

Fig 1: Hubnumber of a Hypergraph H

(i) possible hub-hyperpaths between v_1 and v_6 are:
($\mathbb{H}_{P1}$): v_1 e_1 v_3 e_3 v_5 e_4 v_6
($\mathbb{H}_{P2}$):v_1 e_1 v_2 e_2 v_3 e_3 v_5 e_4 v_6
(ii) possible $\mathcal{S}$'s in a hypergraph $\mathbb{H}$ are:
$\{v_3, v_5\}, \{v_2, v_3, v_5\}$
(iii) $\min \mathcal{S} = \{v_3, v_5\}$ and
(iv) hubnumber of a hypergraph $\mathbb{H}$, $h_n(\mathbb{H})$=2.

Theorem 4.5. *A hubset of an empty hypergraph is empty.*

Proof. To prove that the hubset of an empty hypergraph ($\mathbb{H}_e$) is empty, then need to show that there are no $\mathbb{H}_\mathbb{P}$ in the hypergraph $\mathbb{H}$. Recall that a $\mathbb{H}_\mathbb{P}$ is an alternative sequence of $\mathscr{V}$ and $\mathscr{E}$. In $\mathbb{H}_e$, there are no $\mathscr{E}$, so there cannot be any $\mathbb{H}_\mathbb{P}$.

Formally, let $\mathbb{H} = (\mathscr{V}, \mathscr{E})$ be $\mathbb{H}_e$, where $\mathscr{V}$ is the set of vertices and $\mathscr{E}$ is the set of hyperedges. Since $\mathbb{H}$ is empty, we have $\mathscr{E} = \phi$. Suppose for the sake of contradiction that there exists a hub-hyperpath $\mathbb{H}_\mathbb{P}$ in $\mathbb{H}$. Then, by definition of a hubset $\mathcal{S} \subseteq \mathscr{V}$ with $v_i, v_j \in \mathscr{V}$, $v_i, v_j \notin \mathcal{S}$ and $\mathbb{H}_{P(\mathscr{V})} \backslash \{v_i, v_j\} \in \mathcal{S}$. However, since $\mathbb{H}$ has no $\mathscr{E}$, there are no $\mathbb{H}_P$ for $\mathbb{H}$ and hence $\mathbb{H}$ cannot have $\mathcal{S}$. This contradicts our assumption that there is a $\mathbb{H}_\mathbb{P}$ in $\mathbb{H}$.

Therefore, there are no $\mathbb{H}_\mathbb{P}$ in an empty hypergraph and the $\mathcal{S}$ of an empty hypergraph is empty. □

Theorem 4.6. *Hubnumber of an empty hypergraph is* $h_n(\mathbb{H}) = 0$.

Proof. The hub number of an empty hypergraph $\mathbb{H}_e$ is the minimum cardinality of a hubset in $\mathbb{H}_e$. Now, let us consider an empty hypergraph $\mathbb{H}_e$. Consider $v_1, v_2 \in \mathbb{H}_e$ and it contains no hyperedges ie., $\mathscr{E} = \phi$, which means that it does not contain any vertices that can cover $\mathscr{E}$. Thus there is no $\mathbb{H}_\mathbb{P}$ between v_1 and v_2. Hence the $\mathcal{S}$ of an empty hypergraph is empty. Therefore, this shows that the hub number of an empty hypergraph is 0. This can be expressed mathematically as $h_n(\mathbb{H}_e) = 0$ for an empty hypergraph $\mathbb{H}_e$. □

Theorem 4.7. *Hypergraph* $\mathbb{H}$ *is empty if and only if hubnumber of* $\mathbb{H}_e$ *is always zero.*

Proof. Consider an empty hypergraph $\mathbb{H}_e$. Suppose $h_n(\mathbb{H}_e) \geq 1$ with two end vertices $v_1, v_2 \in \mathscr{V}$. Then $\exists$ atleast one $\mathbb{H}_\mathbb{P}$ which connects two end vertices $v_1, v_2 \in \mathbb{H}_e$ when $h_n(\mathbb{H}_e) \geq 1$. However, from the definition of an empty hypergraph $\mathbb{H}_e$ we know that, an $\mathbb{H}_e$ has no edges. Thus, there is no $\mathbb{H}_\mathbb{P}$ between v_1 and v_2. Hence, this contradicts our assumption. Therefore, if the hypergraph $\mathbb{H}$ is empty then $h_n(\mathbb{H}_e) = 0$.

