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A WHITE NOISE APPROACH TO THE SELF-INTERSECTION LOCAL TIMES OF A GAUSSIAN PROCESS

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Abstract. In this paper we show that for any spatial dimension, the renormalized self-intersection local times of a certain Gaussian process defined by indefinite Wiener integral exist as Hida distributions. An explicit expression for the chaos decomposition in terms of Wick tensor powers of white noise is also obtained. We also study a regularization of the self-intersection local times and prove a convergence result in the space of Hida distributions.

Key words: b-Gaussian process, white noise analysis, self-intersection local time.

Abstrak. Di dalam makalah ini dibuktikan bahwa untuk sebarang dimensi spasial renormalisasi dari waktu lokal perpotongan-diri dari sebuah proses Gaussian yang didefinisikan melalui integral Wiener tak tentu merupakan distribusi Hida. Dekomposisi chaos dari distribusi Hida tersebut juga diberikan secara eksplisit. Studi terhadap sebuah regularisasi dari waktu lokal perpotongan-diri juga dilakukan dan dibuktikan sebuah hasil terkait kekonvergenan dari regularisasi tersebut di ruang distribusi Hida.

Kata kunci: proses Gaussian-b, analisis white noise, waktu lokal perpotongan-diri.

1. Introduction

As an infinite-dimensional stochastic distribution theory, white noise analysis provides a natural framework for the study of local times and self-intersection local times of Gaussian processes, see e.g. [4]. The concept of self-intersection local times itself plays important roles in several branches of science. For example, it is

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used in the construction of certain Euclidean quantum field [10]. In the Edwards polymer theory self-intersection local times appeared in the path integral to model the excluded volume effects of the polymer formations [3]. The first idea of analyzing local times and self-intersection local times using a white noise approach goes back at least to the work of Watanabe [11]. He showed that as the dimension of the Brownian motion increases, successive omissions of lowest order chaos in the Wiener-Itô decomposition are sufficient to ensure that the truncated local time is a white noise distribution. A further investigation was given by Da Faria et al in [1]. They gave the chaos decomposition in terms of Wick tensor powers of white noise. Their results were later generalized to fractional Brownian motion for any Hurst parameter $H \in (0,1)$ by Drumond et al [2]. In the present paper we provide another direction of generalization of some results in [1] to a certain class of Gaussian process defined by indefinite Wiener integrals (in the sense of Itô).

First of all, let us fix $0 < T < \infty$. The space of real-valued square-integrable function with respect to the Lebesgue measure on $[0,T]$ will be denoted by $L^2[0,T]$. Let $f \in L^2[0,T]$ and $B = (B_t)_{t\in[0,T]}$ be a standard one-dimensional Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is a fundamental fact from Itô's stochastic integration theory that the stochastic process $X = (X_t)_{t\in[0,T]}$ defined by the indefinite Wiener integral $X_t := \int_0^t f(u)\, dB_u$ is an $L^2(\mathbb{P})$- continuous martingale with respect to the natural filtration of B. It is also a centered Gaussian process with covariance function $\mathbb{E}\left(X_s X_t\right) = \int_0^{s\wedge t} |f(u)|^2\, du$, $s,t \geq 0$, see e.g. [7]. Here $\mathbb{E}$ denotes the expectation with respect to the probability measure $\mathbb{P}$. In this work we further assume that f is bounded and never takes value zero on $[0,T]$. We call the corresponding stochastic process as *b-Gaussian process*. By choosing f to be the constant function 1, we see that our new class of Gaussian processes contains Brownian motion as an example. Moreover, by d-dimensional b-Gaussian process we mean the random vector $\left(X^1, \ldots, X^d\right)$ where $X^1, \ldots, X^d$ are d independent copies of a one-dimensional b-Gaussian process. Motivated by similar works on self-intersection local times of Brownian motion and fractional Brownian motion, see e.g. da Faria et al [1], Drumond et al [2] and Watanabe [11], we consider the *self-intersection local time* of b-Gaussian process X, which is informally defined as

$$\int_0^T \int_0^T \delta\left(X_t - X_s\right)\, ds\, dt, \tag{1}$$

where δ denotes the Dirac delta distribution at 0. The (generalized) random variable (1) is intended to measure the amount of time in which the sample path of a b-Gaussian process X spends intersecting itself within the time interval $[0,T]$. A priori the expression (1) has no mathematical meaning since Lebesgue integration of Dirac delta distribution is not defined. One common way to give a mathematically rigorous meaning to such an expression, as in [1] and [2], is by approximation using a Dirac sequence. More precisely, we interpret (1) as the limiting object of the approximated self-intersection local time $L_{X,\varepsilon}(T)$ of b-Gaussian process X defined as

$$L_{X,\varepsilon}(T) := \int_0^T \int_0^t p_\varepsilon\left(X_t - X_s\right)\, ds\, dt, \quad \varepsilon > 0,$$

as $\varepsilon \to 0$, where p_ε is the heat kernel given by

$$p_\varepsilon(x) := \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{x^2}{2\varepsilon}\right), \quad x \in \mathbb{R}.$$

This approximation procedure will make the limiting object, which we denote by $L_X(T)$, more and more singular as the dimension of the process X increases. Hence, we need to do a *renormalization*, i.e. cancelation of the divergent terms, to obtain a well-defined and sufficiently regular object.

