J. Indones. Math. Soc.
Vol. 29, No. 03 (2023), pp. 347–371.

# A UNIFIED DISTANCE APPROACH FOR RANKING FUZZY NUMBERS AND ITS COMPARATIVE REVIEWS

Shiv Prasad[a], Shatabdi Sinha[b]

**University Department of Mathematics,**
**Binod Bihari Mahto Koyalanchal University**
**Dhanbad- 828130, Jharkhand, India**
**[a]shpd1512@gmail.com, [b]shatabdi778@gmail.com**

**Abstract.** Even though a large number of research studies have been presented in recent years for ranking and comparing fuzzy numbers, the majority of existing techniques suffer from plenty of shortcomings. These shortcomings include counter-intuitiveness, the inability to distinguish the fuzzy number and its partnered image, and the inconsistent ability to distinguish symmetric fuzzy numbers and fuzzy numbers that represent the compensation of areas. To overcome the cited drawbacks, this paper suggests a unified distance that multiplies the centroid value (weighted mean value) of the fuzzy number on the horizontal axis and a linear sum of the distances of the centroid points of the left and right fuzziness areas from the original point through an indicator. The indicator reflects the attitude of the left and right fuzziness of the fuzzy number, we can call it the indicator of fuzziness. To use this technique, the membership functions of the fuzzy numbers need not be linear. That is the proposed approach can also rank the fuzzy numbers with non-linear membership functions. The suggested technique is highly convenient and reliable to discriminate the symmetric fuzzy numbers and the fuzzy numbers having compensation of areas. The advantages of the proposed approach are illustrated through examples that are common for a wide range of numerical studies and comparisons with several representative approaches, that existed in the literature.

*Key words and Phrases*: Fuzzy number, Ranking, Unified distance, Centroid value, Indicator of fuzziness.

## 1. Introduction

Zadeh [39] introduced and evolved the fuzzy set as a useful tool for the mathematical representation of uncertainty and ambiguity to manage imprecise knowledge efficiently. One of the most practical uses of fuzzy sets is in decision-making,

*2020 Mathematics Subject Classification*: 03E72; 94D05
Received: 09-12-2022, accepted: 02-11-2023.

which is a logical and reasonable process. The fuzzy sets describe the suitable information about the values and preferences of a decision-maker which are ambiguous. Thus, a decision is made by ordering fuzzy numbers that represent the imprecise numerical value of alternatives. However, selecting the best option from a set of alternatives in a fuzzy environment is intricate and laborious. Several methods for ordering fuzzy numbers have been suggested in the literature over the last few decades. The pioneering work of ranking imprecise quantities characterized as fuzzy sets for selecting an optimal alternative was first proposed by Jain [20]. Dubois and Prade [17] demonstrated the notion of fuzzy number and the associated fuzzy arithmetic. Yager [33], [34], [35] evolved the concept of ranking fuzzy numbers as fuzzy subsets over the unit interval based on the centroid point, the linear distance between the sets, and the mean value, respectively. Since then, the literature acknowledges a large number of suggestions for ordering fuzzy numbers using various notions. S-H Chen [8] proposed the maximizing set and minimizing set approach. Bortolan and Degani [7] compared and reviewed some of the methods of ranking fuzzy numbers. There are numerous approaches for ranking fuzzy numbers based on specific notions which are suitable in certain situations. Liou and Wang [21] introduced an indexing technique based on total integral values and also considers the decision-maker's choice. Choobineh and Li [11] suggested an indexing technique based on the left and right areas of the fuzzy number. Fortemps and Roubens [18] presented a ranking technique using the compensation of areas. Using the centroid point and original point, Cheng [9] suggested the distance technique while Chu and Tsao [12] presented the area method for ranking fuzzy numbers. Deng et al. [15] suggested a ranking method for fuzzy numbers based on the radius of gyration. Abbasbandy and Asady [1] proposed a sign distance approach by using the parametric form of the fuzzy number. Asady et al. [5] suggested a distance minimization method for ranking fuzzy numbers. Garcia and Lamata [31] endorsed an alteration in the index of Liou and Wang [21] by using the mode area integral and the index of modality. Wang and Lee [32] suggested a revision in Chu and Tsao [12] based on the importance of the degree of the representative location of fuzzy numbers on the real line. Abbasbandy et al. [2] proposed a magnitude for the fuzzy numbers for their ranking. Asady [4] proposed a revised procedure for distance minimization [5]. Yu et al. [37] presented an approach based on the epsilon-deviation degree. Yu et al. [36] advised ranking general fuzzy quantities in fuzzy decision-making by utilizing the left and right transfer factors and areas. Rao et al. [29] proposed an area method using the circumcenter of the centroid to rank the fuzzy numbers. Nasseri et al. [23] presented a very good idea based on the angle between the reference functions. A new parametric method based on alpha-cut is proposed by Shureshjani and Darehmiraki [3]. Zhang et al. [40] presented a new method for ranking fuzzy numbers and its application to group decision making. To address the drawbacks of Liou and Wang [21], Yu and Dat [38] suggested a better approach for ranking fuzzy quantities with integral values. A noble approach for comparing fuzzy numbers was put out by Rezvani [30]. Nguyen [25] offered a unified index by multiplying two different discriminatory components of the fuzzy number. The ranking approach suggested by Rao et al. [29] was examined by Nasseri et al. [24]. Chutia and Chutia [14] presented a method based on the value and ambiguity of the fuzzy number with the defuzzifiers positioned at various heights. A modified epsilon-deviation degree approach was presented by Chutia [13] who also looked at several drawbacks. Chi and Yu [10] used the centroid point and suggested a ranking index to order generalized fuzzy numbers. The ranking of fuzzy numbers based on weighted distance was proposed by Q-S Mao [22]. To get over Chen [8]'s drawbacks, Rao [28] offered a fresh approach to discriminate the fuzzy numbers. Hajjari [19] suggested an index to determine the similarity of the generalized trapezoidal fuzzy quantities. A fuzzy relation-based probability was suggested by Dombi and Jonas [16] to compare the fuzzy quantities with trapezoidal reference functions. Barazandeh and Ghazanfari [6] suggested a novel strategy for ranking generalized fuzzy numbers by taking into account the left and right heights differently. Prasad and Sinha [27] and [26] suggested ranking fuzzy numbers with unified integral values and the mean value of points, respectively.

Among the different ranking methods discussed above, we found that the highly cited index approach of Liou and Wang [21], Yu and Dat [38], and a recent approach of the unified index by Nguyen [25] display inconsistency in ranking fuzzy numbers which are symmetrical about a line and represent compensation of areas. Numerical illustrations are demonstrated in Ex. 5.7. The ranking approaches of Liou and Wang [21], and Yu and Dat [38] give indistinguishable results at some level of optimism, whereas Chutia and Chutia [14], and Nguyen [25] display counter-intuitive ranking conclusions for the fuzzy numbers having a different degree of representative locations on the real axis. Numerically illustrated in Ex. 5.6. To overcome these shortcomings, this paper suggests a unified distance that multiplies the centroid value (weighted mean value) of the fuzzy number on the horizontal axis and a linear sum of the distances of the centroid points of the left and right fuzziness areas from the original point through an indicator of fuzziness which reflects the attitude of the left and right fuzziness of the fuzzy number. In the linear sum, the distance of the centroid point of the right fuzziness area from the original point is used to replicate the favourable attitude and the distance of the centroid point of the left fuzziness region is used to replicate the adverse attitude.

Apart from the above introduction, the onward task of the article is planned into the following six sections. Section 2 comprises a brief appraisal of the basic notion of the fuzzy number under the headline "Preliminaries". The proposed unified distance and the ranking scheme of the fuzzy numbers and their attributes are organized in Section 3. Section 4 contains simpler formulas suggesting the unified distance for triangular and trapezoidal fuzzy numbers. Section 5 comprises the comparative reviews to illustrate the consistency and intuitiveness strength of the proposed approach and to validate the superiority over some existing methods in the literature. The concluding remarks are furnished in section 6 at the last.

## 2. Preliminaries

To review some basic definitions and notations pertinent to the present investigation, Prasad and Sinha [26], [27] are followed.

2.1. **Generalized L-R type fuzzy number.** A fuzzy set $A$ in the set of real numbers $\mathbb{R}$ with its reference function $f_A(x)$, satisfies the listed below conditions for $a, b, \sigma, \delta \in \mathbb{R}$, (a $\leq$ b) is known as a generalized L-R type fuzzy number,

(1) $f_A(x)$ is a piece-wise continuous mapping from the real line $\mathbb{R}$ onto the interval $[0, \omega]$ where $\omega$ is a constant lying in the unit interval $[0, 1]$,
(2) $f_A(x) = 0$, for all $x \in ]-\infty, a-\sigma]$,
(3) $f_A(x)$ is continuously growing on $[a-\sigma, a]$,
(4) $f_A(x) = \omega$, for all $x \in [a, b]$,
(5) $f_A(x)$ is continuously diminishing on $[b, b+\delta]$,
(6) $f_A(x) = 0$, for all $x \in \big[b+\delta, \infty\big[$.

The generalized L-R type fuzzy number in definition 2.1 is conveniently represented by $A = (a, b, \sigma, \delta; \omega)$, and its membership function $f_A(x)$ is stated as

$$f_A(x) = \begin{cases} f_A^L(x) & ;\ x \in [a-\sigma,\ a] \\ \omega & ;\ x \in [a,\ b] \\ f_A^R(x) & ;\ x \in [b,\ b+\delta] \\ 0 & ;\ \text{otherwise} \end{cases} \tag{1}$$

where $f_A^L(x) : [a-\sigma,\ a] \longrightarrow [0,\ \omega]$ is continuously increasing, called the left membership function and $f_A^R(x) : [b,\ b+\delta] \longrightarrow [0,\ \omega]$ is continuously decreasing, called the right membership functions of $A = (a,\ b,\ \sigma,\ \delta;\ \omega)$.

2.2. **Image of a generalized L-R type fuzzy number.** The image of a generalized L-R type fuzzy number $A = (a, b, \sigma, \delta; \omega)$ with respect to the axis of membership is denoted by $A^{'}$ and defined as $A^{'} = (-b, -a, \delta, \sigma; \omega)$ with its membership $f_{A'}(x)$ stated as

$$f_{A'}(x) = \begin{cases} f_{A'}^L(x) & ;\ x \in [-b-\delta,\ -b] \\ \omega & ;\ x \in [-b,\ -a] \\ f_{A'}^R(x) & ;\ x \in [-a,\ -a+\sigma] \\ 0 & ;\ \text{otherwise} \end{cases} \tag{2}$$

where $f_{A'}^L(x) : [-b-\delta,\ -b] \longrightarrow [0,\ \omega]$ is continuously increasing left membership function and $f_{A'}^R(x) : [-a,\ -a+\sigma] \longrightarrow [0,\ \omega]$ is continuously decreasing right membership function of $A^{'} = (-b,\ -a,\ \delta,\ \sigma;\ \omega)$.

