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ON HORADAM-LUCAS SEQUENCE

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Abstract. Horadam introduced a generalized sequence of numbers, describing its key features and the special sub-sequences obtained from specific choices of initial parameters. This sequence and its sub-sequences are known as the Horadam, generalized Fibonacci, and generalized Lucas numbers, respectively. In the present study, we propose another new sequence, which satisfies a second-order recurrence relation. Further, we prove the Binet's formula, some famous identities, and summation formulas for this new sequence. In particular, we demonstrate the interrelationships between our new sequence and the Horadam sequence.

Keywords and Phrases: Horadam sequence, Generalized Fibonacci number, Generalized Lucas number, Honsberger formula.

1. Introduction

In [1], Horadam considered a generalized form of the classic Fibonacci numbers, changing the initial terms $F_0 = 0$ and $F_1 = 1$ to a and b, respectively. Then in [2], Horadam defined the second-order linear recurrence sequence $w_n\left(a, b; p, q\right)$ as

$$w_n = pw_{n-1} - qw_{n-2} \tag{1}$$

with $w_0 = a$ and $w_1 = b$. This generalizes many sequences of integers; e.g., the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, Generalized Fibonacci and Generalized Lucas sequences. The Binet's formula for the Horadam sequence is

$$w_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}. \tag{2}$$

Here, the author used the notations

$$\alpha = \frac{p+d}{2},\ \beta = \frac{p-d}{2},\ A = b - a\beta,\ B = b - a\alpha \text{ and } d = \sqrt{p^2 - 4q}. \tag{3}$$

One can readily show that

$$\alpha + \beta = p,\ \alpha\beta = q,\ \alpha - \beta = d,$$

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$$A+B=2b-ap,\ A-B=ad,\ \text{and}\ AB=b^2-abp+a^2q=E.$$

In working with this sequence, it is useful to consider the following special cases:

$$w_n(0,1;p,q)=u_n(p,q)$$
$$w_n(2,p;p,q)=v_n(p,q).$$

We note that the Binet's formulas for these special cases are

$$u_n=\frac{\alpha^n-\beta^n}{d},$$
$$v_n=\alpha^n+\beta^n.$$

For positive integers n, Horadam [3] has given the following formulas for w_n, u_n, and v_n :

$$w_{-n}=q^{-n}\frac{au_n-bu_{n-1}}{au_n+(b-pa)u_{n-1}},\ u_{-n}=q^{-n+1}u_{n-2}\,\text{and}\,v_{-n}=q^{-n}v_n.$$

In [4], Horadam presented some geometric interpretations of the Horadam sequence including the Pythagorean property. In [5], Morales defined the 2×2 matrix

$$U(p,q)=\begin{bmatrix} p & -q \\ 1 & 0 \end{bmatrix}$$

and showed that

$$U^n(p,q)=\begin{bmatrix} u_{n+1} & -qu_n \\ u_n & -qu_{n-1} \end{bmatrix}.$$

For brevity, we denote the matrix $U(p,q)$ by R unless stated otherwise.

In this paper, we define a new generalization $h_n(a,b;p,q)$ of the well-known second-order linear sequences, i.e., of the Fibonacci, Lucas, Pell, Jacobsthal, Generalized Fibonacci, and Generalized Lucas sequences. We present many results from this new generalization, including the Binet's formula, the d'Ocagne's and Gelin-Cesáro identities, and some summation formulas. Further, we give some special identities of $h_n(a,b;p,q)$ via matrix techniques.

The paper is organized as follows. Section 2 introduces our main definition and the related special cases. While Section 3 presents some elementary properties and identities corresponding to our generalized definition, Section 4 develops a matrix approach to the respective generalized sequences to obtain determinantal results.

2. Main Results

We can give the definition of our generalized sequence as follows.

Definition 2.1. *Let n be any integer. Then, for $n\geqslant 2$, we define*

$$h_n=ph_{n-1}-qh_{n-2}, \tag{4}$$

with the initial conditions $h_0=2b-ap$ and $h_1=bp-2aq$.

TABLE 1. Sequences corresponding to different choices of a, b, p, and q.

a	b	p	q	Horadam sequence	Horadam-Lucas sequence
0	1	1	-1	Fibonacci numbers	Lucas numbers
0	1	2	-1	Pell numbers	Pell-Lucas numbers
0	1	1	-2	Jacobsthal numbers	Jacobsthal-Lucas numbers

Note that the definitions in (1) and (4) allow us to investigate, respectively, the primary sequences, e.g. Fibonacci and Pell numbers, and the secondary sequences, e.g. the Lucas and Pell-Lucas numbers, at the same time. Briefly, then h_n may be considered to be a companion sequence to w_n. This is summarized in Table 2.

