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TOTAL EDGE IRREGULARITY STRENGTH OF THE CARTESIAN PRODUCT OF BIPARTITE GRAPHS AND PATHS

Rachel Wulan Nirmalasari Wijaya[1], Joe Ryan[2], Thomas Kalinowski[3]

[1] **Universitas Kristen Satya Wacana, Indonesia, rachel.wijaya@uksw.edu**
[2] **The University of Newcastle, NSW, Australia, joe.ryan@newcastle.edu.au**
[3] **Hochschule Mittweida, Germany, kalinows@hs-mittweida.de**

Abstract. For a simple graph $G = (V(G), E(G))$, a total labeling ∂ is called an edge irregular total k-labeling of G if $\partial : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that for any two different edges uv and $u'v'$ in $E(G)$, we have $wt_\partial(uv) \neq wt_\partial(u'v')$ where $wt_\partial(uv) = \partial(u) + \partial(v) + \partial(uv)$. The minimum k for which G has an edge irregular total k-labeling is called the total edge irregularity strength, denoted by $\mathrm{tes}(G)$. It is known that $\left\lceil \frac{|E(G)|+2}{3} \right\rceil$ is a lower bound for the total edge irregularity strength of a graph G. In this paper we prove that if G is a bipartite graph for which this bound is tight then this is also true for Cartesian product of G with any path.

Key words and Phrases: total edge irregularity strength, Cartesian product, bipartite graph, path

1. Introduction

The graphs considered in this paper are finite, undirected and simple. For a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. The *Cartesian product* of two graphs G_1 and G_2, denoted by $G_1 \times G_2$, is a simple graph with vertex set $V(G_1) \times V(G_2)$, in which (u_1, v_1) is adjacent to (u_2, v_2) if and only if either $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1u_2 \in E(G_1)$ [8]. Figure 1 shows the Cartesian product of two paths on 2 and 3 vertices.

A *graph labeling*, as introduced in [13], is a mapping that carries graph elements to numbers (usually positive or non-negative integers), called labels. The most common choices of domain are the vertex set (*vertex labeling*), the edge set (*edge labeling*), and the union of the vertex set and edge set (*total labeling*). Over the years, a large variety of different types of graph labelings have been studied, see [10] for an extensive survey.

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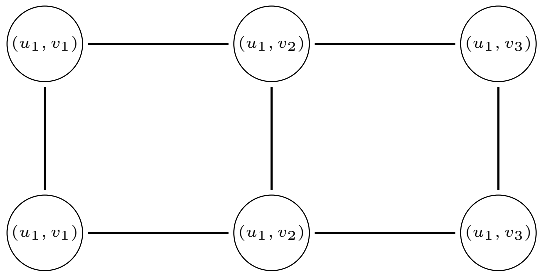


FIGURE 1. $P_2 \times P_3$

In 1988 Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba [9] proposed the problem of *irregular labeling.* For a graph $G = (V(G), E(G))$ with no isolated vertices, the *weight* of a vertex v under an edge labeling $f : E \to \{1, 2, \dots, k\}$ is

$$w_f(v) = \sum_{u \in N(v)} f(uv),$$

where $N(v)$ is the set of neighbours of a vertex v. The edge labeling f is called *irregular assignment* of G if the weight of two different vertices u and v in $V(G)$ satisfies $w_f(u) \neq w_f(v)$. The *irregularity strength* $s(G)$ of G is the minimum k for which the graph G has an irregular assignment.

In this paper, we consider total labelings. For a graph G the *weight* of an edge uv under a total labeling ∂ is

$$wt_\partial(uv) = \partial(u) + \partial(v) + \partial(uv).$$

A total labeling $\partial : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ is called *edge irregular total k-labeling* of G if the weight of two distinct edges uv and $u'v'$ in $E(G)$ satisfies $wt_\partial(uv) \neq wt_\partial(u'v')$. The minimum k for which G has an edge irregular total k-labeling is called the *total edge irregularity strength of* G denoted by $\operatorname{tes}(G)$.

The concept of the total edge irregularity strength was introduced by Bača, Jendroľ, Miller and Ryan [5]. In the same paper, they proved the total edge irregularity strength for any graph $G = (V(G), E(G))$ with a non-empty set $E(G)$ is

$$\left\lceil \frac{|E(G)| + 2}{3} \right\rceil \leq \operatorname{tes}(G) \leq |E(G)|$$

and if the maximum degree $\Delta = \Delta(G) \leq (|E(G)| - 1)/2$, then

$$\left\lceil \frac{\Delta + 1}{2} \right\rceil \leq \operatorname{tes}(G) \leq |E(G)| - \Delta.$$

They also proved the total edge irregularity strength of path, cycle, wheel, and friendship graph.