2.3. **Generalized trapezoidal fuzzy number.** An L-R type fuzzy number $A = (a,\ b,\ \sigma,\ \delta;\ \omega)$ is called generalized trapezoidal fuzzy quantity if its reference function $f_A(x)$ is stated as

$$f_A(x) = \begin{cases} \frac{\omega(x-a+\sigma)}{\sigma} & ;\ x \in [a-\sigma,\ a] \\ \omega & ;\ x \in [a,\ b] \\ \frac{\omega(x-b-\delta)}{-\delta} & ;\ x \in [b,\ b+\delta] \\ 0 & ;\ \text{otherwise} \end{cases} \tag{3}$$

2.4. **Image of trapezoidal fuzzy number.** The image of trapezoidal fuzzy number $A = (a, b, \sigma, \delta; \omega)$ with respect to the membership axis is denoted by $A'$ and termed as $A' = (-b, -a, \delta, \sigma; \omega)$ with its reference function $f_{A'}(x)$ stated as

$$f_{A'}(x) = \begin{cases} \frac{\omega(x+b+\delta)}{\delta} & ;\ x \in [-b-\delta,\ -b] \\ \omega & ;\ x \in [-b,\ -a] \\ \frac{\omega(x+a-\sigma)}{-\sigma} & ;\ x \in [-a,\ -a+\sigma] \\ 0 & ;\ \text{otherwise} \end{cases} \tag{4}$$

2.5. **Generalized triangular fuzzy number.** A generalized L-R type fuzzy number $A = (a,\ b,\ \sigma,\ \delta;\ \omega)$ is said to be generalized triangular if $a = b$, simply symbolized as $A = (a,\ \sigma,\ \delta;\ \omega)$ or $A = (a,\ a,\ \sigma,\ \delta;\ \omega)$ and its membership function $f_A(x)$ is given by

$$f_A(x) = \begin{cases} \frac{\omega(x-a+\sigma)}{\sigma} & ;\ x \in [a-\sigma,\ a] \\ \omega & ;\ x = a \\ \frac{\omega(x-a-\delta)}{-\delta} & ;\ x \in [a,\ a+\delta] \\ 0 & ;\ \text{otherwise} \end{cases} \tag{5}$$

2.6. **Image of triangular fuzzy number.** The image of triangular fuzzy number $A = (a, \sigma, \delta; \omega)$ with respect to the membership axis is denoted by $A^{'}$ and termed as $A^{'} = (-a, \delta, \sigma; \omega)$ with its membership function $f_{A'}(x)$ stated as

$$f_{A'}(x) = \begin{cases} \frac{\omega(x+a+\delta)}{\delta} & ;\ x \in [-a-\delta,\ -a] \\ \omega & ;\ x = -a \\ \frac{\omega(x+a-\sigma)}{-\sigma} & ;\ x \in [-a,\ -a+\sigma] \\ 0 & ;\ \text{otherwise} \end{cases} \tag{6}$$

## 3. Unified Distance and Proposed Ranking Algorithm

In this section, a unified distance that multiplies the centroid value (weighted mean value) of the fuzzy number on the horizontal axis and the linear sum of the distances of centroid points of left-right fuzziness areas from the original point with an indicator that reflects the attitude of fuzziness derived after a brief overview of basic terms.

3.1. **Centroid value (weighted mean) of the fuzzy number.** Let $\overline{x}_A$ denote the centroid value of the fuzzy number $A = (a,\ b,\ \sigma,\ \delta;\ \omega)$ on the horizontal axis, then its value is given by Cheng [9] as

$$\overline{x}_A = \frac{\int_{a-\sigma}^{b+\delta} x\ f_A\,(x)\,dx}{\int_{a-\sigma}^{b+\delta} f_A\,(x)\,dx}. \tag{7}$$

3.2. **Centroid value (weighted mean) of the image of the fuzzy number.** Again we symbolize the centroid value of the image $A^{'} = (-b,\ -a,\ \delta,\ \sigma;\ \omega)$ of the fuzzy number $A = (a,\ b,\ \sigma,\ \delta;\ \omega)$ by $\overline{x}_{A'}$, then we have

$$\overline{x}_{A'} = \frac{\int_{-b-\delta}^{-a+\sigma} x\ f_{A'}\,(x)\,dx}{\int_{-b-\delta}^{-a+\sigma} f_{A'}\,(x)\,dx}. \tag{8}$$

3.3. **Centroid Points of the Left and Right Fuzziness Areas of the fuzzy number.** Let $C_A^L\left(\overline{x}_A^L,\ \overline{y}_A^L\right)$ and $C_A^R\left(\overline{x}_A^R,\ \overline{y}_A^R\right)$ represent, respectively the centroid point of the left and the right fuzziness areas of an L-R type fuzzy number $A = (a,\ b,\ \sigma,\ \delta;\ \omega)$, then these points are derived as follows

$$\overline{x}_A^L = \frac{\int_{a-\sigma}^{a} x\ f_A^L\,(x)\,dx}{\int_{a-\sigma}^{a} f_A^L\,(x)\,dx} \tag{9}$$

$$\overline{y}_A^L = \frac{\int_0^\omega a\ y\ dy - \int_0^\omega y\ g_A^L(y)\ dy}{\int_0^\omega a\ dy - \int_0^\omega g_A^L(y)\ dy} \tag{10}$$

$$\overline{x}_A^R = \frac{\int_b^{b+\delta} x\ f_A^R\,(x)\,dx}{\int_b^{b+\delta} f_A^R\,(x)\,dx} \tag{11}$$

$$\overline{y}_A^R = \frac{\int_0^\omega y\ g_A^R\,(y)\,dy - \int_0^\omega y\ b\ dy}{\int_0^\omega g_A^R\,(y)\,dy - \int_0^\omega b\ dy} \tag{12}$$

where $g_A^L(y)$ and $g_A^R\,(y)$ are the inverse of the left-right reference functions $f_A^L\,(x)$ and $f_A^R\,(x)$, respectively. The visual representation of these points are shown in Fig. 1.

The distances of centroid points of the left-right fuzziness areas of an L-R type fuzzy number $A = (a,\ b,\ \sigma,\ \delta;\ \omega)$ from the original point are obtained with the use of Eq. (9) to Eq. (12) as,

$$OC_A^L = \sqrt{\overline{x}_A^{L^2} + \overline{y}_A^{L^2}}, \tag{13}$$

$$OC_A^R = \sqrt{\overline{x}_A^{R^2} + \overline{y}_A^{R^2}}. \tag{14}$$

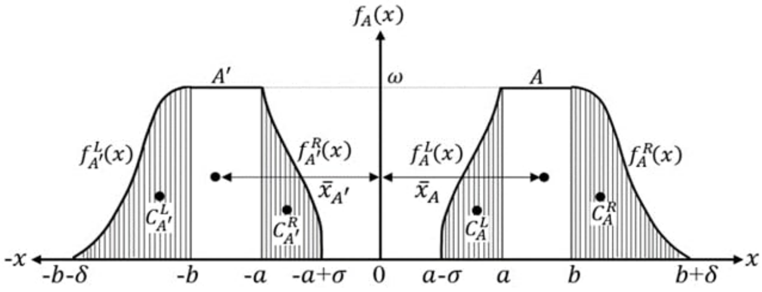


FIGURE 1. Visual representation of centroid points of the fuzziness areas and the centroid value of the fuzzy number on the horizontal axis.

3.4. **Centroid Points of the Left and Right Fuzziness Areas of the image.** Let $C^L_{A'}\left(\overline{x}^L_{A'}, \overline{y}^L_{A'}\right)$ and $C^R_{A'}\left(\overline{x}^R_{A'}, \overline{y}^R_{A'}\right)$ represent, respectively the centroid points of the left and the right fuzziness areas of the image $A' = (-b,\ -a,\ \delta,\ \sigma;\ \omega)$ of an L-R type fuzzy number $A = (a,\ b,\ \sigma,\ \delta;\ \omega)$, then these points are derived as follows:

$$\overline{x}^L_{A'} = \frac{\int_{-b-\delta}^{-b} x\ f^L_{A'}(x)\,dx}{\int_{-b-\delta}^{-b} f^L_{A'}(x)\,dx} \tag{15}$$

$$\overline{y}^L_{A'} = \frac{\int_0^{\omega} y\ g^L_{A'}(y)\,dy + \int_0^{\omega} b\ y\ dy}{\int_0^{\omega} g^L_{A'}(y)\,dy + \int_0^{\omega} b\ dy} \tag{16}$$

$$\overline{x}^R_{A'} = \frac{\int_{-a}^{-a+\sigma} x\ f^R_{A'}(x)\,dx}{\int_{-a}^{-a+\sigma} f^R_{A'}(x)\,dx} \tag{17}$$

$$\overline{y}^R_{A'} = \frac{\int_0^{\omega} y\ g^R_{A'}(y)\,dy + \int_0^{\omega} a\ y\ dy}{\int_0^{\omega} g^R_{A'}(y)\,dy + \int_0^{\omega} a\ dy} \tag{18}$$

where $g^L_{A'}(y)$ and $g^R_{A'}(y)$ are the inverse of the reference functions $f^L_{A'}(x)$, and $f^R_{A'}(x)$, respectively. The visual representation of these centroid points is also shown in Fig. 1.

