J. Indones. Math. Soc.
Vol. 30, No. 03 (2024), pp. 352–360.

FIXED POINT THEOREMS FOR (ψ, ϕ, ω)-WEAK CONTRACTIONS IN COMPLETE METRIC SPACES

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Abstract. This paper defines the mapping called (ψ, ϕ, ω)-weak contractions. We then use this definition to prove the existence of a fixed point. The mapping we defined above is a modified mapping by Liu and Chai. We use the concept of ω-distance to prove the fixed point theorem. Since every ω-distance is a metric, the resulting theorem is also satisfied for every metric.

Key words: fixed point, (ψ, ϕ, ω)-weak contractions, ω-distance

1. INTRODUCTION

The fixed point theory is one of the topics in mathematical analysis that can be applied in certain areas of mathematics. One of them is used to prove the existence of the solution for any system of differential equations. There are a lot of researchers that are interested in this area [1–15], one of them is Kada [1]. Kada introduced the fixed point theory using the ω distance. The interesting concept of ω-distance stated that every metric is ω-distance, but not vice versa. There are a lot of fixed point theorems developed by using ω-distance [1–12]. Another researcher working in the fixed point theory is Lakzian and Samet [13]. They proved the existence of a fixed point for (ψ, ϕ)-weakly contractive mappings in rectangular metric spaces. Liu and Chai then continued this result citeLiu by generalizing the concept of (ψ, ϕ)-weakly contractive mapping. Xue and Lv [15] expanded the

2020 Mathematics Subject Classification: 46A19, 54E35
Received: 19-02-2023, accepted: 10-08-2024.

theorem from [14] by changing and weakening the condition of the theorem. In this paper, we use the ω-distance [1] and mapping given by Liu and Chai [14] to prove the fixed point theorem. Next, we denote Ψ as a collection of function $h : [0, \sim) \to [0, \sim)$ that satisfied the following conditions.

(1) h is a continuous function.
(2) h is a nondecreasing function.
(3) $h(y) = 0$ if and only if $y = 0$.

We also denote Φ as a collection of function $\lambda : [0, \sim) \to [0, \sim)$ which satisfied these following conditions:

(1) $\liminf_{t \to k} \lambda(t) > 0$ for any $k > 0$.
(2) $\lambda(y) = 0$ if and only if $y = 0$.

Below, we define ω-distance introduced by Kada [1].

Definition 1.1. *Let (A, d) be a metric space. A map $s : A \times A \to [0, \infty)$ is called a ω-distance on A if*

(1) $s(a, b) \leq s(a, c) + s(c, b)$, $\forall a, b, c \in A$.
(2) For every $a \in A$, $s(a, \cdot) : A \to [0, \infty)$ is a lower semicontinuous function.
(3) For each $\varepsilon > 0$, there exists $\delta > 0$ such that $\forall a, b, c \in A$ where $s(c, a) \leq \delta$ and $s(c, b) \leq \delta$ implies $d(a, b) \leq \varepsilon$.

Every metric is a ω-distance, but the converse is not necessarily true. The following property is given regarding the ω-distance that will be used in proving the fixed-point theorem using the ω-distance.

Lemma 1.2. *Let (A, d) be a metric space and s be a ω-distance on A. If $\{a_n\}, \{b_n\}$ are sequences in A and $\{\alpha_n\}, \{\beta_n\} \subseteq [0, \infty)$ converge to 0, and $a, b, c \in A$ then the following statements are true.*

(1) If $s(a_n, b) \leq \alpha_n$ and $s(a_n, c) \leq \beta_n$ for every $n \in \mathbb{N}$ then $b = c$. Furthermore, if $s(a, b) = 0$ and $s(a, c) = 0$, then $b = c$.
(2) If $s(a_n, b_n) \leq \alpha_n$ and $s(a_n, c) \leq \beta_n$ for each $n \in \mathbb{N}$ then $\lim_{n \to \infty} b_n = c$.
(3) If $s(a_n, a_m) \leq \alpha_n$ for every $n, m \in \mathbb{N}$ with $m > n$, then $\{a_n\}$ is a Cauchy sequence.
(4) If $s(b, a_n) \leq \alpha_n$ for every $n \in \mathbb{N}$, then $\{a_n\}$ is a Cauchy sequence.

2. MAIN RESULTS

Here, we define the (ψ, ϕ, ω)-weak contractions. Later, we prove the existence of a fixed point for this mapping.