In the following years, the exact value of total edge irregularity strength has been proven for hexagonal grid graphs [4]; barbell graph for $n \geq 3$ [1]; staircase graphs, double staircase graphs, and mirror-staircase graphs [16]; generalized arithmetic staircase graphs and generalized double-staircase graphs [17].

Ivančo and Jendrol' [11] conjecture that any graph G with maximum degree $\Delta(G)$ other than K_5 satisfies

$$\operatorname{tes}(G) = \max\left\{\left\lceil\frac{\Delta+1}{2}\right\rceil, \left\lceil\frac{|E(G)|+2}{3}\right\rceil\right\}.$$

In the same paper, they proved that this conjecture is true for all trees.

This conjecture has been proven for complete graphs and complete bipartite graphs [12], the categorical product of two paths [2], the categorical product of a cycle and a path [14], the categorical product of two cycles [3], the Cartesian product of a cycle and a path [7], the subdivision of a star [15], and the torodial polyhexes [6].

2. Main Result

In this paper we consider any bipartite graph H with the total edge irregularity strength

$$\operatorname{tes}(H) = \left\lceil\frac{|E(H)|+2}{3}\right\rceil$$

and prove that the Cartesian product of graph H with a path with n vertices P_n, denoted by $H \times P_n$ for $n \geq 1$ is

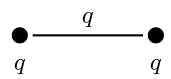
$$\operatorname{tes}(H \times P_n) = \left\lceil\frac{|E(H \times P_n)|+2}{3}\right\rceil.$$

Before we proceed to our main theorem, we give some definitions as follows.

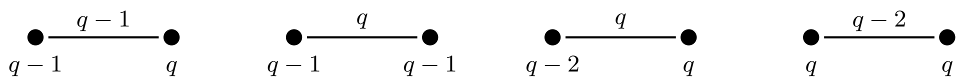
Definition 2.1. *The* extreme edge *of an edge irregular total labeling is the unique edge of maximum weight.*

Remark 2.1. For any graph G with $\operatorname{tes}(G) = \left\lceil\frac{|E(G)|+2}{3}\right\rceil$, the weight of the extreme edge is $|E(G)|+2$. Applying this to H and setting $q = \left\lceil\frac{|E(H)|+2}{3}\right\rceil$ the possible labelings for the extreme edge are as follows:

(1) $|E(G)|+2 \equiv 0 \pmod 3$



(2) $|E(G)|+2 \equiv 1 \pmod 3$



(3) $|E(G)|+2 \equiv 2 \pmod 3$

Let $\gamma = |V(H)|+|E(H)| = 3s+r$ with $r \in \{0,1,2\}$. Note that $|E(H\times P_n)| = (n-1)|V(H)| + n|E(H)|$. Set

$$q_n = \left\lceil \frac{|E(H \times P_n)| + 2}{3} \right\rceil \qquad \text{for } n = 1, 2, \ldots.$$

In other words, q_n is the largest label we want to use in our labeling of $H \times P_n$.

Recall that

$$\begin{aligned} \gamma &= |V(H)| + |E(H)| = 3s + r, \\ q_n &= \left\lceil \frac{|E(H \times P_n)| + 2}{3} \right\rceil = \left\lceil \frac{(n-1)|V(H)| + n|E(H)| + 2}{3} \right\rceil \\ &= \left\lceil \frac{n\gamma - |V(H)| + 2}{3} \right\rceil. \end{aligned}$$

Lemma 2.2. *Let $n \geq 2$ and $m' = (n-1)\gamma - |V(H)|$. Then*

$$q_n = \begin{cases} q_{n-1} + s & \textit{if } r = 0, \\ q_{n-1} + s & \textit{if } r = 1 \textit{ and } m' \equiv 0 \textit{ or } 2 \pmod 3, \\ q_{n-1} + s & \textit{if } r = 2 \textit{ and } m' \equiv 2 \pmod 3, \\ q_{n-1} + s + 1 & \textit{if } r = 1 \textit{ and } m' \equiv 1 \pmod 3, \\ q_{n-1} + s + 1 & \textit{if } r = 2 \textit{ and } m' \equiv 0 \textit{ or } 1 \pmod 3. \end{cases}$$