The distances of centroid points of the left-right fuzziness areas of the image $A^{'}$ of an L-R type fuzzy number $A = (a,\ b,\ \sigma,\ \delta;\ \omega)$ from the original point can be obtained by using Eq. (15) to Eq. (18) as follows,

$$OC^L_{A'} = \sqrt{{\overline{x}^L_{A'}}^2 + {\overline{y}^L_{A'}}^2}, \tag{19}$$

$$OC^R_{A'} = \sqrt{{\overline{x}^R_{A'}}^2 + {\overline{y}^R_{A'}}^2}. \tag{20}$$

3.5. **Unified distance of a generalized L-R type Fuzzy Number.** The unified distance of a generalized L-R type fuzzy number $A = (a,\ b,\ \sigma,\ \delta;\ \omega)$, that multiplies the centroid value of the fuzzy number on the horizontal axis and a linear sum of the distances of the centroid points of the left and right fuzziness areas with an indicator $\eta \in [0,\ 1]$ of fuzziness, we denote it by $UD_A^\eta$ and define as,

$$UD_A^\eta \ = (\overline{x}_A + \varepsilon)\left[\eta\ OC_A^R + (1-\eta)\ \ OC_A^L\right] \tag{21}$$

where $\varepsilon$ is zero if $\overline{x}_A \neq 0$, otherwise it is a compatible positive rational number, used for comparing the fuzzy numbers which are symmetrical about the membership axis. $\overline{x}_A$, $OC_A^L$ and $OC_A^R$ are defined in Eq. (7), Eq. (13) and Eq. (14), respectively. $\eta \in [0,\ 1]$ is the indicator of fuzziness which reflects the attitude of fuzziness. $\eta$ also represents an indicator of optimism which reflects the attitude of a decision-maker. The value of $\eta$ larger than 0.5 represents the favourabe attitude of fuzziness and less than 0.5 represents the adverse attitude of fuzziness, whereas $\eta = 0.5$ reflects a neutral attitude of fuzziness. The values $\eta = 0$ and $\eta = 1$ reflect the fully adverse and fully favourbale attitudes, respectively.

3.6. **Ranking Scheme.** The unified distance $(UD_A^\eta)$ defined in Eq.(21) is used to rank the generalized L-R type fuzzy numbers $A_i = (a_i, b_i, \sigma_i, \delta_i; \omega_i)$ and $A_j = (a_j, b_j, \sigma_j, \delta_j;\ \ \omega_j);\ \ i,\ j = \ldots n$ as follows,

$$\begin{array}{ll} (1) & UD_{A_i}^\eta > UD_{A_j}^\eta \implies A_i > A_j \\ (2) & UD_{A_i}^\eta < UD_{A_j}^\eta \implies A_i < A_j \\ (3) & UD_{A_i}^\eta = UD_{A_j}^\eta \implies A_i \sim\ A_j \end{array} \tag{22}$$

We now verify the reliability properties of the unified distance $(UD_A^\eta)$ for discriminating the fuzzy numbers as well as their partnered image.

**Proposition 3.1.** If $A^{'} = (-b,\ -a,\ \delta,\ \sigma;\ \omega)$ be the partnered image of $A = (a,\ b,\ \sigma\ \delta;\ \omega)$, then,

(1) $\overline{x}_A = -\overline{x}_{A'}$
(2) $OC_A^L = OC_{A'}^R$ and $OC_A^R = OC_{A'}^L$
(3) $UD_A^\eta = -UD_{A'}^{(1-\eta)}$ and $UD_A^{(1-\eta)} = -UD_{A'}^\eta$

*Proof.* (1) Using Eq. (7) and Eq. (8), we have

$$\overline{x}_A = \frac{\int_{a-\sigma}^{b+\delta} x\ f_A\left(x\right)\ dx}{\int_{a-\sigma}^{b+\delta}\ f_A\left(x\right)\ dx} = -\frac{\int_{-b-\delta}^{-a+\sigma} x\ f_{A'}\left(x\right)\ dx}{\int_{-b-\delta}^{-a+\sigma}\ f_{A'}\left(x\right)\ dx} = -\overline{x}_{A'}$$

(2) Using Eq.(9) to Eq.(20), we have

$$\begin{aligned}\overline{x}_A^L &= \frac{\int_{a-\sigma}^{a} x\ f_A^L(x)\,dx}{\int_{a-\sigma}^{a} f_A^L(x)\,dx} \\ &= \frac{\int_{a-\sigma}^{a} x f_{A'}^R(-x)\,dx}{\int_{a-\sigma}^{a} f_{A'}^R(-x)\,dx} \\ &= -\frac{\int_{-a}^{-a+\sigma} x\ f_{A'}^R(x)\,dx}{\int_{-a}^{-a+\sigma} f_{A'}^R(x)\,dx} = -\overline{x}_{A'}^R \end{aligned} \tag{23}$$

$$\begin{aligned}\overline{x}_A^R &= \frac{\int_{b}^{b+\delta} x\ f_A^R(x)\,dx}{\int_{b}^{b+\delta} f_A^R(x)\,dx} \\ &= \frac{\int_{b}^{b+\delta} x f_{A'}^L(-x)\,dx}{\int_{b}^{b+\delta} f_{A'}^L(-x)\,dx} \\ &= -\frac{\int_{-b-\delta}^{-b} x\ f_{A'}^L(x)\,dx}{\int_{-b-\delta}^{-b} f_{A'}^L(x)\,dx} = -\overline{x}_{A'}^L \end{aligned} \tag{24}$$

$$\begin{aligned}\overline{y}_A^L &= \frac{\int_0^w (a - g_A^L(y))\ y\ dy}{\int_0^w (a - g_A^L(y))\ dy} \\ &= \frac{\int_0^w (a - (-g_{A'}^R(y)))\ y\ dy}{\int_0^w (a - (-g_{A'}^R(y)))\ dy} \\ &= \frac{\int_0^w (a + g_{A'}^R(y))\ y\ dy}{\int_0^w (a + g_{A'}^R(y))\ dy} = \overline{y}_{A'}^R \end{aligned} \tag{25}$$

$$\begin{aligned}\overline{y}_A^R &= \frac{\int_0^w (g_A^R(y) - b)\ y\ dy}{\int_0^w (g_A^R(y) - b)\ dy} \\ &= \frac{\int_0^w (-g_{A'}^L(y) - b)\ y\ dy}{\int_0^w (-g_{A'}^L(y) - b)\ dy} \\ &= \frac{\int_0^w (g_{A'}^L(y) + b)\ y\ dy}{\int_0^w (g_{A'}^L(y) + b)\ dy} = \overline{y}_{A'}^L \end{aligned} \tag{26}$$

Hence, we have

$$\sqrt{{\overline{x}_A^L}^2 + {\overline{y}_A^L}^2} = \sqrt{{\overline{x}_{A'}^R}^2 + {\overline{y}_{A'}^R}^2}$$

and

$$\sqrt{{\overline{x}_A^R}^2 + {\overline{y}_A^R}^2} = \sqrt{{\overline{x}_{A'}^L}^2 + {\overline{y}_{A'}^L}^2}$$

$$\Longrightarrow OC_A^L = OC_{A'}^R \quad \text{and} \quad OC_A^R = OC_{A'}^L$$

(3) From Eq.(21), we have

$$\begin{aligned} UD_A^{\eta} &= \overline{x}_A \left[\eta\, OC_A^R + (1-\eta)\ OC_A^L\right] \\ &= -\overline{x}_{A'} \left[\eta\, OC_{A'}^L + (1-\eta)\ OC_{A'}^R\right] \\ &= -\overline{x}_{A'} \left[(1-\eta)\ OC_{A'}^R + (1-(1-\eta))\ OC_{A'}^L\right] \ = \ -UD_{A'}^{(1-\eta)} \end{aligned}$$

Also,

$$\begin{aligned} UD_A^{(1-\eta)} &= \overline{x}_A \left[(1-\eta)\ OC_A^R + (1-(1-\eta))\ OC_A^L\right] \\ &= -\overline{x}_{A'} \left[(1-\eta)\ OC_{A'}^L + \eta\, OC_{A'}^R\right] \\ &= -\overline{x}_{A'} \left[\eta\, OC_{A'}^R + (1-\eta)\ OC_{A'}^L\right] \ = \ -UD_{A'}^{\eta} \end{aligned}$$

□

**Proposition 3.2.** If $A_i^{'} = (-b_i,\ -a_i\ , \delta_i\ , \sigma_i\ ; \omega_i)$ and $A_j^{'} = (-b_j,\ -a_j\ , \delta_j\ , \sigma_j\ ; \omega_j)$ are the partnered images of $A_i = (a_i,\ b_i,\ \sigma_i\ \delta_i;\ \omega_i)$ and $A_j = (a_j,\ b_j,\ \sigma_j\ \delta_j;\ \omega_j)$ respectively, then

(1) $UD_{A_i}^{\eta} > UD_{A_j}^{\eta}$ if and only if $UD_{A_i^{'}}^{(1-\eta)} < UD_{A_j^{'}}^{(1-\eta)}$

(2) $UD_{A_i}^{\eta} < UD_{A_j}^{\eta}$ if and only if $UD_{A_i^{'}}^{(1-\eta)} > UD_{A_j^{'}}^{(1-\eta)}$

*Proof.* (1) By Proposition 3.1

$$\begin{aligned} UD_{A_i}^{\eta} &> UD_{A_j}^{\eta} \\ \Longleftrightarrow -UD_{A_i^{'}}^{(1-\eta)} &> -UD_{A_j^{'}}^{(1-\eta)} \\ \Longleftrightarrow \ UD_{A_i^{'}}^{(1-\eta)} &< UD_{A_j^{'}}^{(1-\eta)} \end{aligned}$$

(2) By Proposition 3.2

$$\begin{aligned} UD_{A_i}^{\eta} &< UD_{A_j}^{\eta} \\ \Longleftrightarrow -UD_{A_i^{'}}^{(1-\eta)} &< -UD_{A_j^{'}}^{(1-\eta)} \\ \Longleftrightarrow \ UD_{A_i^{'}}^{(1-\eta)} &> UD_{A_j^{'}}^{(1-\eta)} \end{aligned}$$

□

## 4. UNIFIED DISTANCE OF GENERALIZED TRIANGULAR AND TRAPEZOIDAL FUZZY NUMBER

In this part, we derive the unified distance formulae for triangular and also for trapezoidal fuzzy numbers in shortened form.