Definition 2.1. *Let (A,d) be a metric space, q be a ω-distance, $\psi \in \Psi$, and $\varphi \in \Phi$. The function $T : A \to A$ is called a (ψ, ϕ, ω)-weak contractions if*

$$\psi(q(T(b),T(c))) \leq \begin{pmatrix} \psi\left(a_1 q\left(b,c\right) + a_2 q\left(b, T\left(b\right)\right) + a_3 q\left(c, T\left(c\right)\right)\right) \\ -\varphi\left(a_1 q\left(b,c\right) + a_2 q\left(b, T\left(b\right)\right) + a_3 q\left(c, T\left(c\right)\right)\right) \end{pmatrix}$$

for every $b, c \in A$, where $a_1, a_2, a_3 \geq 0$ with $a_1 + a_2 + a_3 \leq 1$.

Example 2.2. *Let $X = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$ and a metric d with $d(x,y) = |x-y|$. Define $\psi(t) = t$, $\varphi(t) = \frac{t}{10}$, and $T : X \to X$ with*

$$T(x) = \begin{cases} \frac{1}{4} & , x = \frac{1}{2} \\ 0 & , x \in \{0, \frac{1}{3}, \frac{1}{4}\}. \end{cases}$$

Define a ω-distance with $q(x,y) = y$ for all $x, y \in X$. As a result, for $a_1 = 1, a_2 = 0$, and $a_3 = 0$, we get that T satisfies the Definition 2.1.

Theorem 2.3. *Let (C,d) be a complete metric space, q be a ω-distance, $\psi \in \Psi$, and $\varphi \in \Phi$. If $T : C \to C$ is a (ψ, ϕ, ω)-weak contractions and for every $c_1, c_2 \in C$ with $q(T(c_1), T(c_2)) > 0$ results in $q(c_1, c_2) > 0$ and for every $c \in C$ where $c \neq T(c)$ applies $\inf\{q(u,c) + q(u, T(u)) : u \in C\} > 0$, then T has a unique fixed point $c^* \in C$. Furthermore $q(c^*, c^*) = 0$.*

PROOF. Suppose that $c_0 \in C$. The sequence c_n is formed where $c_1 = T(c_0)$, $c_2 = T(c_1) = T^2(c_0)$, $\ldots$, $c_n = T(c_{n-1}) = T^n(c_0)$, for every $n \in \mathbb{N}$.
Consider the following possibilities.
Case 1. If there exist $k \in \mathbb{N} \cup \{0\}$ so that $q(c_k, c_{k+1}) = 0$, then $q(c_{k+1}, c_{k+2}) = 0$, because if we assume that $q(c_{k+1}, c_{k+2}) > 0$, then according to premise, we get $q(c_k, c_{k+1}) > 0$, which lead to a contradiction. Consequently, $q(c_k, c_{k+2}) \leq q(c_k, c_{k+1}) + q(c_{k+1}, c_{k+2}) = 0$. Furthermore, since $q(c_k, c_{k+1}) = 0$ and $q(c_k, c_{k+2}) = 0$, then from Lemma 1.2 we get $c_{k+1} = c_{k+2}$, or in other words $c_{k+1} = T(c_{k+1})$.
Case 2. If $q(c_n, c_{n+1}) > 0$ for every $n \in N$, because $T : C \to C$ is a (ψ, ϕ, ω)-weak contractions then

$$\begin{aligned} \psi(q(c_n,c_{n+1})) &= \psi(q(T(c_{n-1}), T(c_n))) \\ &\leq \begin{pmatrix} \psi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, T(c_{n-1})) + a_3 q(c_n, T(c_n))) \\ -\varphi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, T(c_{n-1})) + a_3 q(c_n, T(c_n))) \end{pmatrix} \\ &\leq \begin{pmatrix} \psi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, (c_{n+1}))) \\ -\varphi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) \end{pmatrix}. \end{aligned} \tag{1}$$

Assuming $q(c_{n-1}, c_n) \leq q(c_n, c_{n+1})$ for some $n \in \mathbb{N}$, based on (1) and since ψ is a non-increasing function, we get

$$\begin{aligned}
\psi(q(c_n, c_{n+1})) &\le \begin{pmatrix} \psi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) \\ -\varphi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) \end{pmatrix} \\
&\le \begin{pmatrix} \psi(a_1 q(c_n, c_{n+1}) + a_2 q(c_n, c_{n+1}) + a_3 q(c_n, c_{n+1})) \\ -\varphi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) \end{pmatrix} \\
&\le \begin{pmatrix} \psi((a_1 + a_2 + a_3) q(c_n, c_{n+1})) \\ -\varphi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) \end{pmatrix} \\
&\le \begin{pmatrix} \psi(q(c_n, c_{n+1})) \\ -\varphi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) \end{pmatrix}.
\end{aligned} \tag{2}$$