Now we describe briefly our main results. Under some conditions on the spatial dimension of the b-Gaussian process X and the number of subtracted terms in the truncated Donsker's delta function, we are able to show the existence of the renormalized (or truncated) self-intersection local time $L_X(T)$ as a well-defined object in some white noise distribution space. Moreover, we derive the chaos decomposition of $L_X(T)$ in terms of Wick tensor powers of white noise. This decomposition corresponds to that in terms of multiple Wiener-Itô integrals when one works in the classical stochastic analysis using Wiener space as the underlying probability space and Brownian motion as the basic random variable. Finally, we also analyze a regularization corresponding to the Gaussian approximation described above and prove a convergence result. The organization of the paper is as follows. In section 2 we summarize some of the standard facts from the theory of white noise analysis. Section 3 contains a detailed exposition of the main results and their proofs.

2. Basics of White Noise Analysis

In order to make the paper self-contained, we summarize some fundamental concepts of white noise analysis used throughout this paper. For a more comprehensive explanation including various applications of white noise theory, see for example, the books of Hida et al [4], Kuo [6] and Obata [9]. Let $(\mathcal{S}'_d(\mathbb{R}), \mathcal{C}, \mu)$ be the $\mathbb{R}^d$-valued white noise space, i.e., $\mathcal{S}'_d(\mathbb{R})$ is the space of $\mathbb{R}^d$-valued tempered distributions, $\mathcal{C}$ is the Borel σ-algebra generated by weak topology on $\mathcal{S}'_d(\mathbb{R})$, and the white noise probability measure μ is uniquely determined through the Bochner-Minlos theorem (see e.g. [6]) by fixing the characteristic function

$$C(\vec{f}) := \int_{\mathcal{S}'_d(\mathbb{R})} \exp\left(i\langle\vec{\omega}, \vec{f}\rangle\right) d\mu(\vec{\omega}) = \exp\left(-\frac{1}{2}|\vec{f}|_0^2\right)$$

for all $\mathbb{R}^d$-valued Schwartz test function $\vec{f} \in \mathcal{S}_d(\mathbb{R})$. Here $|\cdot|_0$ denotes the usual norm in the real Hilbert space $L^2_d(\mathbb{R})$ of all $\mathbb{R}^d$-valued Lebesgue square-integrable functions, and $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $\mathcal{S}'_d(\mathbb{R})$ and $\mathcal{S}_d(\mathbb{R})$. The dual pairing is considered as the bilinear extension of the inner product on $L^2_d(\mathbb{R})$, i.e.

$$\left\langle \vec{g}, \vec{f} \right\rangle = \sum_{j=1}^{d} \int_{\mathbb{R}} g_j(x) f_j(x)\, dx,$$

for all $\vec{g} = (g_1, \ldots, g_d) \in L^2_d(\mathbb{R})$ and $\vec{f} = (f_1, \ldots, f_d) \in \mathcal{S}_d(\mathbb{R})$. We should remark that we have the Gel'fand triple, i.e. the continuous and dense embeddings of spaces

$$\mathcal{S}_d(\mathbb{R}) \hookrightarrow L^2_d(\mathbb{R}) \hookrightarrow \mathcal{S}'_d(\mathbb{R}).$$

Let f be a function in the subset of $L^2[0,T]$ consisting all real-valued bounded functions on $[0,T]$ which has no zeros. In the white noise analysis setting a d-dimensional b-Gaussian process can be represented by a continuous version of the stochastic process $X = (X_t)_{t\in[0,T]}$ with

$$X_t := \left(\left\langle \cdot, \mathbf{1}_{[0,t]} f \right\rangle, \ldots, \left\langle \cdot, \mathbf{1}_{[0,t]} f \right\rangle\right),$$

such that for independent d-tuples of Gaussian white noise $\vec{\omega} = (\omega_1, \ldots, \omega_d) \in \mathcal{S}'_d(\mathbb{R})$

$$X_t(\vec{\omega}) = \left(\left\langle \omega_1, \mathbf{1}_{[0,t]} f \right\rangle, \ldots, \left\langle \omega_d, \mathbf{1}_{[0,t]} f \right\rangle\right),$$

where $\mathbf{1}_A$ denotes the indicator function of a set $A \subset \mathbb{R}$.

Recall that the complex Hilbert space $L^2(\mu) := L^2(\mathcal{S}'_d(\mathbb{R}), \mathcal{C}, \mu)$ is canonically unitary isomorphic to the d-fold tensor product of Fock space of symmetric square-integrable function, i.e.

$$L^2(\mu) \cong \left(\bigoplus_{k=0}^{\infty} L^2_s(\mathbb{R}^k, k! d^k x)\right)^{\otimes d},$$

via the so-called Wiener-Itô-Segal isomorphism. Thus, we have the unique chaos decomposition of an element $F \in L^2(\mu)$,

$$F(\omega_1, \ldots, \omega_d) = \sum_{(m_1,\ldots,m_d)\in\mathbb{N}_0^d} \left\langle :\omega_1^{\otimes m_1} : \otimes \ldots \otimes : \omega_d^{\otimes m_d} :, \vec{f}_{(m_1,\ldots,m_d)} \right\rangle, \qquad (2)$$

with kernel functions $\vec{f}_{(m_1,\ldots,m_d)}$ of the m-th chaos are in the Fock space. Here $:\omega_j^{\otimes m_j}:$ denotes the m_j-th Wick tensor power of $\omega_j \in \mathcal{S}'_1(\mathbb{R})$. We also introduce the following notations

$$\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}_0^d, \quad m = \sum_{j=1}^{d} m_j, \quad \mathbf{m}! = \prod_{j=1}^{d} m_j!,$$

which simplify (2) to

$$F(\vec{\omega}) = \sum_{\mathbf{m}\in\mathbb{N}_0^d} \left\langle : \vec{\omega}^{\otimes \mathbf{m}} :, \vec{f}_{\mathbf{m}} \right\rangle, \ \ \vec{\omega} \in \mathcal{S}'_d(\mathbb{R}).$$

Using, for example, the Wiener-Itô chaos decomposition theorem and the second quantization operator of the Hamiltonian of a harmonic oscillator we can construct the Gel'fand triple

$$(\mathcal{S}) \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})^*$$

where $(\mathcal{S})$ is the space of white noise test functions obtained by taking the intersection of a family of Hilbert subspaces of $L^2(\mu)$. It is equipped with the projective limit topology and has the structure of nuclear Frechet space. The space of white noise distributions $(\mathcal{S})^*$ is defined as the topological dual space of $(\mathcal{S})$. Elements of $(\mathcal{S})$ and $(\mathcal{S})^*$ are also known as *Hida test functions* and *Hida distributions*, respectively.