4.1. **Generalized triangular fuzzy number.** In the case of triangular fuzzy number $A = (a,\ a,\ \sigma,\ \delta; \omega)$, the centroid points $C_A^L\left(\overline{x}_A^L,\ \overline{y}_A^L\right)$ and $C_A^R\left(\overline{x}_A^R,\ \overline{y}_A^R\right)$ of the left and the right fuzziness areas and the centroid value of the fuzzy number on the horizontal axis are derived by using formulas in Eq. (5), Eq. (7) and Eq. (9) to Eq. (12) as follows:

$$\overline{x}_A^L = \frac{\int_{a-\sigma}^{a} x\ f_A^L(x)\,dx}{\int_{a-\sigma}^{a} f_A^L(x)\,dx} = \frac{3a-\sigma}{3}$$

$$\overline{y}_A^L = \frac{\int_0^\omega a\ y\ dy - \int_0^\omega y\ g_A^L(y)\ dy}{\int_0^\omega a\ dy - \int_0^\omega g_A^L(y)\ dy} = \frac{\omega}{3}$$

$$\overline{x}_A^R = \frac{\int_{a}^{a+\delta} x\ f_A^R(x)\,dx}{\int_{a}^{a+\delta} f_A^R(x)\,dx} = \frac{3a+\delta}{3}$$

$$\overline{y}_A^R = \frac{\int_0^\omega y\ g_A^R(y)\,dy - \int_0^\omega y\ a\ dy}{\int_0^\omega g_A^R(y)\,dy - \int_0^\omega a\ dy} = \frac{\omega}{3}$$

$$\overline{x}_A = \frac{\int_{a-\sigma}^{a+\delta} x\ f_A(x)\ dx}{\int_{a-\sigma}^{a+\delta} f_A(x)\ dx} = \frac{3a-\sigma+\delta}{3}$$

Therefore, the distance of the centroid points of the left-right fuzziness areas of the triangular fuzzy number from the original point is obtained by using Eq. (13) and Eq. (14) as follows:

$$OC_A^L = \sqrt{\overline{x}_A^{L\,2} + \overline{y}_A^{L\,2}} = \frac{\sqrt{(3a-\sigma)^2+\omega^2}}{3}$$

$$OC_A^R = \sqrt{\overline{x}_A^{R\,2} + \overline{y}_A^{R\,2}} = \frac{\sqrt{(3a+\delta)^2+\omega^2}}{3}$$

Hence, from Eq. (21), the unified distance of the triangular fuzzy number with the index of fuzziness/optimism ($\eta$) is given by,

$$\begin{aligned} UD_A^\eta \quad &= \quad \left(\frac{3a-\sigma+\delta}{3} + \varepsilon\right) \\ &\times \quad \left[\eta\left(\frac{\sqrt{(3a+\delta)^2+\omega^2}}{3}\right) + (1-\eta)\left(\frac{\sqrt{(3a-\sigma)^2+\omega^2}}{3}\right)\right] \end{aligned} \tag{27}$$

4.2. **Generalized trapezoidal fuzzy number.** In the case of trapezoidal fuzzy number $A = (a,\ b,\ \sigma,\ \delta; \omega)$, the centroid points $C_A^L\left(\overline{x}_A^L,\ \overline{y}_A^L\right)$ and $C_A^R\left(\overline{x}_A^R,\ \overline{y}_A^R\right)$ of the left and right fuzziness areas and centroid value of the fuzzy number on the horizontal axis are derived by using formulas in Eq.(3), Eq. (7) and Eq. (9) to Eq. (12) as follows:

$$\overline{x}_A^L = \frac{\int_{a-\sigma}^{a} x\ f_A^L(x)\,dx}{\int_{a-\sigma}^{a}\ f_A^L(x)\,dx} = \frac{3a-\sigma}{3}$$
$$\overline{y}_A^L = \frac{\int_0^{\omega} a\ y\ dy - \int_0^{\omega} y\ g_A^L(y)\ dy}{\int_0^{\omega} a\ dy - \int_0^{\omega} g_A^L(y)\ dy} = \frac{\omega}{3}$$
$$\overline{x}_A^R = \frac{\int_b^{b+\delta} x\ f_A^R(x)\,dx}{\int_b^{b+\delta} f_A^R(x)\,dx} = \frac{3b+\delta}{3}$$
$$\overline{y}_A^R = \frac{\int_0^{\omega} y\ g_A^R(y)\,dy - \int_0^{\omega} y\ b\ dy}{\int_0^{\omega} g_A^R(y)\,dy - \int_0^{\omega} b\ dy} = \frac{\omega}{3}$$
$$\overline{x}_A = \frac{\int_{a-\sigma}^{b+\delta} x\ f_A(x)\ dx}{\int_{a-\sigma}^{b+\delta}\ f_A(x)\ dx} = \frac{\left[\left\{b^2+(b+\delta)(2b+\delta)\right\}-\left\{a^2+(a-\sigma)(2a-\sigma)\right\}\right]}{3\left\{2(b-a)+(\delta+\sigma)\right\}}$$

Therefore, the distance of the centroid points of the left-right fuzziness areas of the triangular fuzzy number from the original point is obtained by using Eq. (13) and Eq. (14) as follows:

$$OC_A^L = \sqrt{\overline{x}_A^{L\,2} + \overline{y}_A^{L\,2}} = \frac{\sqrt{(3a-\sigma)^2+\omega^2}}{3};\quad OC_A^R = \sqrt{\overline{x}_A^{R2} + \overline{y}_A^{R2}} = \frac{\sqrt{(3b+\delta)^2+\omega^2}}{3}$$

Hence, from Eq. (21), the unified distance of the trapezoidal fuzzy number with the index of fuzziness/optimism ($\eta$) is given by,

$$UD_A^{\eta} = \left(\frac{\left[\left\{b^2+(b+\delta)(2b+\delta)\right\}-\left\{a^2+(a-\sigma)(2a-\sigma)\right\}\right]}{3\left\{2(b-a)+(\delta+\sigma)\right\}}+\varepsilon\right)\times \left[\eta\left(\frac{\sqrt{(3b+\delta)^2+\omega^2}}{3}\right)+(1-\eta)\left(\frac{\sqrt{(3a-\sigma)^2+\omega^2}}{3}\right)\right] \tag{28}$$

**Remark 4.1.** Based on Proposition 3.1 and 3.2, the unified distance attains the steadiness to differentiate the fuzzy quantities and their partnered image. Therefore, the values of unified distance for images are not demonstrated in the comparing tables

## 5. COMPARATIVE REVIEWS

In this section, we compare the ranking results of the proposed approach with some representative methods available in the literature using fuzzy numbers samples which are common for an extensive range of comparative studies.

**Example 5.1.** Consider the ranking of a pair of normalized triangular fuzzy numbers $A_1 = (4,\ 4,\ 3,\ 1)$ and $A_2 = (3,\ 3,\ 1,\ 3)$ which are overlapped and have the compensation of areas as pictured in Fig. 2. Their partnered image $A_1^{'} = (-4,\ -4,\ 1,\ 3)$ and $A_2^{'} = (-3,\ -3,\ 3,\ 1)$ are leftward of the axis of membership. Fuzzy numbers are picked-up from Nguyen [25]. There is a challenging situation for the logic to differentiate these two fuzzy numbers due to their flipping and sliding nature. Based on Eq. (27), the unified distance values for both the triangular fuzzy numbers are found at a different level of fuzziness $\eta \in [0,\ 1]$ and demonstrated in Table 1. Based on the ranking scheme in sec. 3.6, the ranking outcomes are $A_1 > A_2$ for the index of fuzziness $0 \le \eta \le 0.4$ and $A_1 < A_2$ for $0.5 \le \eta \le 1$.The total integral values approach [21] and improved integral values method [38] are inconsistent to advocate any preference and conclude $A_1 \sim A_2$ .Yu and Dat [38] further compared these fuzzy numbers using median values and found $A_1 < A_2$.The unified index method by Nguyen [25] advocates $A_1 < A_2$ irrespective of the level of optimism, while Chutia and Chutia [14] infer $A_1 > A_2$ irrespective of the level of optimism ($\eta$). Different approaches, [35], [18], [11], [12], [1], [5], and [23] are inconsistent to make any preference and advocate $A_1 \sim A_2$. Thus, the proposed method can be used to differentiate the fuzzy numbers very accurately.

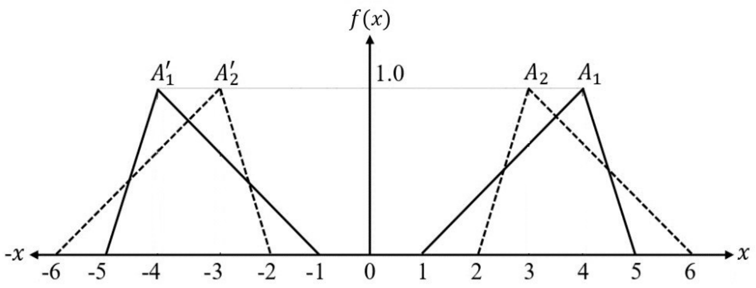


FIGURE 2. Visualization of the fuzzy quantities and their partnered images of Ex. 5.1

**Example 5.2.** Considering the following three triangular normalized fuzzy quantities, $A_1 = (6,\ 6,\ 1,\ 1)$, $A_2 = (6,\ 6,\ 0.1,\ 1)$ and $A_3 = (6,\ 6,\ 0,\ 1)$. They demonstrated congruent vertex and equal right fuzziness as visualized in Fig. 3. The partnered images $A_1^{'} = (-6,\ -6,\ 1,\ 1)$, $A_2^{'} = (-6,\ -6,\ 1,\ 0.1)$ and $A_3^{'} = (-6, -6,\ 1,\ 0)$ are leftward of the axis of membership. Based on the left fuzziness data, the reasonable ranking is $A_1 < A_2 < A_3$. Hence, this example is appropriate to explain the intuitive performance of the strategic method. Based on the formulae in Eq. (27), the unified distance values of the fuzzy numbers at different degrees of fuzziness are found and demonstrated in Table 2. Based on the ranking technique in Sec. 3.6, the ranking outcome is $A_1 < A_2 < A_3$ an arbitrary value of $\eta \in [0,\ 1]$. The ranking outcome of the total integral values approach [21] and improved integral values method [38] are in support except at the level $\eta = 1$, where they infer $A_1 \sim A_2 \sim A_3$. The unified index method of Nguyen [25] advocates the same ranking outcomes, irrespective of the level of fuzziness/optimism $\eta \in [0,\ 1]$. The index

TABLE 1. Ranking outcomes at different fuzziness/decision levels for the fuzzy quantities in Ex. 5.1

| $\eta$ | $UD^{\eta}_{A_1}$ | $UD^{\eta}_{A_2}$ | *Ranking outcomes* |
|---|---|---|---|
| 0.0 | 10.062 | 9.8540 | $A_1 > A_2$ |
| 0.1 | 10.504 | 10.340 | $A_1 > A_2$ |
| 0.2 | 10.947 | 10.827 | $A_1 > A_2$ |
| 0.3 | 11.389 | 11.313 | $A_1 > A_2$ |
| 0.4 | 11.832 | 11.799 | $A_1 > A_2$ |
| 0.5 | 12.274 | 12.286 | $A_1 < A_2$ |
| 0.6 | 12.717 | 12.772 | $A_1 < A_2$ |
| 0.7 | 13.160 | 13.259 | $A_1 < A_2$ |
| 0.8 | 13.602 | 13.745 | $A_1 < A_2$ |
| 0.9 | 14.044 | 14.231 | $A_1 < A_2$ |
| 1.0 | 14.487 | 14.718 | $A_1 < A_2$ |

approach of Chutia and Chutia [14] is consistent with the proposed method and yields the same ranking results. Different methods [35], [8], [9], [1], [5], and [23] are also consistent with the proposed method and produce the same ranking results. Thus, this example approves the strong discrimination strength of the proposed approach.