Based on (2), we have

$$\varphi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) = 0$$

or

$$a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1}) = 0.$$

Furthermore, since $q(c_n, c_{n+1}) > 0$ for every $n \in \mathbb{N}$ and $a_1, a_2, a_3 \ge 0$, we get that $a_1 = a_2 = a_3 = 0$. As a result $\psi(q(c_n, c_{n+1})) = 0$ or $q(c_n, c_{n+1}) = 0$, it leads to a contradiction with $q(c_n, c_{n+1}) > 0$ for every $n \in \mathbb{N}$. So, we have $q(c_n, c_{n+1}) < q(c_{n-1}, c_n)$.

Since $q(c_n, c_{n+1}) < q(c_{n-1}, c_n)$ for every $n \in \mathbb{N}$ and $\{q(c_n, c_{n+1})\}$ have 0 as a lower bound, then there exist $r \ge 0$ such that $\lim_{n\to\infty} q(c_n, c_{n+1}) = r$. In the same way, since ψ is non-increasing function, then $\psi(q(c_n, c_{n+1})) \le \psi(q(c_{n-1}, c_n))$. Consequently, there exist $r^* \ge 0$ such that $\lim_{n\to\infty} \psi(q(c_n, c_{n+1})) = r^*$.

Next, we prove that $r = 0$ using contradiction. By assuming $r > 0$, we get

$$\lim_{n\to\infty} (a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) = (a_1 + a_2 + a_3) r > 0.$$

Consequently,

$$\liminf_{n\to\infty} \varphi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) > 0. \tag{3}$$

Since $q(c_n, c_{n+1}) < q(c_{n-1}, c_n)$ and based on (1), we get

$$\begin{aligned}
\psi(q(c_n, c_{n+1})) &\le \begin{pmatrix} \psi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) \\ -\varphi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) \end{pmatrix} \\
&\le \begin{pmatrix} \psi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_{n-1}, c_n)) \\ -\varphi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) \end{pmatrix} \\
&\le \begin{pmatrix} \psi(q(c_{n-1}, c_n)) \\ -\varphi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) \end{pmatrix}.
\end{aligned} \tag{4}$$

Applying the inferior limit for $n \to \infty$ at (4) we get

$$r^* \le \begin{pmatrix} r^* \\ -\liminf_{n\to\infty} \varphi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) \end{pmatrix}$$

or

$$\liminf_{n\to\infty} \varphi(a_1 q(c_{n-1}, c_n) + a_2 q(c_{n-1}, c_n) + a_3 q(c_n, c_{n+1})) \le 0,$$

which contradicts with (3). So, we have $r = 0$, or in the other words

$$\lim_{n\to\infty} q(c_n, c_{n+1}) = 0. \tag{5}$$

Next, we show that $\{c_n\}$ is a Cauchy's sequence. We will show $\{c_n\}$ is Cauchy's sequence using Lemma 1.2, that is $\lim\limits_{n\to\infty} q(c_n, c_{n+m}) = 0$ for every $m \in \mathbb{N}$. We use contradiction. Assume that $\{c_n\}$ is not a Cauchy's sequence. Then, there exist $\alpha > 0$, a subsequence $\{c_{n_i}\}$ and $\{c_{m_i}\}$ of $\{c_n\}$ with $n_i > m_i > i$ such that

$$q(c_{m_i}, c_{n_i}) \geq \alpha \text{ and } q(c_{m_i}, c_{n_{i-1}}) < \alpha \text{ for every } i \in \mathbb{N}. \tag{6}$$

As a result,

$$\alpha \leq q(c_{m_i}, c_{n_i}) \leq q(c_{m_i}, c_{n_{i-1}}) + q(c_{n_{i-1}}, c_{n_i}) < \alpha + q(c_{n_{i-1}}, c_{n_i}). \tag{7}$$

By applying the limit for $i \to \infty$ and using (5), we get

$$\lim_{i\to\infty} q(c_{m_i}, c_{n_i}) = \alpha. \tag{8}$$

Next step:

$$q(c_{m_{i-1}}, c_{n_{i-1}}) \leq q(c_{m_{i-1}}, c_{m_i}) + q(c_{m_i}, c_{n_{i-1}}) < q(c_{m_{i-1}}, c_{m_i}) + \alpha.$$