Similarly, consider a hypergraph $\mathbb{H}$ with $h_n(\mathbb{H}) = 0$. It means that there is no $\mathcal{S}$ in that hypergraph. Therefore, the $\mathbb{H}_\mathbb{P}$ does not exist between two v_1 and v_2. In case of absence of $\mathscr{E}$'s in $\mathbb{H}$, there is no $\mathbb{H}_\mathbb{P}$. From this, we have $\mathbb{H}$ is an empty hypergraph $\mathbb{H}_e$. Hence proved. □

Theorem 4.8. *Hubnumber of a 2-regular hypergraph is zero.*

Proof. A 2-regular hypergraph is a hypergraph in which every vertices have degree 2. In other words, each vertex is incident with exactly two hyperedges. Now, let us suppose that $h_n(2\mathbb{H}_r) = k$. This means that every $\mathscr{E}$ in $\mathbb{H}$ can be covered by at most k vertices. However, since every hyperedge in a $2\mathbb{H}_r$ contains exactly two $\mathscr{V}$'s, therefore we can cover every hyperedge with at most two $\mathscr{V}$'s.

Thus, here $k \geq 2$, since each hyperedge can be covered with at most two $\mathscr{V}$. However, since each vertex in a $2\mathbb{H}_r$ is contained in exactly 2 hyperedges, it follows that every vertex covers at most 2 hyperedges. Therefore, $h_n(2\mathbb{H}_r)$ cannot be greater than 2.

Combining these two results, we have that $h_n(2\mathbb{H}_r)$ is either 0, 1 or 2. But since every $\mathscr{V}$ in a $2\mathbb{H}_r$ is contained in exactly 2 $\mathscr{E}$'s, it follows that cover all $\mathscr{E}$ of $\mathbb{H}$ with just one $\mathscr{V}$ and therefore $k = 1$. Thus, if $\mathbb{H}$ is 2-regular, then its hub number is 0 or 1, but since it cannot be 1, we conclude that $h_n(2\mathbb{H}_r)$=0. □

Theorem 4.9. $\mathbb{H}$ *is a complete hypergraph* $\mathbb{H}_c$ *if and only if* $h_n(\mathbb{H}_c) = 0$.

Proof. **Necessary part:** If $\mathbb{H}$ is a complete hypergraph $\mathbb{H}_c$, then $h_n(\mathbb{H}_c) = 0$.

A complete hypergraph $\mathbb{H}_c$ is a hypergraph in which every pair of distinct vertices is contained in a $\mathscr{E}$. In other words, every possible $\mathscr{E}$ is present in the $\mathbb{H}$. Now, suppose that $\mathbb{H}$ is a $\mathbb{H}_c$. Then every $\mathscr{E}$ in $\mathbb{H}$ contains every vertex in $\mathbb{H}$, since every possible $\mathscr{E}$ is present. Thus, to cover each $\mathscr{E}$, then simply choose all the vertices in $\mathbb{H}$. Since every $\mathscr{E}$ is connected with all vertices, then $h_n(\mathbb{H}_c) = 0$.

Sufficient part: If $h_n(\mathbb{H}) = 0$, then $\mathbb{H}$ is a complete hypergraph.