The rest of this section is devoted to the characterization of a Hida distribution via the so-called S-transform, which can be considered as an analogue of the Gauss-Laplace transform on infinite-dimensional spaces. The *S-transform* of an element $\Phi \in (\mathcal{S})^*$ is defined as

$$(S\Phi)\left(\vec{f}\right) := \left\langle\left\langle \Phi, :\exp\left(\left\langle \cdot, \vec{f}\right\rangle\right): \right\rangle\right\rangle, \quad \vec{f} \in \mathcal{S}_d(\mathbb{R}),$$

where

$$:\exp\left(\left\langle \cdot, \vec{f}\right\rangle\right) ::= \sum_{\mathbf{m}\in\mathbb{N}_0^d} \left\langle :\cdot^{\otimes\mathbf{m}}:, \vec{f}^{\otimes\mathbf{m}}\right\rangle = C(\vec{f})\exp\left(\left\langle \cdot, \vec{f}\right\rangle\right),$$

is the so-called Wick exponential and $\langle\langle\cdot,\cdot\rangle\rangle$ denotes the dual pairing between $(\mathcal{S})^*$ and $(\mathcal{S})$. We define this dual pairing as the bilinear extension of the sesquilinear inner product on $L^2(\mu)$. The decomposition $S\Phi(\vec{f}) = \sum_{\mathbf{m}\in\mathbb{N}_0^d} \left\langle F_{\mathbf{m}}, \vec{f}^{\otimes\mathbf{m}}\right\rangle$ extends the chaos decomposition to $\Phi \in (\mathcal{S})^*$ with distribution-valued kernels $F_{\mathbf{m}}$ such that $\langle\langle\Phi,\varphi\rangle\rangle = \sum_{\mathbf{m}\in\mathbb{N}_0^d} \mathbf{m}! \left\langle F_{\mathbf{m}}, \vec{\varphi}_{\mathbf{m}}\right\rangle$, for every Hida test function $\varphi \in (\mathcal{S})$ with kernel functions $\vec{\varphi}_{\mathbf{m}}$. The S-transform provides a quite useful way to identify a Hida distribution $\Phi \in (\mathcal{S})^*$, in particular, when it is very hard or impossible to find the explicit form for the Wiener-Itô chaos decomposition of Φ.

Theorem 2.1. [5] *A function $F : \mathcal{S}_d(\mathbb{R}) \to \mathbb{C}$ is the S-transform of a unique Hida distribution in $(\mathcal{S})^*$ if and only if it satisfies the conditions:*

(1) *F is ray analytic, i.e., for every $\vec{f}, \vec{g} \in \mathcal{S}_d(\mathbb{R})$ the mapping $\mathbb{R} \ni \lambda \mapsto F\left(\lambda\vec{f} + \vec{g}\right)$ has an entire extension to $\lambda \in \mathbb{C}$, and*
(2) *F has growth of second order, i.e., there exist constants $K_1, K_2 > 0$ and a continuous seminorm $\|\cdot\|$ on $\mathcal{S}_d(\mathbb{R})$ such that for all $z \in \mathbb{C}$, $\vec{f} \in \mathcal{S}_d(\mathbb{R})$*

$$\left|F(z\vec{f})\right| \leq K_1 \exp\left(K_2|z|^2 \left\|\vec{f}\right\|^2\right).$$

There are two important consequences of the above characterization theorem. The first one deals with the Bochner integration of a family of Hida distributions which depend on an additional parameter and the second one concerns the convergence of sequences of Hida distributions. For details and proofs see [5].

Corollary 2.2. [5] *Let $(\Omega, \mathcal{A}, \nu)$ be a measure space and $\lambda \mapsto \Phi_\lambda$ be a mapping from Ω to $(\mathcal{S})^*$. If the S-transform of Φ_λ fulfils the following two conditions:*

(1) *the mapping $\lambda \mapsto S(\Phi_\lambda)(\vec{f})$ is measurable for all $\vec{f} \in \mathcal{S}_d(\mathbb{R})$, and*
(2) *there exist $C_1(\lambda) \in L^1(\Omega, \mathcal{A}, \nu)$, $C_2(\lambda) \in L^\infty(\Omega, \mathcal{A}, \nu)$ and a continuous seminorm $\|\cdot\|$ on $\mathcal{S}_d(\mathbb{R})$ such that for all $z \in \mathbb{C}$, $\vec{f} \in \mathcal{S}_d(\mathbb{R})$*

$$\left|S(\Phi_\lambda)(z\vec{f})\right| \leq C_1(\lambda) \exp\left(C_2(\lambda)|z|^2 \left\|\vec{f}\right\|^2\right),$$

then Φ_λ is Bochner integrable with respect to some Hilbertian norm which topologizing $(\mathcal{S})^*$. Hence $\int_\Omega \Phi_\lambda \, d\nu(\lambda) \in (\mathcal{S})^*$, and furthermore

$$S\left(\int_\Omega \Phi_\lambda \, d\nu(\lambda)\right) = \int_\Omega S(\Phi_\lambda) \, d\nu(\lambda).$$

Corollary 2.3. [5] *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathcal{S})^*$ such that*