We apply the limit for $i \to \infty$ and refer to (5), we get

$$\lim_{i\to\infty} q(c_{m_{i-1}}, c_{n_{i-1}}) \leq \alpha. \tag{9}$$

Note that

$$\psi(q(c_{m_i}, c_{n_i})) = \psi(q(T(c_{m_{i-1}}), T(c_{n_{i-1}})))$$
$$\leq \begin{pmatrix} \psi(a_1 q(c_{m_{i-1}}, c_{n_{i-1}}) + a_2 q(c_{m_{i-1}}, c_{m_i}) + a_3 q(c_{n_{i-1}}, c_{n_i})) \\ -\varphi(a_1 q(c_{m_{i-1}}, c_{n_{i-1}}) + a_2 q(c_{m_{i-1}}, c_{m_i}) + a_3 q(c_{n_{i-1}}, c_{n_i})) \end{pmatrix}. \tag{10}$$

Apply an inferior limit for $i \to \infty$ and use (5), (8), and (9), we get

$$\psi(\alpha) \leq \begin{pmatrix} \psi(a_1\alpha) \\ -\liminf\limits_{i\to\infty} \varphi(a_1 q(c_{m_{i-1}}, c_{n_{i-1}}) + a_2 q(c_{m_{i-1}}, c_{m_i}) + a_3 q(c_{n_{i-1}}, c_{n_i})) \end{pmatrix}$$
$$\leq \begin{pmatrix} \psi(\alpha) \\ -\liminf\limits_{i\to\infty} \varphi(a_1 q(c_{m_{i-1}}, c_{n_{i-1}}) + a_2 q(c_{m_{i-1}}, c_{m_i}) + a_3 q(c_{n_{i-1}}, c_{n_i})) \end{pmatrix}.$$

The result is

$$\liminf_{i\to\infty} \varphi(a_1 q(c_{m_{i-1}}, c_{n_{i-1}}) + a_2 q(c_{m_{i-1}}, c_{m_i}) + a_3 q(c_{n_{i-1}}, c_{n_i})) \leq 0. \tag{11}$$

On the other hand

$$\lim_{i\to\infty} (a_1 q(c_{m_{i-1}}, c_{n_{i-1}}) + a_2 q(c_{m_{i-1}}, c_{m_i}) + a_3 q(c_{n_{i-1}}, c_{n_i})) = a_1\alpha. \tag{12}$$

If $a_1 = 0$, then based on (10) we get $\lim\limits_{i\to\infty} \psi(q(c_{m_i}, c_{n_i})) = 0$, which contradicts with (8).

If $a_1 > 0$ then from (12) we get

$$\lim_{i\to\infty} (a_1 q(c_{m_{i-1}}, c_{n_{i-1}}) + a_2 q(c_{m_{i-1}}, c_{m_i}) + a_3 q(c_{n_{i-1}}, c_{n_i})) > 0.$$

It contradicts (11). So we have $\lim_{i\to\infty} q(c_n, c_{n+m}) = 0$ for every $m \in \mathbb{N}$ or in the other words $\{c_n\}$ is a Cauchy sequence.
Since C is a complete metric space and $\{c_n\}$ is a Cauchy sequence, there exists $c^* \in C$ such that $\lim_{n\to\infty} c_n = c^*$. Next, we prove that $T(c^*) = c^*$. By using contradiction, assume that $T(c^*) \neq c^*$. We get that

$$\begin{aligned} 0 &< \inf\{q(y, c^*) + q(y, T(y)) : y \in C\} \\ &\le \inf\{q(c_n, c^*) + q(c_n, T(c_n)) : n \in \mathbb{N}\} \\ &= \inf\{q(c_n, c^*) + q(c_n, c_{n+1}) : n \in \mathbb{N}\}. \end{aligned} \tag{13}$$