Suppose that $h_n(\mathbb{H}) = 0$. This means that there are no intermediate vertices in hub-hyperpath. In other words, every hyperedge of $\mathbb{H}$ must contain every vertex in $\mathbb{H}$, since there are no intermediate vertices in $\mathbb{H}_{\mathbb{P}}$. To see why this is true, suppose that there exists a $\mathscr{E}$ in $\mathbb{H}$ that does not contain some vertex v_i in $\mathbb{H}$. Then, there is a $\mathbb{H}_{\mathbb{P}}$ in $\mathbb{H}$. But this contradicts the fact that $h_n(\mathbb{H}) = 0$, since we have no intermediate vertices in $\mathbb{H}_{\mathbb{P}}$. Therefore, every $\mathscr{E}$ of $\mathbb{H}$ must contain every vertex in $\mathbb{H}$. Thus, it has been proved that if $h_n(\mathbb{H}) = 0$, then every hyperedge of $\mathbb{H}$ contains every vertex in $\mathbb{H}$, which means that $\mathbb{H}$ is a complete hypergraph. Hence $\mathbb{H}$ is a complete hypergraph($\mathbb{H}_c$) if and only if $h_n(\mathbb{H}) = 0$. □

Theorem 4.10. *Let $\mathcal{S}$ be a subset of $\mathscr{V}$ in $\mathbb{H}$. Then $\mathbb{H}\backslash\mathcal{S}$ is complete if and only if $\mathcal{S}$ is a hubset of $\mathbb{H}$*

Proof. Consider $\mathcal{S}$ be a hubset of $\mathbb{H}$. Let $v_i, v_j \in \mathscr{V}\backslash\mathcal{S}$, and consider that they are not the same. If they are adjacent, $\mathbb{H}_{\mathbb{P}}$ in $\mathbb{H}$ between v_i and v_j with all of the intermediate vertices in $\mathcal{S}$ must exist. As a result of this $\mathbb{H}_{\mathbb{P}}$'s contraction to a $\mathscr{E}$, v_i and v_j are adjacent in $\mathbb{H}\backslash\mathcal{S}$. They remains adjacent in the contraction if they were already adjacent.

Assuming $\mathbb{H}\backslash\mathcal{S}$ is a complete graph, and take $v_i, v_j \in \mathscr{V}(\mathbb{H}\backslash\mathcal{S})$. Then, in the initial graph $\mathbb{H}$, v_i and $v_j \in \mathscr{V}$.If they are neither adjacent nor identical; then, given that they are adjacent in $(\mathbb{H}\backslash\mathcal{S})$, there must be a hub-hyperpath connecting them in $\mathbb{H}$ which was contracted into $\mathscr{E}$. However, such a hub-hyperpath will only go through $\mathscr{V}$ from $\mathcal{S}$. From we choose v_i and v_j at arbitrary, $\mathcal{S}$ must have a hub set of $\mathbb{H}$. □

Corollary 4.11. *Let $\mathbb{H}$ be a hypergraph with hub set $\mathcal{S}$, and let $v_i \in \mathcal{S}$. Then $\mathcal{S}\backslash v_i$ is a hub set in $\mathbb{H}\backslash v_i$.*

5. HUB NUMBER OF HYPERGRAPH VS ITS DUAL

Definition 5.1. The *hubset of a dual hypergraph* $\mathbb{H}^* = (\mathscr{V}^* = (e_1, e_2, ..., e_n), \mathscr{E}^* = (v_1, v_2, ...v_n)$ is said to be a set $\mathcal{S} \subseteq \mathscr{V}^*$ in a hypergraph $\mathbb{H} = (\mathscr{V}, \mathscr{E})$ where $\mathbb{H}_{\mathscr{V}_i} = \{e_j/v_i \in \mathbb{H}_{\mathscr{E}_j}$ in $\mathbb{H}$ and a hub-hyperpath $\mathbb{H}_P$ connecting the two end vertices $v_i, v_j \in \mathscr{V}^*$ in $\mathbb{H}$ where $v_i, v_j \notin \mathcal{S}$ & $\mathbb{H}_{P(\mathscr{V})}\backslash\{v_i, v_j\} \in \mathcal{S}$.

Definition 5.2. The *hubnumber of a dual hypergraph* $\mathbb{H}^*$, is denoted by $h_n(\mathbb{H}^*)$ is the minimum cardinality of a hubset in $\mathbb{H}^*$.

Example 5.3. It is obvious that the incidence matrix of $\mathbb{H}^*$ is the transpose of the incidence matrix of $\mathbb{H}$ and so, that $(\mathbb{H}^*)^* = \mathbb{H}$. Consider the following hypergraph $\mathbb{H}$.