(1) *for all $\vec{f} \in \mathcal{S}_d(\mathbb{R})$, $\left(S(\Phi_n)(\vec{f})\right)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{C}$, and*
(2) *there exist constants $K_1, K_2 > 0$ and a continuous seminorm $\|\cdot\|$ on $\mathcal{S}_d(\mathbb{R})$ such that $\left|S(\Phi_n)(z\vec{f})\right| \leq K_1 \exp\left(K_2 |z|^2 \left\|\vec{f}\right\|^2\right)$, for all $z \in \mathbb{C}$, $\vec{f} \in \mathcal{S}_d(\mathbb{R})$, $n \in \mathbb{N}$.*

Then $(\Phi_n)_{n\in\mathbb{N}}$ converges strongly in $(\mathcal{S})^*$ to a unique Hida distribution $\Phi \in (\mathcal{S})^*$.

3. Main Results

In several applications, we need to "pin" a b-Gaussian process at some point $c \in \mathbb{R}^d$. For this purpose, we consider the *Donsker's delta function* of b-Gaussian process which is defined as the informal composition of the Dirac delta distribution $\delta_d \in \mathcal{S}'(\mathbb{R}^d)$ with a d-dimensional b-Gaussian process $(X_t)_{t\in[0,T]}$, i.e., $\delta_d(X_t - c)$. We can give a precise meaning to the Donskers's delta function as a Hida distribution.

Proposition 3.1. *Let $X = (X_t)_{t\in[0,T]}$ be a d-dimensional b-Gaussian process and $c \in \mathbb{R}^d$. The Bochner integral*

$$\delta_d(X_t - X_s - c) := \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp\left(i\lambda\left(X_t - X_s - c\right)\right) d\lambda, \quad t \neq s$$

is a Hida distribution with S-transform given by

$$\begin{aligned} &S(\delta_d(X_t - X_s - c))(\vec{f}) \\ &= \left(\frac{1}{2\pi \int_{s\wedge t}^{s\vee t} |f(u)|^2 \, du}\right)^{d/2} \\ &\quad \times \exp\left(-\frac{1}{2\int_{s\wedge t}^{s\vee t} |f(u)|^2 \, du} \sum_{j=1}^d \left(\int_{s\wedge t}^{s\vee t} f_j(u) f(u) \, du - c_j\right)^2\right), \end{aligned} \tag{3}$$

for all $\vec{f} = (f_1, \ldots, f_d) \in \mathcal{S}_d(\mathbb{R})$.

Proof. Without loss of generality, we may assume $t > s$. Define a mapping $F_\lambda : \mathcal{S}_d(\mathbb{R}) \to \mathbb{C}$ by $F_\lambda := S\left(\exp\left(i\lambda(X_t - X_s - c)\right)\right)$. Then we have for any $\vec{f} \in \mathcal{S}_d(\mathbb{R})$

$$F_\lambda(\vec{f}) = \left\langle\left\langle \exp\left(i\lambda\left(\left\langle \cdot, \mathbf{1}_{[0,t]} f\right\rangle - \left\langle \cdot, \mathbf{1}_{[0,s]} f\right\rangle - c\right)\right), : \exp\left(\left\langle \cdot, \vec{f}\right\rangle\right) : \right\rangle\right\rangle$$

$$= \exp\left(-\frac{1}{2}\left|\vec{f}\right|_0^2\right)\exp\left(-i\lambda c\right)\int_{\mathcal{S}'_d(\mathbb{R})}\exp\left(\left\langle\vec{\omega}, i\lambda\mathbf{1}_{[s,t]}f+\vec{f}\right\rangle\right)\,d\mu(\vec{\omega})$$
$$= \exp\left(-\frac{1}{2}\left|\lambda\right|^2\int_s^t |f(u)|^2\,du\right)\exp\left(i\lambda\left(\left\langle\vec{f},\mathbf{1}_{[s,t]}f\right\rangle - c\right)\right).$$

The mapping $\lambda \mapsto F_\lambda(\vec{f})$ is measurable for all $\vec{f}\in\mathcal{S}_d(\mathbb{R})$. Furthermore, let $z\in\mathbb{C}$ and $\vec{f}\in\mathcal{S}_d(\mathbb{R})$, then

$$\begin{aligned}\left|F_\lambda(z\vec{f})\right| &\le \exp\left(-\frac{1}{2}|\lambda|^2\int_s^t |f(u)|^2\,du\right)\exp\left(|\lambda||z|\left|\left\langle\vec{f},\mathbf{1}_{[s,t]}f\right\rangle\right|\right)\\ &\le \exp\left(-\frac{1}{2}|\lambda|^2\int_s^t |f(u)|^2\,du\right)\exp\left(|\lambda||z|\beta(t-s)\sum_{j=1}^d\sup_{u\in\mathbb{R}}|f_j(u)|\right)\\ &\le \exp\left(-\frac{1}{4}|\lambda|^2\int_s^t |f(u)|^2\,du\right)\exp\left(\frac{\beta^2(t-s)^2}{\int_s^t|f(u)|^2\,du}|z|^2\|\vec{f}\|_\infty^2\right)\\ &\le \exp\left(-\frac{1}{4}|\lambda|^2\alpha^2(t-s)\right)\exp\left(\frac{\beta^2}{\alpha^2}T|z|^2\|\vec{f}\|_\infty^2\right),\end{aligned}$$

for some positive constants α and β, and $\|\cdot\|_\infty$ is a continuous seminorm on $\mathcal{S}_d(\mathbb{R})$ defined as

$$\|\vec{f}\|_\infty := \sum_{j=1}^d \sup_{u\in\mathbb{R}}|f_j(u)|.$$

The first factor is an integrable function of λ, and the second factor is constant with respect to λ. Hence, according to the Corollary 2.2 $\delta_d\left(X_t - X_s - c\right)\in(\mathcal{S})^*$. Now, we integrate F_λ over $\mathbb{R}^d$ to obtain an explicit expression for the S-transform.