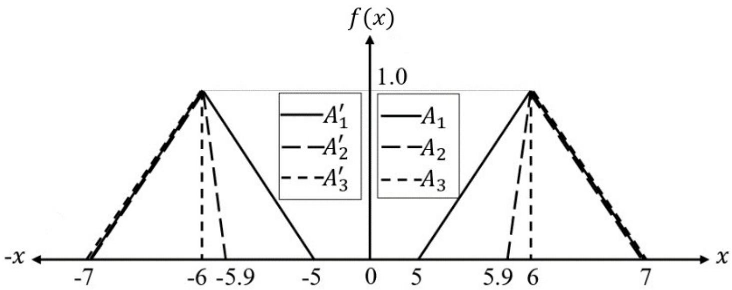


FIGURE 3. Visualization of the fuzzy quantities and their partnered images of Ex. 5.2

**Example 5.3.** Considering the three triangular normalized fuzzy numbers $A_1 = (3,\ 3,\ 2,\ 2,),\ \ A_2 = (3,\ 3,\ 1,\ 1)$ and $A_3 = (4,\ 4,\ 3,\ 2)$ .The associated images

Table 2. Ranking outcomes at different fuzziness/decision levels for the fuzzy quantities in Ex. 5.2

| $\eta$ | $UD^{\eta}_{A_1}$ | $UD^{\eta}_{A_2}$ | $UD^{\eta}_{A_3}$ | *Ranking outcomes* |
|---|---|---|---|---|
| 0.0 | 34.059 | 37.649 | 38.059 | $A_1 < A_2 < A_3$ |
| 0.1 | 34.458 | 37.879 | 38.270 | $A_1 < A_2 < A_3$ |
| 0.2 | 34.858 | 38.110 | 38.480 | $A_1 < A_2 < A_3$ |
| 0.3 | 35.257 | 38.341 | 38.691 | $A_1 < A_2 < A_3$ |
| 0.4 | 35.656 | 38.571 | 38.902 | $A_1 < A_2 < A_3$ |
| 0.5 | 36.056 | 38.802 | 39.113 | $A_1 < A_2 < A_3$ |
| 0.6 | 36.455 | 39.033 | 39.323 | $A_1 < A_2 < A_3$ |
| 0.7 | 36.855 | 39.263 | 39.534 | $A_1 < A_2 < A_3$ |
| 0.8 | 37.254 | 39.494 | 39.745 | $A_1 < A_2 < A_3$ |
| 0.9 | 37.653 | 39.725 | 39.956 | $A_1 < A_2 < A_3$ |
| 1.0 | 38.053 | 39.955 | 40.166 | $A_1 < A_2 < A_3$ |

$A_1^{'} = (-3, -3,\ 2,\ 2)$, $A_2^{'}(-3,\ -3,\ 1,\ 1)$ and $A_3^{'} = (-4,\ -4,\ 2,\ 3)$ are leftward of the reference axis. Their reference functions are pictured in Fig. 4. The vertex and right fuzziness of $A_3$ are in the right of $A_1$ and $A_2$. Therefore, logical perception prefers $A_3$ to $A_1$ and $A_2$. The intuitive perception is not precise to differentiate $A_1$ and $A_2$ because of symmetry about the line $x = 3$ and balanced left and right fuzziness areas. Using formulae in Eq. (27), the unified distance values of these triangular fuzzy numbers are obtained and demonstrated in Table 3. Based on the ranking algorithm, we find that $A_3$ leads $A_1$ and $A_2$, which approves intuitive perception. The order of $A_1$ and $A_2$ are as follows: $A_1 < A_2$ for $0 \leq \eta \leq 0.4$ and $A_1 > A_2$ for $0.5 \leq \eta \leq 1$. These results are reasonable and quite logical because the degree of fuzziness $\eta$ demonstrates the attitude of left and right fuzziness of the fuzzy number. The unified index by Nguyen [25] is consistent with the proposed approach and yields similar ranking results. The total integral values approach [21] and the modified integral values method [1] realize most of the results except at a completely pessimistic level $\eta = 0$, where they advocate unreasonably, $A_2 \sim A_3$. Chutia and Chutia [14] advocates $A_3 > A_1 > A_2$, consistent with the proposed approach irrespective of the level $\eta \in [0.5,\ 1]$. Thus, the proposed method can be utilized to rank the fuzzy numbers with confidence.

**Example 5.4.** Considering a normalized triangular fuzzy number $A_1 = (5,\ 5,\ 4,\ 0)$, overlapped on a normalized trapezoidal fuzzy number $A_2 = (3,\ 5,\ 1,\ 0)$. They exhibited the congruent vertex and equal right spreads as visualized in Fig. 5. The

TABLE 3. Ranking outcomes at different fuzziness/decision levels for the fuzzy quantities in Ex. 5.3

| $\eta$ | $UD^{\eta}_{A_1}$ | $UD^{\eta}_{A_2}$ | $UD^{\eta}_{A_3}$ | *Ranking outcomes* |
|---|---|---|---|---|
| 0.0 | 7.0710 | 8.0622 | 11.068 | $A_1 < A_2 < A_3$ |
| 0.1 | 7.4684 | 8.2610 | 11.677 | $A_1 < A_2 < A_3$ |
| 0.2 | 7.8659 | 8.4598 | 12.285 | $A_1 < A_2 < A_3$ |
| 0.3 | 8.2633 | 8.6585 | 12.894 | $A_1 < A_2 < A_3$ |
| 0.4 | 8.6608 | 8.8573 | 13.503 | $A_1 < A_2 < A_3$ |
| 0.5 | 9.0582 | 9.0561 | 14.111 | $A_2 < A_1 < A_3$ |
| 0.6 | 9.4556 | 9.2549 | 14.720 | $A_2 < A_1 < A_3$ |
| 0.7 | 9.8531 | 9.4537 | 15.329 | $A_2 < A_1 < A_3$ |
| 0.8 | 10.251 | 9.6524 | 15.938 | $A_2 < A_1 < A_3$ |
| 0.9 | 10.648 | 9.8512 | 16.546 | $A_2 < A_1 < A_3$ |
| 1.0 | 11.045 | 10.050 | 17.155 | $A_2 < A_1 < A_3$ |

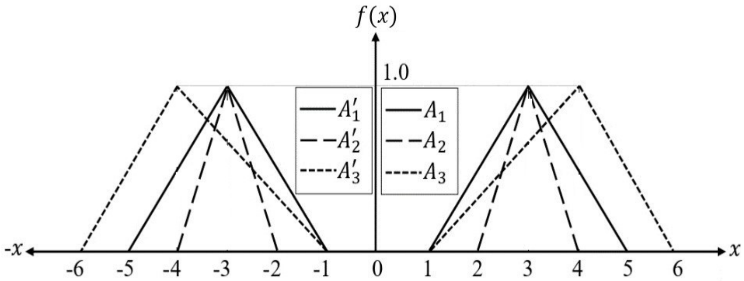


FIGURE 4. Visualization of the fuzzy quantities and their partnered images of Ex. 5.3

associated images $A_1^{'}$ and $A_2^{'}$ are leftward of the reference axis. The intuitive perception is not very clear for these fuzzy numbers. Many ranking measures that existed in the literature have publicized conflicting results. Methods, [9] and [15] advocate $A_1 < A_2$, whereas [8], [12], [1], [5] and [23] demonstrates $A_1 > A_2$. Now, we use the ranking formulae in Eq. (27) and Eq. (28), the unified distance values for the fuzzy numbers are found and shown in Table 4. Based on the ranking procedure in Sec. 3.6, the ranking outcomes are $A_1 > A_2$ for $\eta \in [0,\ 0.9]$ and $A_1 < A_2$ at the completely favourable level ($\eta = 1$). The total integral values method [21] and modified integral values method [38] are in support except at a completely optimistic level ($\eta = 1$), where they advocate $A_1 \sim A_2$ .The unified index approach of Nguyen [25] gives almost the same ranking consequences irrespective of the level of optimism. The value and ambiguity approach of Chutia and Chutia [14] is also consistent with the proposed method. Hence, this sample evaluates the effectiveness of the suggested method.

TABLE 4. Ranking outcomes at different fuzziness/decision levels for the fuzzy quantities of Ex. 5.4

| $\eta$ | $UD^{\eta}_{A_1}$ | $UD^{\eta}_{A_2}$ | *Ranking outcomes* |
|---|---|---|---|
| 0.0 | 13.500 | 10.033 | $A_1 > A_2$ |
| 0.1 | 13.987 | 10.900 | $A_1 > A_2$ |
| 0.2 | 14.475 | 11.768 | $A_1 > A_2$ |
| 0.3 | 14.962 | 12.635 | $A_1 > A_2$ |
| 0.4 | 15.450 | 13.503 | $A_1 > A_2$ |
| 0.5 | 15.937 | 14.370 | $A_1 > A_2$ |
| 0.6 | 16.425 | 15.238 | $A_1 > A_2$ |
| 0.7 | 16.912 | 16.105 | $A_1 > A_2$ |
| 0.8 | 17.399 | 16.973 | $A_1 > A_2$ |
| 0.9 | 17.887 | 17.840 | $A_1 > A_2$ |
| 1.0 | 18.374 | 18.708 | $A_1 < A_2$ |

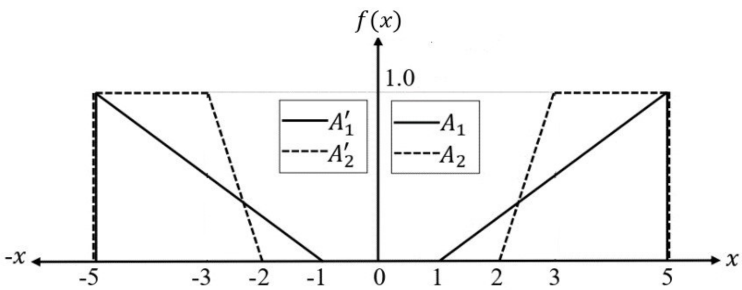


FIGURE 5. Visualization of the fuzzy quantities and their partnered images of Ex. 5.4