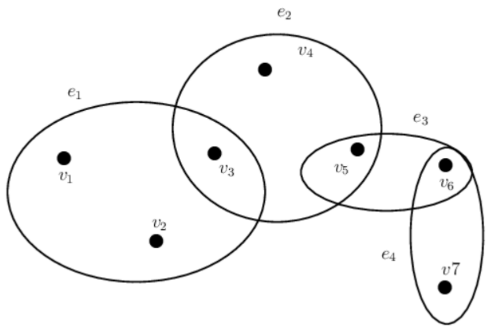

Fig 2: Hypergraph(H)

Then, incidence matrix of $\mathbb{H}$ is, $\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$

Hub-hyperpath of $\mathbb{H}$ is $v_1 \quad e_1 \quad v_3 \quad e_2 \quad v_5 \quad e_3 \quad v_6 \quad e_4 \quad v_7$. Then hub number of this graph $h_n(\mathbb{H}) = 3$.

Dual of this graph drawn from the transpose of the incidence matrix which is,

$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

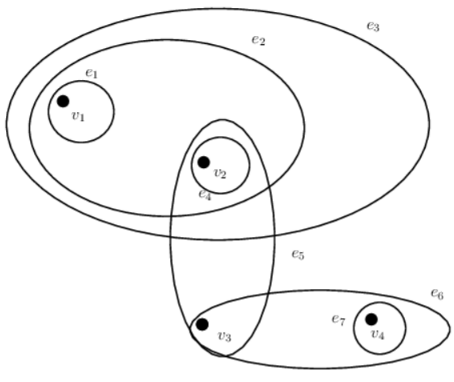

Fig 3: Dual of $H(H^*)$

Hub-hyperpath of $\mathbb{H}^*$ is $v_1 \quad e_2 \quad v_2 \quad e_5 \quad v_3 \quad e_6 \quad v_4$. Then hub number of this graph $h_n(\mathbb{H}^*) = 2$.
Compairing these hypergraphs $\mathbb{H}$ and $\mathbb{H}^*$, hub number is reduced from $\mathbb{H}$ and $\mathbb{H}^*$.

Another example is, consider fig.1 as $\mathbb{H}$, then the incidence matrix described as follows,

$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

The incidence matrix of $\mathbb{H}^*$ is,

$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

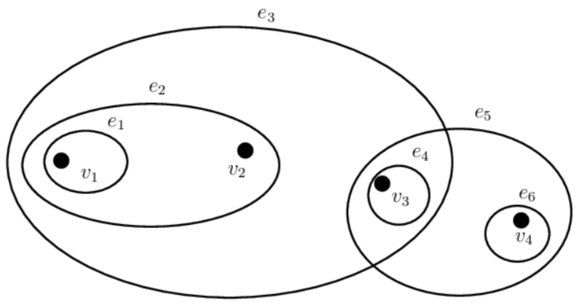

fig 4: Dual of H (H^*)

From, this example it is concluded that, there is no change in hubnumber hypergraph $\mathbb{H}$ and its dual $\mathbb{H}^*$.

Let $\mathbb{H} = (\mathscr{V}, \mathscr{E})$ be a hypergraph, and let $\mathbb{H}^* = (\mathscr{V}^*, \mathscr{E}^*)$ be its dual hypergraph, where $\mathscr{V}^*$ is the set of hyperedges in $\mathbb{H}$. Then, $h_n(\mathbb{H})$ is defined as the minimum cardinality of a hubset in $\mathbb{H}$, while $h_n(\mathbb{H}^*)$ is the minimum cardinality of a hubset in $\mathbb{H}^*$.

Then, compare the hub numbers of $\mathbb{H}$ and $\mathbb{H}^*$ as follows: The hub number of $\mathbb{H}$ is at least as large as the hub number of $\mathbb{H}^*$: It follows from the definitions that every $\mathscr{V}$ in $\mathbb{H}^*$ is contained in at least one hyperedge in $\mathbb{H}$, and conversely, every $\mathscr{E}$ in $\mathbb{H}$ is contained in at least one vertex in $\mathbb{H}^*$. Therefore, number of intermediate vertices in $\mathbb{H}_{\mathbb{P}}$ in $\mathbb{H}$ is less than the number of intermediate vertices in hub-hyperpath $\mathbb{H}^*$, which implies that the hub number of $\mathbb{H}$ is at least as large as the hub number of $\mathbb{H}^*$.