Next, we show that $\lim_{n\to\infty} q(c_n, c^*) = 0$. Suppose that $\varepsilon_0 > 0$. Since $\lim_{n\to\infty} q(c_n, c_{n+m}) = 0$ for every $m \in \mathbb{N}$, then there exists $n_0 \in \mathbb{N}$ so that for any $k, l \ge n_0$ with $l > k$ implies $q(c_k, c_l) < \varepsilon_0$. Furthermore, because of $\lim_{n\to\infty} c_n = c^*$ and $q(c_k, \cdot)$ lower semicontinuous, then $q(c_k, c^*) \le \liminf_{m\to\infty} q(c_k, c_m) \le \varepsilon_0$. So, for every positive number of ε_0, there exist $n_0 \in \mathbb{N}$ such that for every natural number $k \ge n_0$, we have $q(c_k, c^*) \le \varepsilon_0$. This statement indicates that

$$\lim_{n\to\infty} q(c_n, c^*) = 0. \tag{14}$$

Based on (5), (13), and (14), we get $0 < \inf\{q(c_n, c^*) + q(c_n, c_{n+1}) : n \in \mathbb{N}\} = 0$, a contradiction. So, it must be $T(c^*) = c^*$.
Next, we show that $q(c^*, c^*) = 0$. By assuming $q(c^*, c^*) > 0$ and using the fact that T is a (ψ, ϕ, ω)-weak contractions then

$$\begin{aligned} \psi(q(c^*, c^*)) &= \psi(q(T(c^*), T(c^*))) \\ &\le \begin{pmatrix} \psi(a_1 q(c^*, c^*) + a_2 q(c^*, T(c^*)) + a_3 q(c^*, T(c^*))) \\ -\varphi(a_1 q(c^*, c^*) + a_2 q(c^*, T(c^*)) + a_3 q(c^*, T(c^*))) \end{pmatrix} \\ &\le \begin{pmatrix} \psi(a_1 q(c^*, c^*) + a_2 q(c^*, c^*) + a_3 q(c^*, c^*)) \\ -\varphi(a_1 q(c^*, c^*) + a_2 q(c^*, c^*) + a_3 q(c^*, c^*)) \end{pmatrix} \\ &\le \begin{pmatrix} \psi(q(c^*, c^*)) \\ -\varphi((a_1 + a_2 + a_3) q(c^*, c^*)) \end{pmatrix}. \end{aligned} \tag{15}$$

As a result, $\varphi((a_1 + a_2 + a_3) q(c^*, c^*)) = 0$, so we get $a_1 = a_2 = a_3 = 0$. Based on (15) we get $\psi(q(c^*, c^*)) = 0$ or in the other words $q(c^*, c^*) = 0$, a contradiction. So, we have $q(c^*, c^*) = 0$. Next, we prove the uniqueness of the fixed point. Suppose there exist $b^* \in C$ such that $T(b^*) = b^*$ and $q(c^*, b^*) > 0$. In the same way as inequality in (15), we obtain $q(b^*, b^*) = 0$. Since T is a (ψ, ϕ, ω)-weak contractions, then

$$
\begin{aligned}
\psi(q(c^*, b^*)) &= \psi(q(T(c^*), T(b^*))) \\
&\leq \left(\begin{array}{c} \psi(a_1 q(c^*, b^*) + a_2 q(c^*, T(c^*)) + a_3 q(b^*, T(b^*))) \\ -\varphi(a_1 q(c^*, b^*) + a_2 q(c^*, T(c^*)) + a_3 q(b^*, T(b^*))) \end{array} \right) \\
&\leq \left(\begin{array}{c} \psi(a_1 q(c^*, b^*) + a_2 q(c^*, c^*) + a_3 q(b^*, b^*)) \\ -\varphi(a_1 q(c^*, b^*) + a_2 q(c^*, c^*) + a_3 q(b^*, b^*)) \end{array} \right) \qquad (16) \\
&\leq (\psi(a_1 q(c^*, b^*)) - \varphi(a_1 q(c^*, b^*))) \\
&\leq (\psi(q(c^*, b^*)) - \varphi(a_1 q(c^*, b^*))).
\end{aligned}
$$

Consequently, $\varphi(a_1 q(c^*, b^*)) = 0$ or $a_1 = 0$. Based on (16), because $a_1 = 0$ then $\psi(q(c^*, b^*)) = 0$ or in the other words $q(c^*, b^*) = 0$, a contradiction. So, it must be $q(c^*, b^*) = 0$. Since $q(c^*, c^*) = 0$ and $q(c^*, b^*) = 0$, then $c^* = b^*$. So, T has a unique fixed point. ■

By taking $a_3 = 1, a_1 = a_2 = 0$ or $a_2 = 1, a_1 = a_3 = 0$ or $a_1 = 1, a_2 = a_3 = 0$, we get the following results.