$$\begin{aligned}&S\left(\delta_d\left(X_t-X_s-c\right)\right)(\vec{f})\\ &=\left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^d} S\left(\exp\left(i\lambda(X_t-X_s-c)\right)\right)(\vec{f})\,d\lambda\\ &=\left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^d}\exp\left(-\frac{1}{2}\left|\lambda\right|^2\int_s^t|f(u)|^2\,du\right)\exp\left(i\lambda\left(\left\langle\vec{f},\mathbf{1}_{[s,t]}f\right\rangle - c\right)\right)d\lambda\\ &=\left(\frac{1}{2\pi}\right)^d\left(\frac{2\pi}{\int_s^t|f(u)|^2\,du}\right)^{d/2}\prod_{j=1}^d\exp\left(\frac{\left(i\left(\int_s^t f_j(u)f(u)\,du - c_j\right)\right)^2}{2\int_s^t|f(u)|^2\,du}\right)\\ &=\left(\frac{1}{2\pi\int_s^t|f(u)|^2\,du}\right)^{d/2}\exp\left(-\frac{1}{2\int_s^t|f(u)|^2\,du}\sum_{j=1}^d\left(\int_s^t f_j(u)f(u)\,du - c_j\right)^2\right).\blacksquare\end{aligned}$$

Now we prove our main results on self-intersection local times $L_X(T)$ and their subtracted counterparts $L_X^{(N)}(T)$. In the sequel we fix the following notations: $\Delta := \{(s,t)\in\mathbb{R}^2 : 0<s<t<T\}$ and $d^2(s,t)$ is the Lebesgue measure on Δ. For

simplicity, we also take $c = 0$. Moreover, we define the truncated exponential series

$$\exp^{(N)}(x) := \sum_{m=N}^{\infty} \frac{x^m}{m!}$$

and the truncated Donsker's delta function

$$\delta_d^{(N)}(X_t - X_s)$$

via

$$S(\delta_d^{(N)}(X_t - X_s))(\vec{f}) = \left(\frac{1}{2\pi \int_s^t |f(u)|^2\,du}\right)^{d/2} \exp^{(N)}\left(-\frac{1}{2\int_s^t |f(u)|^2\,du} \sum_{j=1}^{d} \left(\int_s^t f_j(u) f(u)\,du\right)^2\right),$$

for every $\vec{f} \in \mathcal{S}_d(\mathbb{R})$. Using Theorem 2.1, one can verify easily that $\delta_d^{(N)}(X_t - X_s)$ is a well-defined element from $(\mathcal{S})^*$.

Theorem 3.2. *Let $X = (X_t)_{t\in[0,T]}$ be a d-dimensional b-Gaussian process. For any pair of integers $d \geq 1$ and $N \geq 0$ such that $2N > d - 2$, the Bochner integral*

$$L_X^{(N)}(T) := \int_{\Delta} \delta_d^{(N)}(X_t - X_s)\; d^2(s,t)$$

is a Hida distribution.

PROOF. From the definition of the truncated Donsker's delta function we see immediately that $S\left(\delta_d^{(N)}(X_t - X_s)\right)(\vec{f})$ is a measurable function for every $\vec{f} \in \mathcal{S}_d(\mathbb{R})$. Furthermore, for every $z \in \mathbb{C}$ and $\vec{f} \in \mathcal{S}_d(\mathbb{R})$

$$\begin{aligned}
&\left|S\left(\delta_d^{(N)}(X_t - X_s)\right)(z\vec{f})\right| \\
&\leq \left(\frac{1}{2\pi \int_s^t |f(u)|^2\,du}\right)^{d/2} \exp^{(N)}\left(\frac{\beta^2 (t-s)^2}{2\int_s^t |f(u)|^2\,du} |z|^2 \left\|\vec{f}\right\|_\infty^2\right) \\
&\leq \left(\frac{1}{2\pi\alpha^2}\right)^{d/2} \frac{1}{(t-s)^{d/2}} \exp^{(N)}\left(\frac{\beta^2}{2\alpha^2}(t-s)|z|^2 \left\|\vec{f}\right\|_\infty^2\right) \\
&\leq \left(\frac{1}{2\pi\alpha^2}\right)^{d/2} \left(\frac{1}{T}\right)^{N} (t-s)^{N-d/2} \exp\left(\frac{\beta^2 T}{2\alpha^2}|z|^2 \left\|\vec{f}\right\|_\infty^2\right),
\end{aligned}$$

where $(t-s)^{N-d/2}$ is integrable with respect to $d^2(s,t)$ on Δ if and only if $N - d/2 > -1$. Therefore we can conclude, using Corollary 2.2, that $L_X^{(N)}(T) \in (\mathcal{S})^*$. ■

Moreover, we are able to derive the chaos decomposition for the (truncated) self-intersection local times $L_X^{(N)}(T)$.

Proposition 3.3. *Let $X = (X_t)_{t\in[0,T]}$ be a d-dimensional b-Gaussian process. For any pair of integers $d \geq 1$ and $N \geq 0$ such that $2N > d-2$, the kernel functions $F_{2\mathbf{m}}$ of $L_X^{(N)}(T)$ are given by*

$$F_{2\mathbf{m}}(u_1, \dots, u_{2m}) = \frac{\left(-\frac{1}{2}\right)^m}{\mathbf{m}!} \left(\frac{1}{2\pi}\right)^{d/2} \int_\Delta \frac{\prod_{l=1}^{2m} \left(\mathbf{1}_{[s,t]} f\right)(u_l)}{\left(\int_s^t |f(u)|^2\, du\right)^{m+d/2}}\, d^2(s,t)$$

for each $\mathbf{m} \in \mathbb{N}_0^d$ such that $m \geq N$. All other odd kernel functions $F_{\mathbf{m}}$ vanish.