We call the recurrence relation in (4) a Horadam-Lucas sequence due to Horadam's great contribution to the subject of this paper. Depending on the choice of a, b, p, and q, we already know that many second-order sequences can be constructed. Further, for each of these sequences, we must consider a different choice of the corresponding parameters. But, our definition is probably the most special one. This statement is supported by the following theorem. Then it's time to give the Binet's formula for the Horadam-Lucas sequence, which we shall extensively use later.

Theorem 2.2 (Binet's formula). *For every integer n, we have the Binet's formula*

$$h_n = A\alpha^n + B\beta^n, \tag{5}$$

where $A = b - a\beta$ and $B = b - a\alpha$, as given by Horadam [2].

Proof. Eq. (4) is a second-order linear homogeneous difference equation with constant coefficients which has the form

$$x_n = px_{n-1} - qx_{n-2}. \tag{6}$$

We assume that there is a solution to Eq. (4) of the form

$$x_n = \lambda^n, \tag{7}$$

where λ is a constant to be determined. Substituting Eq. (7) into Eq. (6) yields

$$\lambda^n = p\lambda^{n-1} - q\lambda^{n-2}.$$

In particular, for $n = 2$, we have

$$\lambda^2 - p\lambda + q = 0, \tag{8}$$

which was given by Horadam [3]. The roots of Eq. (8) are

$$\lambda_1 = (p + d)/2 = \alpha \text{ and } \lambda_2 = (p - d)/2 = \beta. \tag{9}$$

The notations used here were given in Eq. (3). As a result, we have found two independent solutions to Eq. (6) in the form of Eq. (7). Hence, a linear combination of these two solutions is also a solution of Eq. (4), namely

$$h_n = c_1\alpha^n + c_2\beta^n.$$

Considering the initial conditions for our definition, we can write

$$\begin{aligned} c_1 + c_2 &= 2b - ap \\ c_1\alpha + c_2\beta &= bp - 2aq. \end{aligned} \tag{10}$$

If the system of equations in (10) is solved, we obtain the two solutions

$$c_1 = b - a\beta \text{ and } c_2 = b - a\alpha$$

and the result follows. □

Particular cases of the Horadam-Lucas sequence are as follows:

- For $a = 1$, $b = p$, we can write
$$h_n\left(1, p; p, q\right) = A\alpha^n + B\beta^n = v_{n+1}.$$
- For $a = 2$, $b = p$, we can write
$$h_n\left(1, p; p, q\right) = A\alpha^n + B\beta^n = d^2u_n.$$

With the above-mentioned, the presentation of the definitions related to our generalized sequence is thus exhausted. In this context, we can now expand the current study to discover fundamental identities.

3. Special Identities of Horadam-Lucas Sequence

In this section, we present some special properties of the generalized sequence defined in (4). We first define the generating function given in the form

$$h\left(x\right) = \sum_{n=0}^{\infty} h_n x^n. \tag{11}$$

Then we can state the following theorem.

Theorem 3.1. *The generating functions of the Horadam-Lucas sequence are given by*

$$h\left(x\right) = \frac{h_0 + \left(h_1 - ph_0\right)x}{1 - px + qx^2}. \tag{12}$$

Proof. Summing these equations after setting up $h\left(x\right)$, $-pxh\left(x\right)$ and $qx^2h\left(x\right)$ readily yields the first result. □

Theorem 3.2 (De Moivre's Formula)**.** *Let* $x_n = h_{n+1} - qh_{n-1}$ *and* k *be any integer. Then, we have*

$$\left(\frac{x_n + h_n d}{2Ad}\right)^k = \frac{x_{kn} + h_{kn} d}{2Ad}. \tag{13}$$

Proof. From Eq. (5), we can write

$$h_{n+1} = (A\alpha)\,\alpha^n + (B\beta)\,\beta^n$$
$$h_n = A\alpha^n + B\beta^n.$$

Solving this system of equations permits us to obtain

$$\alpha^n = \frac{h_{n+1} - \beta h_n}{Ad} \text{ and } \beta^n = -\frac{h_{n+1} - \alpha h_n}{Bd}.$$

Since $(\alpha^n)^k = \alpha^{(nk)}$, with a little computation, the result follows. □

Note that Eq (13) has a similar form with the famous de Moivre's formula.