Corollary 2.4. *Let (C, d) be a complete metric space, s be a ω-distance, $\psi \in \Psi$, and $\varphi \in \Phi$. If $T : C \to C$ satisfies $\psi(s(T(a), T(b))) \leq \psi(s(b, T(b)) - \varphi(s(b, T(b)))$ for each $a, b \in C$, and for each $u_1, u_2 \in C$ where $s(T(u_1), T(u_2)) > 0$ results in $s(u_1, u_2) > 0$ and for every $c \in C$ with $c \neq T(c)$ implies $\inf\{q(y, c) + q(y, T(y)) : y \in C\} > 0$ then T has a unique fixed point $c^* \in C$.*

Corollary 2.5. *Let (C, d) be a complete metric space, s be a ω-distance, $\psi \in \Psi$, and $\varphi \in \Phi$. If $T : C \to C$ satisfies $\psi(s(T(a), T(b))) \leq \psi(s(a, T(a)) - \varphi(s(a, T(a)))$ for every $a, b \in C$, and for every $u_1, u_2 \in C$ where $q(T(u_1), T(u_2)) > 0$ results in $q(u_1, u_2) > 0$ and for every $c \in C$ with $c \neq T(c)$ implies $\inf\{q(y, c) + q(y, T(y)) : y \in C\} > 0$ then T has a unique fixed point $c^* \in C$.*

Corollary 2.6. *Let (C, d) be a complete metric space, s be a ω-distance, $\psi \in \Psi$, and $\varphi \in \Phi$. If $T : C \to C$ satisfies $\psi(s(T(a), T(b))) \leq \psi(s(a, b)) - \varphi(s(a, b))$ for every $a, b \in C$, and for every $u_1, u_2 \in C$ where $s(T(u_1), T(u_2)) > 0$ results in $s(u_1, u_2) > 0$ and for every $c \in C$ with $c \neq T(c)$ holds $\inf\{s(y, c) + s(y, T(y)) : y \in C\} > 0$ then T has a unique fixed point $c^* \in C$.*

Corollary 2.7. *Let (C, d) be a complete metric space, s be a ω-distance, $\psi \in \Psi$, and $\varphi \in \Phi$. If $T : C \to C$ satisfies*

$$
\psi(s(T(a), T(b))) \leq \left(\begin{array}{c} \psi\left(\max\{s(a, b), s(a, T(a)), s(b, T(b))\}\right) \\ -\varphi\left(\max\{s(a, b), s(a, T(a)), s(b, T(b))\}\right) \end{array} \right)
$$

for any $a, b \in C$, and for any $u_1, u_2 \in C$ where $s(T(u_1), T(u_2)) > 0$ results in $s(u_1, u_2) > 0$ and for any $c \in C$ where $c \neq T(c)$ implies $\inf\{s(y, c) + s(y, T(y)) : y \in C\} > 0$ then T has a unique fixed point $c^* \in C$.

Since every ω-distance is a metric, the result of Theorem 2.3 can be obtained as follows

Corollary 2.8. *Let* (C, d) *be a complete metric space,* $\psi \in \Psi$*, and* $\varphi \in \Phi$*,. If* $T : C \to C$ *satisfies*

$$\psi(d(T(a), T(b))) \leq \begin{pmatrix} \psi(a_1 d(a,b) + a_2 d(a, T(a)) + a_3 d(b, T(b))) \\ -\varphi(a_1 d(a,b) + a_2 d(a, T(a)) + a_3 d(b, T(b))) \end{pmatrix}$$

for every $a, b \in C$*, and* $a_1, a_2, a_3 \geq 0$ *with* $a_1 + a_2 + a_3 \leq 1$ *and for every* $uc \in C$ *where* $c \neq T(c)$ *implies* $\inf\{d(u,c) + d(u, T(u)) : u \in C\} > 0$ *then* T *has a unique fixed point* $c^* \in C$.

Corollary 2.9. *Let* (C, d) *be a complete metric space,* $\psi \in \Psi$*, and* $\varphi \in \Phi$*. If* $T : C \to C$ *satisfies*

$$\psi(d(T(a), T(b))) \leq \begin{pmatrix} \psi(max\ \{d(a,b), d(a, T(a)), d(b, T(b))\}) \\ -\varphi(max\ \{d(a,b), d(a, T(a)), d(b, T(b))\}) \end{pmatrix}$$

for every $a, b \in C$*, and for every* $c \in C$ *where* $c \neq T(c)$ *implies* $\inf\{d(u,c) + d(u, T(u)) : u \in C\} > 0$ *then* T *has a unique fixed point* $c^* \in C$.

Acknowledgement. The authors are grateful for the funding provided by Universitas Nusa Cendana.

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