Proof. Let $\vec{f} = (f_1, \dots, f_d) \in \mathcal{S}_d(\mathbb{R})$. The S-transform of $L_X^{(N)}(T)$ is obtained as follow:

$$\begin{aligned} S\left(L_X^{(N)}(T)\right)(\vec{f}) = \int_\Delta & \frac{1}{\left(2\pi \int_s^t |f(u)|^2\, du\right)^{d/2}} \\ & \times \exp^{(N)}\left(-\frac{1}{2\int_s^t |f(u)|^2\, du} \sum_{j=1}^d \left(\int_s^t f_j(u) f(u)\, du\right)^2\right) d^2(s,t) \\ = \left(\frac{1}{2\pi}\right)^{d/2} & \int_\Delta \sum_{m=N}^\infty \left(-\frac{1}{2}\right)^m \frac{1}{\left(\int_s^t |f(u)|^2\, du\right)^{m+d/2}} \\ & \times \sum_{\substack{m_1,\dots,m_d \\ m_1+\dots+m_d=m}} \frac{1}{\mathbf{m}!} \prod_{j=1}^d \left(\int_s^t f_j(u) f(u)\, du\right)^{2m_j} d^2(s,t). \end{aligned}$$

Remember the general form of the chaos decomposition

$$L_X^{(N)}(T) = \sum_{\mathbf{m}\in\mathbb{N}_0^d} \langle : \vec{\omega}^{\otimes \mathbf{m}} :, F_{\mathbf{m}} \rangle \quad \text{and} \quad S\left(L_X^{(N)}(T)\right)(\vec{f}) = \sum_{\mathbf{m}\in\mathbb{N}_0^d} \left\langle F_{\mathbf{m}}, \vec{f}^{\otimes \mathbf{m}} \right\rangle.$$

Hence, we can read off the kernel functions $F_{\mathbf{m}}$ for $L_X^{(N)}(T)$:

$$F_{\mathbf{m}} = \frac{\left(-\frac{1}{2}\right)^m}{\mathbf{m}!} \left(\frac{1}{2\pi}\right)^{d/2} \int_\Delta \frac{\left(\mathbf{1}_{[s,t]} f\right)^{\otimes 2m}}{\left(\int_s^t |f(u)|^2\, du\right)^{m+d/2}}\, d^2(s,t).$$

More precisely, for every $\mathbf{m} \in \mathbb{N}_0^d$ such that $m \geq N$ and $u_1, \dots, u_{2m} \in \mathbb{R}$ it holds

$$F_{2\mathbf{m}}(u_1, \dots, u_{2m}) = \frac{\left(-\frac{1}{2}\right)^m}{\mathbf{m}!} \left(\frac{1}{2\pi}\right)^{d/2} \int_\Delta \frac{\prod_{l=1}^{2m} \left(\mathbf{1}_{[s,t]} f\right)(u_l)}{\left(\int_s^t |f(u)|^2\, du\right)^{m+d/2}}\, d^2(s,t),$$

while all other odd kernels $F_{\mathbf{m}}$ are identically equal to zero. ■

Theorem 3.2 asserts that for one-dimensional b-Gaussian process all self-intersection local times $L_X^{(N)}(T)$ are well-defined as Hida distributions. For $d \geq 2$, self-intersection local times only become well-defined after omission of the divergent terms. In particular, we obtain the generalized expectation of renormalized self-intersection local times $L_X^{(N)}(T)$ which is given by

$$\mathbb{E}_\mu(L_X^{(N)}(T)) = F_{\mathbf{0}} = \left(\frac{1}{2\pi}\right)^{d/2} \int_\Delta \frac{1}{\left(\int_s^t |f(u)|^2\,du\right)^{d/2}}\, d^2(s,t).$$

It is immediate that the generalized expectation is finite only in dimension one, and for higher dimension ($d \geq 2$) the expectation blows up. Now we extend the result in Theorem 3.2 to local times of intersection of higher order $m \in \mathbb{N}$. The basic motivation for this investigation comes from the situation when we want to count the amount of time in which the sample path of a b-Gaussian process spends intersect itself m-times within the time interval $[0, T]$. The following theorem gives a generalization to a result of Mendonca and Streit [8] on Brownian motion.

Theorem 3.4. *Let $X = (X_t)_{t\in[0,T]}$ be a b-Gaussian process. For any pair of integers $d \geq 1$ and $N \geq 0$ such that $2N > d-2$ the (truncated) m-tuple intersection local time of X*

$$L_{X,m}^{(N)}(T) := \int_{\Delta_m} \delta_d^{(N)}(X_{t_2} - X_{t_2}) \ldots \delta_d^{(N)}(X_{t_m} - X_{t_{m-1}})\, d^m t, \tag{4}$$

where $\Delta_m := \{(t_1, \cdots, t_m) \in \mathbb{R}^m : 0 < t_1 < \cdots < t_m < T\}$ and $d^m t$ denotes the Lebesgue measure on Δ_m, is a Hida distribution.