**Example 5.5.** Considering a trapezoidal normalized fuzzy number $A_1 = (2,\ 4,\ 2,\ 2)$ socialized with two triangular normalized fuzzy numbers $A_2 = (3,\ 3,\ 3.\ 3)$ and

$A_3 = (0,\ 0,\ 1,\ 2)$, occupied from [7], their reference functions are outlined in Fig. 6. The associated images $A_1^{'}$, $A_2^{'}$ and $A_3^{'}$ are leftward of the axis of membership. Here, $A_3$ is leftward of $A_1$ and $A_2$, hence, by intuition $A_3$ is lesser to $A_1$ and $A_2$. $A_1$ and $A_2$ are of the same height, identical left, and right fuzziness, and balanced around the line $x = 3$. Hence, the intuitive perception is unclear to advocate their order. Therefore, our task is to differentiate $A_1$ and $A_2$. Therefore, using formulae in Eq. (27) and Eq. (28), the unified distance values for the triangular and trapezoidal fuzzy numbers are obtained and demonstrated in Table 5. Based on the ranking procedure in Sec. 3.6, $A_3$ found lowest regardless of the index of fuzziness $\eta \in [0,\ 1]$, approves the logical perception. The ranking preference of $A_1$ and $A_2$ are $A_1 < A_2$ for $0 \leq \eta \leq 0.4$ and $A_1 > A_2$ for $0.5 \leq \eta \leq 1$. Liou and Wang [21], Yu and Dat [38] and Nguyen [25] advocate almost similar ranking results. The value and ambiguity approach of Chutia and Chutia [14] yields the ranking result $A_1 > A_2 > A_3$ irrespective of the level of optimism, consistent with the proposed method at optimistic levels ( $0.5 \leq \eta \leq 1$). Hence, this example judged the strength and performance of the proposed approach.

TABLE 5. Ranking outcomes at different fuzziness/decision levels for the fuzzy quantities of Ex. 5.5

| $\eta$ | $UD_{A_1}^{\eta}$ | $UD_{A_2}^{\eta}$ | $UD_{A_3}^{\eta}$ | *Ranking outcomess* |
|---|---|---|---|---|
| 0.0 | 4.1232 | 6.0828 | 0.1571 | $A_3 < A_1 < A_2$ |
| 0.1 | 5.1145 | 6.6787 | 0.1663 | $A_3 < A_1 < A_2$ |
| 0.2 | 6.1057 | 7.2746 | 0.1754 | $A_3 < A_1 < A_2$ |
| 0.3 | 7.0970 | 7.8705 | 0.1845 | $A_3 < A_1 < A_2$ |
| 0.4 | 8.0882 | 8.4664 | 0.1936 | $A_3 < A_1 < A_2$ |
| 0.5 | 9.0795 | 9.0623 | 0.2028 | $A_3 < A_2 < A_1$ |
| 0.6 | 10.071 | 9.6581 | 0.2119 | $A_3 < A_2 < A_1$ |
| 0.7 | 11.062 | 10.254 | 0.2210 | $A_3 < A_2 < A_1$ |
| 0.8 | 12.053 | 10.850 | 0.2302 | $A_3 < A_2 < A_1$ |
| 0.9 | 13.045 | 11.446 | 0.2393 | $A_3 < A_2 < A_1$ |
| 1.0 | 14.036 | 12.042 | 0.2484 | $A_3 < A_2 < A_1$ |

**Example 5.6.** Considering the three fuzzy quantities $A_1 = (7,\ 9,\ 2,\ 1; 1)$, $A_2 = (7,\ 9,\ 1,\ 1; 0.6)$ and $A_3 = (8,\ 9,\ 1,\ 1; 0.4)$ of unlike height and equal right fuzziness, occupied from [13]. The reference functions of these fuzzy numbers are visualized in Fig. 7. Their partnered images $A_1^{'}$, $A_2^{'}$ and $A_3^{'}$ are leftward of the

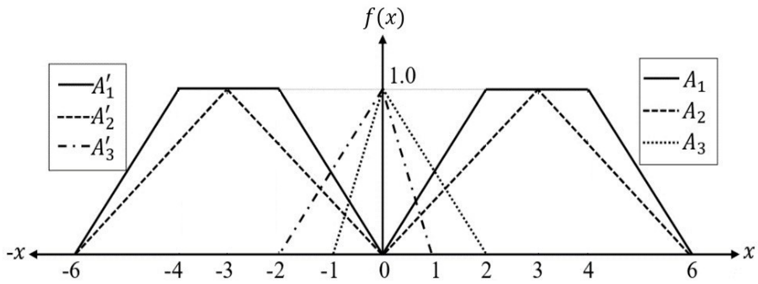


Figure 6. Visualization of the fuzzy quantities and their partnered images of Ex. 5.5

axis of membership. Their intuitive perception of preference is $A_1 < A_2 < A_3$ , based on the degree of the representative location of these fuzzy numbers on the real axis. Based on Eq. (28), the unified distances at different levels of fuzziness for these fuzzy numbers are found and demonstrated in Table 6. Based on the ranking scheme in Sec. 3.6, the ranking outcome is $A_1 < A_2 < A_3$ , irrespective of the levels of fuzziness $\eta \in [0,\ 1]$. The total integral values approach [21] concluded the same ranking orders except for $\eta = 1$, where they indiscriminate them and infer $A_1 \sim A_2 \sim A_3$. The improved integral values approach of Yu and Dat [38] gives indistinguishable criteria as $A_2 < A_1 \sim A_3$ for $\eta = 0$, $A_1 > A_2 \sim A_3$ for $\eta = 0.1$ and $A_1 > A_2 > A_3$ for $0.2 \le \eta \le 1$. The value and ambiguity approach of Chutia and Chutia [14] and the unified index approach of Nguyen [25] demonstrated counter-intuitive outcomes regardless of the levels of optimism $\eta$ and infer $A_1 > A_2 > A_3$. The area method [12] and the radius of gyration method [15] have also used these fuzzy quantities and both produced counter-intuitive ranking order as $A_1 > A_2 > A_3$. The method in [32] presents a review of the area method [12] and suggested an intuitive order. Thus, the proposed approach is a consistent strategy for making intuitive order of the fuzzy numbers.

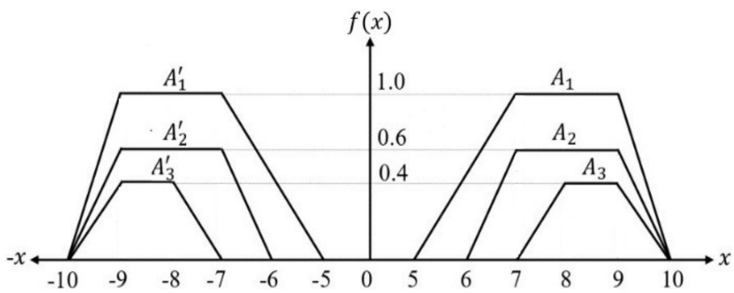


Figure 7. Visualization of the fuzzy quantities and their partnered images of Ex. 5.6

TABLE 6. Ranking outcomes at different fuzziness/decision levels for the fuzzy quantities of Ex. 5.6

| $\eta$ | $UD^{\eta}_{A_1}$ | $UD^{\eta}_{A_2}$ | $UD^{\eta}_{A_3}$ | *Ranking outcomes* |
|---|---|---|---|---|
| 0.0 | 48.925 | 53.358 | 65.176 | $A_1 < A_2 < A_3$ |
| 0.1 | 51.237 | 55.490 | 66.593 | $A_1 < A_2 < A_3$ |
| 0.2 | 53.549 | 57.623 | 68.009 | $A_1 < A_2 < A_3$ |
| 0.3 | 55.861 | 59.756 | 69.426 | $A_1 < A_2 < A_3$ |
| 0.4 | 58.173 | 61.888 | 70.842 | $A_1 < A_2 < A_3$ |
| 0.5 | 60.486 | 64.021 | 72.259 | $A_1 < A_2 < A_3$ |
| 0.6 | 62.798 | 66.153 | 73.675 | $A_1 < A_2 < A_3$ |
| 0.7 | 65.110 | 68.286 | 75.092 | $A_1 < A_2 < A_3$ |
| 0.8 | 67.422 | 70.419 | 76.509 | $A_1 < A_2 < A_3$ |
| 0.9 | 69.734 | 72.551 | 77.925 | $A_1 < A_2 < A_3$ |
| 1.0 | 72.046 | 74.684 | 79.342 | $A_1 < A_2 < A_3$ |

**Example 5.7.** We further consider the two trapezoidal fuzzy numbers $A_1 = (3,\ 5,\ 3,\ 3)$ , and $A_2 = (2,\ 6,\ 1,\ 1)$ intuitively. They are symmetrical about the line $x = 4$ and represent the compensation of areas as visualized in Fig. 8. The images of these fuzzy numbers are displayed on the left of the membership axis in the figure. Based on Eq. (28), the unified distance values for these trapezoidal fuzzy quantities are found and demonstrated in Table 7. Based on the proposed ranking scheme in Sec. 3.6, the outcomes are found, $A_1 > A_2$ for the level of fuzziness $0 \le \eta \le 0.4$ and $A_1 < A_2$ for $0.5 \le \eta \le 1$. The total integral approach of Liou and Wang [21], the improved integral values approach [38], and a unified index method [25] leads to indistinguishable criterion as $A_1 \sim A_2$ $(A_1^{'} \sim A_2^{'})$ irrespective of the levels of optimism ($\eta$). Several existing different approaches that we have studied, [35], [8], [11], [18], [9], [12], [5] and [2] advocate indistinguishable criterion as $A_1 \sim A_2$ .The ranking order of Chutia and Chutia [14] and Chutia [13] disagree with the proposed approach and advocates $A_1 < A_2$ irrespective of the decision level $0 \le \eta < 1$ and $A_1 \sim A_2$ for $\eta = 1$. As a result, the proposed method is a reliable way to generate sensible ranking order.