Proof. Writing $m' = 3s' + r'$ with $r' \in \{0,1,2\}$ we obtain

$$\begin{aligned} q_{n-1} &= \left\lceil \frac{m'+2}{3} \right\rceil = \left\lceil \frac{3s'+r'+2}{3} \right\rceil = \begin{cases} s'+1 & \text{if } r' \in \{0,1\}, \\ s'+2 & \text{if } r' = 2. \end{cases} \\ q_n &= \left\lceil \frac{m'+\gamma+2}{3} \right\rceil \\ &= \left\lceil \frac{3(s'+s)+r'+r+2}{3} \right\rceil = \begin{cases} s'+s+1 & \text{if } r'+r \in \{0,1\}, \\ s'+s+2 & \text{if } r'+r \in \{2,3,4\}. \end{cases} \end{aligned}$$

The result follows. $\square$

Theorem 2.3. *Let H be a bipartite graph with vertex set $V(G) = A \cup B$ and $tes(H) = \left\lceil \frac{|E(H)|+2}{3} \right\rceil$. For the graph $G = H \times P_n$, $n \geq 1$ then*

$$tes(G) = \left\lceil \frac{|E(G)| + 2}{3} \right\rceil$$

and there exists an optimal labeling of G such that the extreme edge has the form $\{(i,n),(j,n)\}$ for $i \in A$ and $j \in B$.

Proof. Recall that $\gamma = |V(H)|+|E(H)| = 3s+r$, $r \in \{0,1,2\}$ and let $G_n = H \times P_n$. Next, we proceed by induction on n.

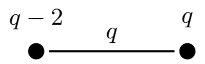
The base case $n = 1$.: Because $G_1 = H \times P_1 = H$, we have $\operatorname{tes}(G_1) = \left\lceil \frac{|E(G_1)|+2}{3} \right\rceil$.

The induction step.: For $n \geq 2$, we assume by induction that $\operatorname{tes}(G_{n-1}) = \left\lceil \frac{|E(G_{n-1})|+2}{3} \right\rceil$, and that there is an optimal labeling f for G_{n-1} with the property that the extreme edge is in the $(n-1)$-th copy of H (that is, it has the form $\{(i, n-1), (j, n-1)\}$). From this we derive

$$\operatorname{tes}(G_n) = \left\lceil \frac{|E(G_n)|+2}{3} \right\rceil$$

and the existence of an optimal labeling F for G_n with the property that the extreme edge is in the n-th copy of H (that is, it has the form $\{(i,n),(j,n)\}$). We construct F as an extension of f, that is, on $V(G_{n-1}) \cup E(G_{n-1})$ the labeling F agrees with f, and it remains to specify the values of F on the n-th copy of H and on the edges connecting the $(n-1)$-th copy with the n-th copy.

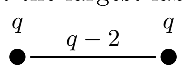
Case 1.: $r = 0$, such that $\gamma = 3s$. We divide the labeling into several cases of extreme edge. Since H is a bipartite graph, the vertex set can be partitioned into two subsets A and B. For every extreme edge in the following sub-cases, we assume the left and right vertex is in the vertex set A and B respectively.

Case 1.1: 

We label the vertices and edges of layer n as follows

$$\begin{aligned}
F(a,n) &= f(a,n-1)+s+1, && \text{for } a \in A,\\
F(b,n) &= f(b,n-1)+s, && \text{for } b \in B,\\
F(e,n) &= f(e,n-1)+s-1, && \text{for } e \in E(H),
\end{aligned}$$

such that the largest label is $q+s$.

Case 1.2: 

We label the vertices and edges of layer n as follows

$$\begin{aligned}
F(a,n) &= f(a,n-1)+s-1, && \text{for } a \in A,\\
F(b,n) &= f(b,n-1)+s, && \text{for } b \in B,\\
F(e,n) &= f(e,n-1)+s+1, && \text{for } e \in E(H),
\end{aligned}$$

such that the largest label is $q+s$.

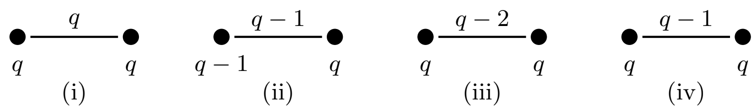
Case 1.3: For the other cases, we label the vertices dan the edges of layer n as follows,

$$F(v,n) = f(v,n-1) + s, \qquad \text{for } v \in V(H),$$
$$F(e,n) = f(e,n-1) + s, \qquad \text{for } e \in E(H),$$

such that the largest label is $q+s$.