PROOF. Let

$$I_m := \prod_{k=1}^{m-1} \delta_d^{(N)}(X_{t_{k+1}} - X_{t_k})$$

denote the integrand in (4). Then

$$S(I_m)(\vec{f}) = \prod_{k=1}^{m-1} \left(\left(\frac{1}{2\pi \int_{t_k}^{t_{k+1}} |f(u)|^2\,du} \right)^{d/2} \times \exp^{(N)}\left(-\frac{1}{2\int_{t_k}^{t_{k+1}} |f(u)|^2\,du} \sum_{j=1}^d \left(\int_{t_k}^{t_{k+1}} f_j(u) f(u)\,du \right)^2 \right)\right),$$

which is a measurable function for every $\vec{f} \in \mathcal{S}_d(\mathbb{R})$. Now, we check the boundedness condition. Let $z \in \mathbb{C}$ and $\vec{f} \in \mathcal{S}_d(\mathbb{R})$. Let us also define a continuous seminorm $|\cdot|_{\infty,k}$ on $\mathcal{S}(\mathbb{R})$ by $|f_j|_{\infty,k} := \sup_{u\in[t_k,t_{k+1}]} |f_j(u)|$ and a continuous seminorm $\|\cdot\|_{\infty,k}$ on $\mathcal{S}_d(\mathbb{R})$ by $\|\vec{f}\|_{\infty,k}^2 := \sum_{j=1}^d |f_j|_{\infty,k}^2$. Then, we have

$$\left|S(I_m)(z\vec{f})\right| = \prod_{k=1}^{m-1} \left(\left(\frac{1}{2\pi \int_{t_k}^{t_{k+1}} |f(u)|^2\,du} \right)^{d/2} \times \exp^{(N)}\left(\frac{1}{2\int_{t_k}^{t_{k+1}} |f(u)|^2\,du} |z|^2 \|\vec{f}\|_{\infty,k}^2 \left(\int_{t_k}^{t_{k+1}} |f(u)|\,du \right)^2 \right)\right)$$

$$\leq \prod_{k=1}^{m-1} \left(\frac{1}{2\pi\alpha^2(t_{k+1}-t_k)} \right)^{d/2} \exp^{(N)} \left(\sum_{k=1}^{m-1} \frac{\beta^2}{2\alpha^2}(t_{k+1}-t_k)|z|^2 \|\vec{f}\|^2_{\infty,k} \right).$$

Finally, by defining another continuous seminorm $\|\cdot\|_*$ on $\mathcal{S}_d(\mathbb{R})$ by

$$\|\vec{f}\|^2_* := \sum_{k=1}^{m-1} \|\vec{f}\|^2_{\infty,k}$$

we obtain that

$$\left| S(I_m)(z\vec{f}) \right| \leq \left(\prod_{k=1}^{m-1} \left(\frac{1}{2\pi\alpha^2} \right)^{d/2} \left(\frac{1}{T} \right)^N (t_{k+1}-t_k)^{N-d/2} \right) \exp\left(p|z|^2 \|\vec{f}\|^2_* \right),$$

for some $p > 0$. To conclude the proof, we notice that for $N - d/2 + 1 > 0$ the coefficient in the front of the exponential is integrable with respect to $d^m t$, that is

$$\int_{\Delta_m} \prod_{k=1}^{m-1} (t_{k+1}-t_k)^{N-d/2} \, d^m t = \frac{(\Gamma(1+N-d/2))^{m-1}}{\Gamma\left(m+1+(m-1)(N-d/2)\right)},$$

where $\Gamma(\cdot)$ denotes the usual Gamma function. Therefore we may apply Corollary 2.2 to establish the existence of the Bochner integral asserted in the theorem. ■

To conclude the section, we present a regularization result corresponding to the renormalization procedure as described in Theorem 3.2. We define the regularized Donsker's delta function of b-Gaussian process as

$$\delta_{d,\varepsilon}(X_t - X_s) := \left(\frac{1}{2\pi\varepsilon} \right)^{d/2} \exp\left(-\frac{|X_t - X_s|^2}{2\varepsilon} \right)$$

and the corresponding regularized self-intersection local time of b-Gaussian process as

$$L_{X,\varepsilon}(T) := \int_\Delta \delta_{d,\varepsilon}(X_t - X_s) \, d^2(s,t).$$

Theorem 3.5. *Let $X = (X_t)_{t\in[0,T]}$ be a d-dimensional b-Gaussian process. For all $\varepsilon > 0$ and $d \geq 1$ the regularized self-intersection local time $L_{X,\varepsilon}(T)$ is a Hida distribution with kernel functions in the chaos decomposition given by*

$$F_{\varepsilon,\mathbf{2m}}(u_1,\ldots,u_{2m}) = \frac{\left(-\frac{1}{2}\right)^m}{\mathbf{m}!} \left(\frac{1}{2\pi} \right)^{d/2} \int_\Delta \frac{\prod_{l=1}^{2m} \left(\mathbf{1}_{[s,t]} f\right)(u_l)}{\left(\varepsilon + \int_s^t |f(u)|^2 \, du\right)^{m+d/2}} \, d^2(s,t)$$

for each $\mathbf{m} \in \mathbb{N}_0^d$, and $F_{\varepsilon,\mathbf{m}}$ is identically to zero if m is an odd number. Moreover, the (truncated) regularized self-intersection local times

$$L^{(N)}_{X,\varepsilon}(T) := \int_\Delta \delta^{(N)}_{d,\varepsilon}\left(X_t - X_s\right) \, d^2(s,t)$$

converges strongly as $\varepsilon \to 0$ in $(\mathcal{S})^*$ to the (truncated) local times $L^{(N)}_X(T)$, provided $2N > d - 2$.