**Example 5.8.** Contemplating the following two sets of crisp quantities which are used in [25]. The first set comprises $A_1 = (1,\ 1,\ 0,\ 0; 0.5)$ and $A_2 = (1,\ 1,\ 0,\ 0; 1.0)$ and the other set comprises $B_1 = (0.1,\ 0.1, 0,\ 0; 0.8)$ and $B_2 = (-0.1,\ -0.1,\ 0,\ 0; 1)$. Fig. 9 presents the visual demonstration of the characteristic functions of these crisp

TABLE 7. Ranking outcomes at different fuzziness/decision levels for the fuzzy quantities of Ex. 5.7

| $\eta$ | $UD^{\eta}_{A_1}$ | $UD^{\eta}_{A_2}$ | *Ranking outcomes* |
|---|---|---|---|
| 0.0 | 8.1104 | 6.7988 | $A_1 > A_2$ |
| 0.1 | 9.7031 | 8.6558 | $A_1 > A_2$ |
| 0.2 | 11.296 | 10.513 | $A_1 > A_2$ |
| 0.3 | 12.888 | 12.370 | $A_1 > A_2$ |
| 0.4 | 14.481 | 14.227 | $A_1 > A_2$ |
| 0.5 | 16.074 | 16.084 | $A_1 < A_2$ |
| 0.6 | 17.666 | 17.941 | $A_1 < A_2$ |
| 0.7 | 19.259 | 19.798 | $A_1 < A_2$ |
| 0.8 | 20.852 | 21.654 | $A_1 < A_2$ |
| 0.9 | 22.445 | 23.511 | $A_1 < A_2$ |
| 1.0 | 24.037 | 25.368 | $A_1 < A_2$ |

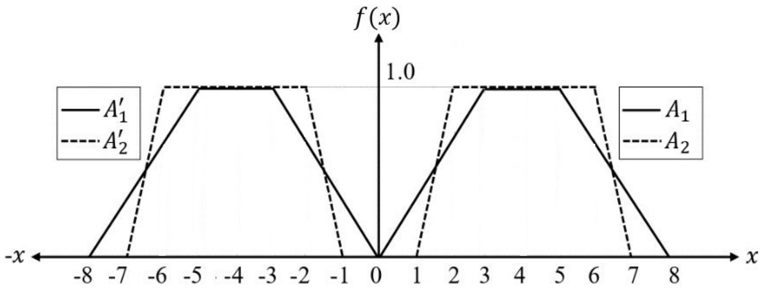


FIGURE 8. Visualization of the fuzzy quantities and their partnered images of Ex. 5.7

data. The unified distance values for these crisp quantities are demonstrated in Table 8, where we found that the indicator of fuzziness $\eta \in [0,\ 1]$ does not affect the values of the unified distance of the crisp quantities. From Table 8, $UD^{\eta}_{A_1}$; $UD^{\eta}_{A_2}$ and $UD^{\eta}_{B_1}$ ; $UD^{\eta}_{B_2}$ are scored as $UD^{\eta}_{A_1} < UD^{\eta}_{A_2}$ and $UD^{\eta}_{B_1} > UD^{\eta}_{B_2}$ respectively, irrespective of the levels of fuzziness $\eta \in [0,\ 1]$. Therefore, based on the proposed ranking scheme in Sec. 3.6, $A_1$ ; $A_2$ and $B_1$ ; $B_2$ are ranked as $A_1 < A_2$ and $B_1 > B_2$, respectively. The variance approach [30], Value and Ambiguity approach [14], and the Unified index [25] all produce identical rankings, demonstrating that the suggested method may also be applied to precise data.

TABLE 8. Ranking outcomes at different fuzziness/decision levels for the fuzzy quantities of Ex. 5.8

| $\eta$ | $UD^{\eta}_{A_1}$ | $UD^{\eta}_{A_2}$ | $UD^{\eta}_{B_1}$ | $UD^{\eta}_{B_2}$ | *Ranking outcomes* |
|---|---|---|---|---|---|
| 0.0 | 1.0307 | 1.1180 | 0.0412 | −0.0501 | $A_1 < A_2,\ B_1 > B_2$ |
| 0.1 | 1.0307 | 1.1180 | 0.0412 | −0.0501 | $A_1 < A_2,\ B_1 > B_2$ |
| 0.2 | 1.0307 | 1.1180 | 0.0412 | −0.0501 | $A_1 < A_2,\ B_1 > B_2$ |
| 0.3 | 1.0307 | 1.1180 | 0.0412 | −0.0501 | $A_1 < A_2,\ B_1 > B_2$ |
| 0.4 | 1.0307 | 1.1180 | 0.0412 | −0.0501 | $A_1 < A_2,\ B_1 > B_2$ |
| 0.5 | 1.0307 | 1.1180 | 0.0412 | −0.0501 | $A_1 < A_2,\ B_1 > B_2$ |
| 0.6 | 1.0307 | 1.1180 | 0.0412 | −0.0501 | $A_1 < A_2,\ B_1 > B_2$ |
| 0.7 | 1.0307 | 1.1180 | 0.0412 | −0.0501 | $A_1 < A_2,\ B_1 > B_2$ |
| 0.8 | 1.0307 | 1.1180 | 0.0412 | −0.0501 | $A_1 < A_2,\ B_1 > B_2$ |
| 0.8 | 1.0307 | 1.1180 | 0.0412 | −0.0501 | $A_1 < A_2,\ B_1 > B_2$ |
| 0.9 | 1.0307 | 1.1180 | 0.0412 | −0.0501 | $A_1 < A_2,\ B_1 > B_2$ |

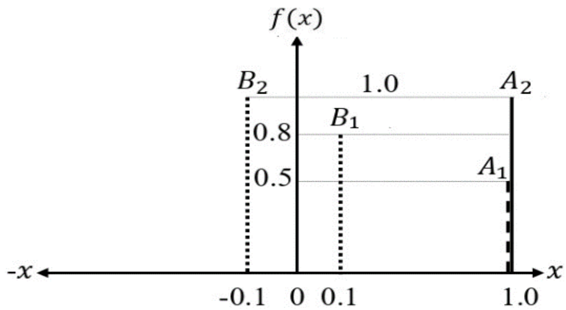


FIGURE 9. Visualization of the crisp quantities of Ex. 5.8

**Example 5.9.** Contemplating a triangular normalized fuzzy quantity $A_1 = (2,\ 2,\ 1,\ 3)$ and a general fuzzy quantity $A_2 = (2,\ 2,\ 1,\ 2; 1)$ with non-rectilinear reference function expressed by

$$f_{A_2}(x) = \begin{cases} \sqrt{1-(x-2)^2} & ;\ 1 \le x \le 2 \\ \sqrt{1-\frac{1}{4}(x-2)^2} & ;\ 2 \le x \le 4 \\ 0 & ;\ \text{otherwise} \end{cases}$$

taken from [21]. Fig. 10 presents the visual demonstration of their membership functions. The intuitive perception realizes on $A_1 > A_2$ based on the right spread. For the fuzzy number $A_2 = (1,\ 2,\ 2,\ 4; 1)$, using formulas in Eq. (9) to Eq. (14), the distances of centroids of the left and right fuzziness areas are obtained as follows,

$$\overline{x}^L_{A_2} = \frac{\int_1^2 x\ f^L_{A_2}(x)\,dx}{\int_1^2 f^L_{A_2}(x)\,dx} = \frac{\int_1^2 x\ \sqrt{1-(x-2)^2}\ dx}{\int_1^2\ \sqrt{1-(x-2)^2}\ dx} = 1.5756$$

$$\overline{y}^L_{A_2} = \frac{\int_0^1 2\ y\ dy - \int_0^1 y\ g^L_{A_2}(y)\,dy}{\int_0^1 2\ dy - \int_0^1 g^L_{A_2}(y)\,dy} = \frac{\int_0^1 2\ y\ dy - \int_0^1 y\ \left(2-\sqrt{1-y^2}\right) dy}{\int_0^1 2\ dy - \int_0^1 \left(2-\sqrt{1-y^2}\right) dy} = 0.4244$$

$$\overline{x}^R_{A_2} = \frac{\int_2^4 x\ f^R_{A_2}(x)\,dx}{\int_2^4 f^R_{A_2}(x)\,dx} = \frac{\int_2^4 x\ \sqrt{1-\ \frac{1}{4}(x-2)^2}\ dx}{\int_2^4\ \sqrt{1-\ \frac{1}{4}(x-2)^2}\ dx} = 2.8488$$

$$\overline{y}^R_{A_2} = \frac{\int_0^1 y\ g^R_{A_2}(y)\,dy - \int_0^1 2y\ dy}{\int_0^1 g^R_{A_2}(y)\,dy - \int_0^1 2\ dy} = \frac{\int_0^1 y\ \left(2+2\sqrt{1-y^2}\right)\ dy - \int_0^1 2y\ dy}{\int_0^1 \left(2+2\sqrt{1-y^2}\right) dy - \int_0^1 2\ dy} = 0.4244$$

$$\overline{x}_{A_2} = \frac{\int_1^4 x\ f_{A_2}(x)\ dx}{\int_1^4\ f_{A_2}(x)\ dx} = \frac{\int_1^2 x\ \sqrt{1-(x-2)^2}\ dx + \int_2^4\ x\sqrt{1-\frac{1}{4}(x-2)^2}\ dx}{\int_1^2\ \sqrt{1-(x-2)^2}\ dx + \int_2^4\ \sqrt{1-\frac{1}{4}(x-2)^2}\ dx} = 2.4244$$

Hence,

$$OC^L_{A_2} = \sqrt{{\overline{x}^L_{A_i}}^2 + {\overline{y}^L_{A_i}}^2} = 1.6318; \qquad OC^R_{A_2} = \sqrt{{\overline{x}^R_{A_i}}^2 + {\overline{y}^R_{A_i}}^2} = 2.8802$$

Using formulae in Eq. (21) for generalized L-R-type fuzzy numbers in concurrence with the above values and the formulae in Eq. (27) for triangular fuzzy numbers, the unified distance values for these two fuzzy quantities are found and demonstrated in Table 9. Based on the proposed ranking scheme in Sec. 3.6, the ranking outcome is $A_1 > A_2$ irrespective of the level of fuzziness $\eta \in [0,\ 1]$, which is in support of intuitive perception. A recent approach by Nguyen [25] advocates the same ranking conclusion. The total integral value approach of Liou and Wang [21] concluded the ranking order as $A_1 > A_2$ for $0 \leq \eta \leq 0.8$ and $A_1 < A_2$ for $0.9 \leq \eta \leq 1$. The value and ambiguity approach of Chutia and Chutia [14] disagreed with the proposed approach and obtain $A_1 < A_2$ at all decision levels $\eta \in [0,\ 1]$. Thus, the suggested strategy is also consistent to discriminate the fuzzy numbers with non-rectilinear membership functions in addition to triangular as well as trapezoidal fuzzy quantities.