The hub number of $\mathbb{H}$ is at most as large as the hub number of $\mathbb{H}^*$: To see why this is true, let $\mathbb{H}_P{}^*$ be a hub-hyperpath between two end vertices in $\mathbb{H}^*$ that continues $\mathbb{H}_{P(\mathscr{V})}{}^*\backslash\{v_i, v_j\}$ are identified as hubset in $\mathbb{H}^*$. Then construct an incidence matrix of $\mathbb{H}^*$ from the incidence matix of $\mathbb{H}$ that is as follows: For each vertex v_i in $\mathbb{H}$, is replaced as hyperedge in $\mathbb{H}^*$. Therefore, the hub number of $\mathbb{H}$ is at most as large as the hub number of $\mathbb{H}^*$.

Putting these two observations together, that the hub number of $\mathbb{H}$ is equal to the hub number of $\mathbb{H}^*$ if and only if $\mathbb{H}$ and $\mathbb{H}^*$ have the same hub number. In general, however, the hub numbers of $\mathbb{H}$ and $\mathbb{H}^*$ need not be equal, since there may be some redundancy in the cover of vertices or hyperedges that arises from the duality relation between $\mathbb{H}$ and $\mathbb{H}^*$.

6. APPLICATIONS

The hub number of a hypergraph is an important concept in network analysis and has several applications, including:

Robustness Analysis: The hub number of a hypergraph is used to measure the robustness of a network. Networks with a high hub number are more robust to vertex failures, as the removal of a small number of vertices does not significantly affect the connectivity of the network. The hub number is used in the analysis of communication networks, transportation networks, power grids and other critical infrastructure systems.

Network Design: The hub number of a hypergraph is used in network design to determine the optimal location of key vertices, such as routers or switches, to ensure that the network is robust to failures. The hub number can also be used in the design of wireless networks to determine the optimal placement of access points.

Graph Algorithms: The hub number of a hypergraph is used in various graph algorithms, such as network decomposition and clustering algorithms. For example, the hub number can be used to identify central vertices in a network, which can be useful in community detection and link prediction.

Social Network Analysis: The hub number of a hypergraph is used in social network analysis to identify the most influential vertices in a social network. Vertices with a high hub number are considered to be more influential than vertices with a low hub number and they are more likely to have a greater impact on the network structure and dynamics.

Overall, the hub number of a hypergraph is a useful tool for analyzing the robustness and structure of complex networks and it has applications in a wide range of fields, including engineering, computer science and social science.

7. CONCLUDING REMARKS

There are several parameters in hypergraphs, including the degree of hypergraph, hypergraph centrality and hypergraph closeness centrality. These parameters help to identify the most important vertices in a hypergraph, which can be useful in many applications, such as social network analysis, transportation planning and biological network analysis. Hub parameters are important measures in hypergraph theory, as they help to identify the most important nodes in a hypergraph. For example, in a transportation network, identifying hub nodes that connect multiple routes can help in improving the efficiency of the system. Similarly, in a social network, identifying hub nodes that connect multiple groups can help in understanding the spread of information or influence within the network. These measures can be used in a wide range of applications to analyze complex networks and understand their properties.

REFERENCES

[1] Bretto, A., *Hypergraph Theory,* Springer International Publishing Switzerland, ISBN 978-3-319-00079-4, 2013.

[2] Berge, C. J., *Hypergraphs,* North-Holland Mathematical Library, Vol.15, ISBN: 0-444-87489-5, Elsevier Science Publishers B.V., 1989.

[3] Walsh, M., *The Hub Number of a Graph,* International Journal of Mathematics and Computer Science, ISSN 1814-0424, **1** (2006), 117-124,.

[4] Khalaf, S. I., Mathad, V., Mahde, S. S., *Hub and global hub numbers of a graph,* Proceedings of the Jangjeon Mathematical Society, **23(2)** 2020, 231 - 239.