PROOF. The first part of the proof again follows by an application of Corollary 2.2 with respect to the Lebesgue measure on Δ. For all $\vec{f} \in \mathcal{S}_d(\mathbb{R})$ we obtain

$$\begin{aligned} & S\left(\delta_{d,\varepsilon}\left(X_t - X_s\right)\right)(\vec{f}) \\ &= \left(\frac{1}{2\pi\left(\varepsilon + \int_s^t |f(u)|^2\,du\right)}\right)^{d/2} \\ &\quad \times \exp\left(-\frac{1}{2\left(\varepsilon + \int_s^t |f(u)|^2\,du\right)} \sum_{j=1}^d \left(\int_s^t f_j(u) f(u)\,du\right)^2\right), \end{aligned}$$

which is evidently measurable. Hence for all $z \in \mathbb{C}$ we have

$$\begin{aligned} & \left|S\left(\delta_{d,\varepsilon}\left(X_t - X_s\right)\right)(z\vec{f})\right| \\ &\leq \left(\frac{1}{2\pi\left(\varepsilon + \int_s^t |f(u)|^2\,du\right)}\right)^{d/2} \exp\left(\frac{(t-s)^2\beta^2}{2\left(\varepsilon + \int_s^t |f(u)|^2\,du\right)}|z|^2 \left\|\vec{f}\right\|_\infty^2\right). \end{aligned}$$

We observe that $\frac{(t-s)^2}{\varepsilon + \int_s^t |f(u)|^2\,du}$ is bounded on Δ and $\left(\frac{1}{2\pi\left(\varepsilon + \int_s^t |f(u)|^2\,du\right)}\right)^{d/2}$ is integrable on Δ. Hence, by Corollary 2.2, we have that $L_{X,\varepsilon}(T) \in (\mathcal{S})^*$. Moreover, for every $\vec{f} \in \mathcal{S}_d(\mathbb{R})$,

$$\begin{aligned} S(L_{X,\varepsilon}(T))(\vec{f}) = \left(\frac{1}{2\pi}\right)^{d/2} \int_\Delta \sum_{m=0}^\infty \left(-\frac{1}{2}\right)^m \frac{1}{\left(\varepsilon + \int_s^t |f(u)|^2\,du\right)^{m+d/2}} \\ \times \sum_{\substack{m_1,\ldots,m_d \\ m_1+\ldots+m_d=m}} \frac{1}{\mathbf{m}!} \prod_{j=1}^d \left(\int_s^t f_j(u) f(u)\,du\right)^{2m_j} d^2(s,t). \end{aligned}$$

It follows from the last expression that the kernel functions $F_{\varepsilon,\mathbf{m}}$ appearing in the chaos decomposition

$$L_{X,\varepsilon}(T) = \sum_{\mathbf{m}\in\mathbb{N}_0^d} \left\langle :\vec{\omega}^{\otimes\mathbf{m}}:, F_{\varepsilon,\mathbf{m}}\right\rangle$$

are of the form

$$F_{\varepsilon,2\mathbf{m}}(u_1,\ldots,u_{2m}) = \frac{\left(-\frac{1}{2}\right)^m}{\mathbf{m}!}\left(\frac{1}{2\pi}\right)^{d/2} \int_\Delta \frac{\prod_{l=1}^{2m}\left(\mathbf{1}_{[s,t]}f\right)(u_l)}{\left(\varepsilon + \int_s^t |f(u)|^2\,du\right)^{m+d/2}}\, d^2(s,t),$$

and are identically equal to zero if m is an odd number. Finally we have to check the convergence of $L_{X,\varepsilon}^{(N)}(T)$ as $\varepsilon \to 0$. For all $z \in \mathbb{C}$ and all $\vec{f} \in \mathcal{S}_d(\mathbb{R})$ we have

$$\left|S\left(L_{X,\varepsilon}^{(N)}(T)\right)(z\vec{f})\right| \leq \int_\Delta \left|S\left(\delta_{d,\varepsilon}^{(N)}\left(X_t - X_s\right)\right)(z\vec{f})\right| d^2(s,t)$$

$$\leq \left(\frac{1}{2\pi\varepsilon}\right)^{d/2} \left(\frac{1}{T}\right)^{N} \exp\left(\frac{T\beta^2}{2\varepsilon}|z|^2 \left\|\vec{f}\right\|_{\infty}^{2}\right),$$

giving the boundedness condition. Furthermore, using similar calculations as in the proof of Theorem 3.2 we obtain that for all $(s,t) \in \Delta$

$$\left|S\left(\delta_{d,\varepsilon}^{(N)}\left(X_t - X_s\right)\right)(\vec{f})\right|$$
$$\leq \left(\frac{1}{2\pi\int_s^t |f(u)|^2\,du}\right)^{d/2} \exp^{(N)}\left(\frac{\beta^2}{2\int_s^t |f(u)|^2\,du}(t-s)^2 \left\|\vec{f}\right\|_{\infty}^{2}\right)$$
$$\leq \left(\frac{1}{2\pi\alpha^2}\right)^{d/2} \left(\frac{1}{T}\right)^{N} (t-s)^{N-d/2} \exp\left(\frac{\beta^2 T}{2\alpha^2}\left\|\vec{f}\right\|_{\infty}^{2}\right).$$

The last upper bound is an integrable function with respect to $d^2(s,t)$. Finally, we can apply Lebesgue's dominated convergence theorem to get the other condition needed for the application of Corollary 2.3. This finishes the proof. ■

4. Concluding Remarks

We have proved under some conditions on the number of divergent terms must be subtracted and the spatial dimension, that self-intersection local times of d-dimensional b-Gaussian process as well as their regularizations, after appropriatelly renormalized, are Hida distributions. The power of the method of truncation based on the fact that the kernel functions in the chaos decomposition of decreasing order are more and more singular in the sense of Lebesgue integrable function. Explicit expressions for the chaos decompositions of the self-intersection local times are also presented. We also remark that white noise approach provides a general idea on renormalization procedures. This idea can be further developed using another tools such as Malliavin calculus to obtain regularity results.

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