TABLE 9. Ranking outcomes at different fuzziness/decision levels for the fuzzy quantities of Ex. 5.9

| $\eta$ | $UD^{\eta}_{A_1}$ | $UD^{\eta}_{A_2}$ | *Ranking outcomes* |
|---|---|---|---|
| 0.0 | 4.5326 | 3.9555 | $A_1 > A_2$ |
| 0.1 | 4.8843 | 4.2581 | $A_1 > A_2$ |
| 0.2 | 5.2360 | 4.5607 | $A_1 > A_2$ |
| 0.3 | 5.5876 | 4.8633 | $A_1 > A_2$ |
| 0.4 | 5.9393 | 5.1659 | $A_1 > A_2$ |
| 0.5 | 6.2910 | 5.4685 | $A_1 > A_2$ |
| 0.6 | 6.6427 | 5.7712 | $A_1 > A_2$ |
| 0.7 | 6.9944 | 6.0738 | $A_1 > A_2$ |
| 0.8 | 7.3461 | 6.3764 | $A_1 > A_2$ |
| 0.9 | 7.6977 | 6.6790 | $A_1 > A_2$ |
| 1.0 | 8.0494 | 6.9816 | $A_1 > A_2$ |

## 6. CONCLUSIONS

The approaches to ranking fuzzy numbers are hindered by counter-intuitiveness, computational intricacy, and paucity of consistency. To lessen these ranking instabilities, this paper suggested a unified distance technique that multiplies two discriminatory tools to boost the power of discrimination to rank the fuzzy numbers. The first multiple is the centroid value (weighted mean value) of the fuzzy number on the horizontal axis and the second is a linear combination of the distances of the centroid points of the left and right fuzziness areas from the original point through

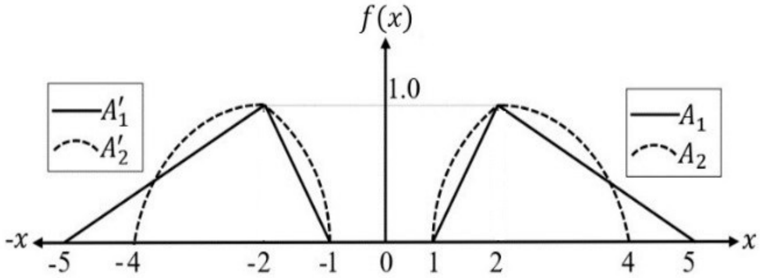


FIGURE 10. Visualization of the fuzzy quantities and their partnered images of Ex. 5.9

an indicator of fuzziness. The indicator reflects the attitude of the left and right fuzziness of the fuzzy number. This method has six advantages in ordering the fuzzy numbers according to theoretical proofs and comparative reviews. At the outset, the ranking results support logical perception. Secondly, it ensures that the computation is simple regardless of the type of fuzzy numbers. Thirdly, the index can help to clarify the ranking disagreements in the literature using the decision maker's choice. Fourthly, the proposed method can overcome the limitations of the other methods that arise due to the compensation of areas. Fifth, it gives a justified ranking preference for the partnered image of the fuzzy numbers. Finally, the suggested approach's consistency properties have no limitation in ordering symmetric fuzzy numbers of equal height. These elements are crucial for the precise matching and retrieval of evidence in the field of pattern recognition.

## REFERENCES


[1] Abbasbandy, S. and Asady, B. "Ranking of fuzzy numbers by sign distance." *Information Sciences*, **176(16)**, (2006), 2405–2416.

[2] Abbasbandy, S. and Hajjari, T. "A new approach for ranking of trapezoidal fuzzy numbers." *Computers Mathematics with Applications*, **57(3)**, (2009), 413–419.

[3] Abbasi Shureshjani, R. and Darehmiraki, M. "A new parametric method for ranking fuzzy numbers." *Indagationes Mathematicae*, **24(3)**, (2013), 518–529.

[4] Asady, B. "Revision of distance minimization method for ranking of fuzzy numbers." *Applied Mathematical Modelling*, **35(3)**, (2011), 1306–1313.

[5] Asady, B. and Zendehnam, A. "Ranking fuzzy numbers by distance minimization." *Applied Mathematical Modelling*, **31(11)**, (2007), 2589–2598.

[6] Barazandeh Hayati, Y. and Ghazanfari, B. "A novel method for ranking generalized fuzzy numbers with two different heights and its application in fuzzy risk analysis." *Iranian Journal of Fuzzy Systems*, **18(2)**, (2021), 81–91.

[7] Bortolan, G. and Degani, R. "A review of some methods for ranking fuzzy subsets." *Fuzzy Sets and Systems*, **15(1)**, (1985), 1–19.

[8] Chen, S.-H. "Ranking fuzzy numbers with maximizing set and minimizing set." *Fuzzy Sets and Systems*, **17(2)**, (1985), 113–129.

[9] Cheng, C.-H. "A new approach for ranking fuzzy numbers by distance method." *Fuzzy Sets and Systems*, **95(3)**, (1998), 307–317.

[10] Chi, H. T. X. and Yu, V. F. "Ranking generalized fuzzy numbers based on centroid and rank index." *Applied Soft Computing*, **68**, (2018), 283–292.

[11] Choobineh, F. and Li, H. "An index for ordering fuzzy numbers." *Fuzzy Sets and Systems*, **54(3)**, (1993), 287–294.

[12] Chu, T.-C. and Tsao, C.-T. "Ranking fuzzy numbers with an area between the centroid point and original point." *Computers Mathematics with Applications*, **43(1)**, (2002), 111–117.

[13] Chutia, R. "Ranking of fuzzy numbers by using value and angle in the epsilon-deviation degree method." *Applied Soft Computing*, **60**, (2017), 706–721.

[14] Chutia, R. and Chutia, B. "A new method of ranking parametric form of fuzzy numbers using value and ambiguity." *Applied Soft Computing*, **52**, (2017), 1154–1168.

[15] Deng, Y., Zhenfu, Z. and Qi, L. "Ranking fuzzy numbers with an area method using radius of gyration." *Computers Mathematics with Applications*, **51(6)**, (2006), 1127–1136.

[16] Dombi, J. and Jónás, T. "Ranking trapezoidal fuzzy numbers using a parametric relation pair." *Fuzzy Sets and Systems*, **399**, (2020), 20–43. Fuzzy Intervals.

[17] Dubois, D. and Prade, H. "Operations on fuzzy numbers." *International Journal of Systems Science*, **9(6)**, (1978), 613–626.

[18] Fortemps, P. and Roubens, M. "Ranking and defuzzification methods based on area compensation." *Fuzzy Sets and Systems*, **82(3)**, (1996), 319–330.

[19] Hajjari, T. "Ranking fuzzy numbers by similarity measure index." ICACS'20. Association for Computing Machinery, New York, NY, USA (2020). 12–16.

[20] Jain, R. "Decisionmaking in the presence of fuzzy variables." *IEEE Transactions on Systems, Man, and Cybernetics*, **6(10)**, (1976), 698–703.

[21] Liou, T.-S. and Wang, M.-J. J. "Ranking fuzzy numbers with integral value." *Fuzzy Sets and Systems*, **50(3)**, (1992), 247–255.

[22] Mao, Q. S. "Ranking fuzzy numbers based on weighted distance." *Journal of Physics: Conference Series*, **1176**, (2019), 032007.

[23] Nasseri, S., Zadeh, M., Kardoost, M. and Behmanesh, E. "Ranking fuzzy quantities based on the angle of the reference functions." *Applied Mathematical Modelling*, **37(22)**, (2013), 9230–9241.

[24] Nasseri, S. H., Taghi-Nezhad, N. and Ebrahimnejad, A. "A note on ranking fuzzy numbers with an area method using circumcenter of centroids." *Fuzzy Information and Engineering*, **9(2)**, (2017), 259–268.

[25] Nguyen, T.-L. "Methods in ranking fuzzy numbers: A unified index and comparative reviews." *Fuzzy Calculus Theory and Its Applications*, **1**.

[26] Prasad, S. and Sinha, A. "Ranking fuzzy numbers with mean value of points and comparative reviews." *Journal of Scientific Research*, **14(3)**, (2022), 755–771.

[27] Prasad, S. and Sinha, A. "Ranking fuzzy numbers with unified integral value and comparative reviews." *Journal of Scientific Research*, **14(1)**, (2022), 131–151.

[28] Rao, P. B. "Defuzzification method for ranking fuzzy numbers based on centroids and maximizing and minimizing set." *Decision Science Letters*, **8**, (2019), 411–428.

[29] Rao, P. P. B. and Shankar, N. R. "Ranking fuzzy numbers with an area method using circumcenter of centroids." *Fuzzy Information and Engineering*, **5(1)**, (2013), 3–18.

[30] Rezvani, S. "Ranking generalized exponential trapezoidal fuzzy numbers based on variance." *Applied Mathematics and Computation*, **262**, (2015), 191–198.

[31] Soccoro Garcia, M. and Teresa Lamata, M. "A modification of the index of liou and wang for ranking fuzzy number." *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, **15(04)**, (2007), 411–424.

[32] Wang, Y.-J. and Lee, H.-S. "The revised method of ranking fuzzy numbers with an area between the centroid and original points." *Computers Mathematics with Applications*, **55(9)**, (2008), 2033–2042.

[33] Yager, R. R. "Ranking fuzzy subsets over the unit interval." "1978 IEEE Conference on Decision and Control including the 17th Symposium on Adaptive Processes," (1978). 1435–1437.

[34] Yager, R. R. "On choosing between fuzzy subsets." **9(2)**, (1980), 151–154.

[35] Yager, R. R. “A procedure for ordering fuzzy subsets of the unit interval.” *Information Sciences*, **24(2)**, (1981), 143–161.

[36] Yu, V. F., Chi, H. T. X., Dat, L. Q., Phuc, P. N. K. and wen Shen, C. “Ranking generalized fuzzy numbers in fuzzy decision making based on the left and right transfer coefficients and areas.” *Applied Mathematical Modelling*, **37(16)**, (2013), 8106–8117.

[37] Yu, V. F., Chi, H. T. X. and wen Shen, C. “Ranking fuzzy numbers based on epsilon-deviation degree.” *Applied Soft Computing*, **13(8)**, (2013), 3621–3627.

[38] Yua, V. F. and Dat, L. Q. “An improved ranking method for fuzzy numbers with integral values.” *Applied Soft Computing*, **14(1)**, (2014), 603–608.

[39] Zadeh, L. A. “Fuzzy sets.” *Information and Control*, **8(3)**, (1965), 338–353.

[40] Zhang, F., Ignatius, J., Lim, C. P. and Zhao, Y. “A new method for ranking fuzzy numbers and its application to group decision making.” *Applied Mathematical Modelling*, **38(4)**, (2014), 1563–1582.