Case 2.: $r = 1$ ($\gamma = 3s+1$). We divide the labeling into several cases of extreme edge. For every extreme edge in the following sub-cases, we assume the left and right vertex is in the vertex set A and B respectively.

Case 2.1: For these four extreme edges,

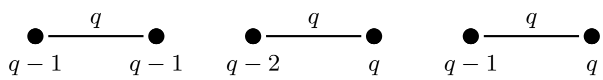


we label the vertices and edges of layer n as follows

$$F(a,n) = f(a,n-1) + s, \qquad \text{for } a \in A,$$
$$F(b,n) = f(b,n-1) + s, \qquad \text{for } b \in B,$$
$$F(e,n) = f(e,n-1) + s + 1, \qquad \text{for } e \in E(H),$$

such that the largest label for (i), is $q+s+1$ and for (ii)-(iv), are $q+s$.

Case 2.2: For these three extreme edges,



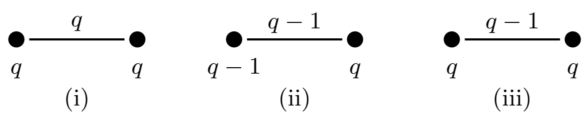
we label the vertices and edges of layer n as follows

$$F(a,n) = f(a,n-1) + s + 1, \qquad \text{for } a \in A,$$
$$F(b,n) = f(b,n-1) + s, \qquad \text{for } b \in B,$$
$$F(e,n) = f(e,n-1) + s, \qquad \text{for } e \in E(H).$$

such that the largest label is $q+s$.

Case 3.: $r = 2$ ($\gamma = 3s+2$). We divide the labeling into several cases of extreme edge. For every extreme edge in the following sub-cases, we assume the left and right vertex is in the vertex set A and B respectively.

Case 3.1: For these extreme edges,

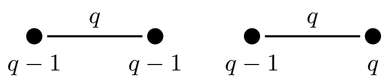


we label the vertices and edges of layer n as follows

$$\begin{aligned}
F(a,n) &= f(a,n-1)+s+1, && \text{for } a \in A,\\
F(b,n) &= f(b,n-1)+s, && \text{for } b \in B,\\
F(e,n) &= f(e,n-1)+s+1, && \text{for } e \in E(H),
\end{aligned}$$

such that the largest label for (i) and (iii), are $q+s+1$ and for (ii), is $q+s$.

Case 3.2: For these extreme edges,

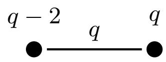


(i) (ii)

we label the vertices and edges of layer n as follows

$$\begin{aligned}
F(a,n) &= f(a,n-1)+s+1, && \text{for } a \in A,\\
F(b,n) &= f(b,n-1)+s+1, && \text{for } b \in B,\\
F(e,n) &= f(e,n-1)+s, && \text{for } e \in E(H),
\end{aligned}$$

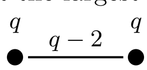
such that the largest label for (i), is $q+s$ and for (ii), is $q+s+1$.

Case 3.3: 

We label the vertices and edges of layer n as follows

$$\begin{aligned}
F(a,n) &= f(a,n-1)+s+2, && \text{for } a \in A,\\
F(b,n) &= f(b,n-1)+s, && \text{for } b \in B,\\
F(e,n) &= f(e,n-1)+s, && \text{for } e \in E(H),
\end{aligned}$$

such that the largest label is $q+s$.

Case 3.4: 

We label the vertices and edges of layer n as follows

$$\begin{aligned}
F(a,n) &= f(a,n-1)+s, && \text{for } a \in A,\\
F(b,n) &= f(b,n-1)+s, && \text{for } b \in B,\\
F(e,n) &= f(e,n-1)+s+2, && \text{for } e \in E(H),
\end{aligned}$$

such that the largest label is $q+s$.

The largest label from Case 1.1 to 3.4 is the value of q_n in Lemma 2.2 with $q = q_{n-1}$.

To label the connecting edge between layers $n-1$ and n, we used the following rules. Let $N = |V(H)|$, and index the vertex set $V(H) = \{v_1, \dots, v_N\}$ in such a way that

$$f(v_1,n-1)+f(v_1,n) \le f(v_2,n-1)+f(v_2,n) \le \ldots \le f(v_N,n-1)+f(v_N,n).$$

Then set for $i = 1, \ldots, N$,

$$f((v_i, n-1), (v_i, n)) = (n-2)\gamma + |E(H)| + 2 + i - |f(v_i, n-1) + f(v_i, n)|,$$

so that the weight of this edge becomes

$$wt((v_i, n-1), (v_i, n)) = (n-2)\gamma + |E(H)| + 2 + i.$$

Assume that the label of $f(v_N, n-1)$ and $f(v_N, n)$ is $\mathrm{tes}(G_{n-1})$ and $\mathrm{tes}(G_n)$ respectively, such that the largest label of the connecting edges is

$$\begin{aligned}
f((v_N, n-1), (v_N, n)) &= (n-2)\gamma + |E(H)| + 2 + N - [f(v_N, n-1) + f(v_N, n)] \\
&= (n-2)\gamma + |E(H)| + 2 + |V(H)| \\
&\quad - \left[\left\lceil \frac{|E(G')| + 2}{3} \right\rceil + \left\lceil \frac{|E(G)| + 2}{3} \right\rceil\right] \\
&= (n-2)\gamma + \gamma + 2 \\
&\quad - \left[\left\lceil \frac{(n-2)\gamma + |E(H)| + 2}{3} \right\rceil + \left\lceil \frac{(n-1)\gamma + |E(H)| + 2}{3} \right\rceil\right] \\
&= (n-1)\gamma + 2 - \left[\left\lceil \frac{(2n-3)\gamma + 2(|E(H)| + 2)}{3} \right\rceil\right] \\
&= \left\lceil \frac{3((n-1)\gamma + 2) - (2n-3)\gamma - 2(|E(H)| + 2)}{3} \right\rceil \\
&= \left\lceil \frac{n\gamma - 2|E(H)| + 2}{3} \right\rceil \\
&= \left\lceil \frac{n(|V(H)| + |E(H)|) - 2|E(H)| + 2}{3} \right\rceil \\
&= \left\lceil \frac{n|V(H)| + (n-2)|E(H)| + 2}{3} \right\rceil \\
&= \left\lceil \frac{(n-1)(|V(H)| + |E(H)|) + |V(H)| - |E(H)| + 2}{3} \right\rceil \\
&= \left\lceil \frac{(n-1)\gamma + |V(H)| - |E(H)| + 2}{3} \right\rceil \\
&= \left\lceil \frac{((n-1)\gamma + |E(H)| + 2) + |V(H)| - 2|E(H)|}{3} \right\rceil \\
&= \left\lceil \frac{(n-1)\gamma + |E(H)| + 2}{3} \right\rceil + \left\lceil \frac{|V(H)| - 2|E(H)|}{3} \right\rceil.
\end{aligned}$$

For a connected bipartite graph H, $|V(H)| \leq 2|E(H)|$, such that

$$f((v_i, n-1), (v_i, n)) \leq \mathrm{tes}(G).$$

Since we have constructed an edge irregular total labeling with the largest label q_n, we conclude that $\mathrm{tes}(G) = q_n$. □

Lemma 2.4 ([5]). *The total edge irregularity strength of paths P_n, $n \geq 2$ is*

$$tes(P_n) = \left\lceil \frac{n+1}{3} \right\rceil.$$

Definition 2.5. *For integers $k \leqslant \ell$ we denote the sets $\{1, 2, \dots, k\}$ and $\{k, k+1, \dots, \ell\}$ by $[k]$ and $[k, \ell]$, respectively. For integers $d \geqslant 2$ and $n_1 \geq n_2 \geq \dots \geq n_d \geqslant 2$, let $\textsc{Grid}(n_1, \dots, n_d)$ denote the $n_1 \times \dots \times n_d$-grid graph, i.e., the Cartesian product of d paths of lengths $n_1, \dots, n_d$. In other words, the vertex set of $G(n_1, \dots, n_d)$ is $V = [n_1] \times \dots \times [n_d]$ and edge set*

$$E = \left\{ \{\boldsymbol{x}, \boldsymbol{y}\} \; : \; \boldsymbol{x}, \boldsymbol{y} \in V \text{ and } \sum_{i=1}^{d} |x_i - y_i| = 1 \right\}.$$

Corollary 2.6. *Let $G = \textsc{Grid}(n_1, \dots, n_d)$ be a d-dimensional grid, then*

$$tes(G) = \left\lceil \frac{|E(G)| + 2}{3} \right\rceil.$$

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