Theorem 3.3 (Pythagorean formula)**.** *Let n be any integer. Then, we have*

$$\left(\frac{p}{q^2} h_n h_{n+3}\right)^2 + \left(2Ph_{n+2}\left(2Ph_{n+2} - h_n\right)\right)^2 = \left(h_n^2 + 2Ph_{n+2}\left(2Ph_{n+2} - h_n\right)\right)^2,$$

where $P = \frac{p^2-q}{2q^2}$.

Proof. Using Eq. (4), we can write

$$\left(p^2 - q\right) h_{n+2} - ph_{n+3} = q^2 h_n$$
$$\left(p^2 - q\right) h_{n+2} + ph_{n+3} = 2\left(p^2 - q\right) h_{n+2} - q^2 h_n.$$

Multiplying these equations side-by-side, we obtain

$$\left(p^2 - q\right)^2 h_{n+2}^2 - p^2 h_{n+3}^2 = 2q^2\left(p^2 - q\right) h_n h_{n+2} - q^4 h_n^2$$

and we rearrange it to obtain

$$\left(ph_{n+3}\right)^2 = \left(\left(p^2 - q\right) h_{n+2}\right)^2 - 2q^2\left(p^2 - q\right) h_n h_{n+2} + \left(q^2 h_n\right)^2.$$

Dividing by q^2 after multiplying the last equation by h_n^2 and then adding the term $\left(p^2 - q\right) h_{n+2}\left(\left(p^2 - q\right) h_{n+2} - 2q^2 h_n\right)$ to each side, we obtain the result. □

From Theorem 3.3, we also obtain the following result.

Corollary 3.4. *All Pythagorean triples can be generated in terms of Horadam-Lucas numbers.*

Theorem 3.5. *For any integer n, we have*

$$h_{n+1}{}^2 - qh_n{}^2 = d^2\left[\left(b^2 - a^2 q\right) u_{2n+1} - aq\,(2b - ap)\, u_{2n}\right]. \tag{14}$$

Proof. To prove this property, we use the Binet's formula for h_n.

$$\begin{aligned} h_{n+1}{}^2 - qh_n{}^2 &= \left(A\alpha^{n+1} + B\beta^{n+1}\right)^2 - q\left(A\alpha^n + B\beta^n\right)^2 \\ &= A^2\alpha^{2n+2} + B^2\beta^{2n+2} + 2Eq^{n+1} - q\left(A^2\alpha^{2n} + B^2\beta^{2n} + 2Eq^n\right) \\ &= A^2\alpha^{2n+2} + B^2\beta^{2n+2} - qA^2\alpha^{2n} - qB^2\beta^{2n} \\ &= A^2\left(\alpha^2 - q\right)\alpha^{2n} + B^2\left(\beta^2 - q\right)\beta^{2n} \end{aligned} \tag{15}$$

Here after some mathematical operations, we can write

$$A^2\left(\alpha^2-q\right)=d\left[\left(b^2-a^2q\right)\alpha-aq\left(2b-ap\right)\right]$$

and

$$B^2\left(\beta^2-q\right)=-d\left[\left(b^2-a^2q\right)\beta-aq\left(2b-ap\right)\right].$$

Substituting these equations into Eq. (15) yields the desired result. □

We next prove the following important theorem, which can be used to obtain a number of special identities.

Theorem 3.6 (Vajda's identity)**.** *Let n, r, and s be any integers. Then,*

$$h_{n+s}h_{n-r}-h_nh_{n-r+s}=Eq^{n-r}\left(v_{r+s}-q^sv_{r-s}\right). \tag{16}$$

Proof.

$$\begin{aligned}
h_nh_{n-r+s}-h_{n+s}h_{n-r}&=\left(A\alpha^n+B\beta^n\right)\left(A\alpha^{n-r+s}+B\beta^{n-r+s}\right)\\
&\quad-\left(A\alpha^{n+s}+B\beta^{n+s}\right)\left(A\alpha^{n-r}+B\beta^{n-r}\right)\\
&=E\left(\alpha^n\beta^{n-r+s}+\alpha^{n-r+s}\beta^n-\alpha^{n-r}\beta^{n+s}-\alpha^{n+s}\beta^{n-r}\right)\\
&=E\left(\alpha^{n-r+s}\beta^{n-r+s}\left(\alpha^{r-s}+\beta^{r-s}\right)-\alpha^{n-r}\beta^{n-r}\left(\alpha^{r+s}+\beta^{r+s}\right)\right)\\
&=E\left(q^{n-r+s}v_{r-s}-q^{n-r}v_{r+s}\right)\\
&=-Eq^{n-r}\left(v_{r+s}-q^sv_{r-s}\right),
\end{aligned}$$

which is the desired result. □

From Vajda's identity, we also have the following special identities:

- For $r=s$, we obtain the Catalan's identity:
$$h_{n+r}h_{n-r}-h_n^2=Eq^{n-r}\left(v_{2r}-2q^r\right) \tag{17}$$
- For $r=s=1$, we find the Cassini's identity:
$$h_{n+1}h_{n-1}-h_n^2=Ed^2q^{n-1} \tag{18}$$
- For $n-r=m$ and $s=1$, we recover the d'Ocagne's identity:
$$h_mh_{n+1}-h_nh_{m+1}=Eq^m\left(v_{n-m+1}-qv_{n-m-1}\right) \tag{19}$$

In addition, we can prove the following theorem.

Theorem 3.7 (Gelin-Cesáro identity)**.** *For any integer n, we have*

$$h_{n-2}h_{n-1}h_{n+1}h_{n+2}-h_n^4=Ed^2q^{n-2}\left[\left(p^2+q\right)h_n^2+Ed^2p^2q^{n-1}\right]. \tag{20}$$

Proof. For $r=2$ in (17), we obtain

$$h_{n+2}h_{n-2}-h_n^2=Ed^2p^2q^{n-2}.$$

Combining the last equation with Cassini's identity, we can write

$$\begin{aligned}
h_{n-2}h_{n-1}h_{n+1}h_{n+2}&=\left(h_n^2+Ed^2q^{n-1}\right)\left(h_n^2+Ed^2p^2q^{n-2}\right)\\
&=h_n^4+\left(Ed^2q^{n-1}+Ed^2p^2q^{n-2}\right)h_n^2+Ed^2q^{n-1}Ed^2p^2q^{n-2}.
\end{aligned}$$

The last equation completes the proof. □

The next theorem provides a number of summation formulas for the Horadam-Lucas numbers.

Theorem 3.8. *Let n be any integer. Then, we have*

$$\sum_{i=1}^{n} h_i = \frac{h_{n+1} - qh_n - h_1 + qh_0}{p - q - 1} \tag{21}$$

and

$$\sum_{i=1}^{n} (-1)^i h_i = \frac{(-1)^n (h_{n+1} + qh_n) - h_1 - qh_0}{p + q + 1}, \tag{22}$$

Proof. We prove only the first summation formula. Let us denote the right-hand side of Eq. (21) by a_n. By the definition of the Horadam-Lucas numbers, we obtain

$$a_t - a_{t-1} = h_t.$$

Applying the idea of "creative telescoping" [6] to Eq. (21), we conclude

$$\sum_{i=1}^{n} h_i = \sum_{t=0}^{n} (a_t - a_{t-1}) = a_n - a_{-1},$$

and since $a_{-1} = 0$, the result follows. □

4. Matrix Approach to Second-order sequences

Note that the terms of the sequences in (1) and (4) may also be stated as matrix recurrence relations. We define

$$W_n = \begin{bmatrix} w_{n+1} & w_n \\ w_n & w_{n-1} \end{bmatrix} \text{ and } H_n = \begin{bmatrix} h_{n+1} & h_n \\ h_n & h_{n-1} \end{bmatrix}. \tag{23}$$

Then we can write

$$W_n = RW_{n-1} \text{ and } H_n = RH_{n-1}. \tag{24}$$

Extending the right-hand side of Eqs. (24) to zero, we obtain

$$W_n = R^n W_0 \text{ and } H_n = R^n H_0, \tag{25}$$

where

$$W_0 = \begin{bmatrix} b & a \\ a & \frac{pa-b}{q} \end{bmatrix} \text{ and } H_0 = \begin{bmatrix} bp - 2aq & 2b - ap \\ 2b - ap & \frac{bp-a(p^2-2q)}{q} \end{bmatrix}.$$

By Eq. (25), we can also obtain the following theorem:

Theorem 4.1 (Honsberger formula)**.** *Let n and m be any integers. Then, we have*

$$h_{n+m} = u_m h_{n+1} - qu_{m-1} h_n. \tag{26}$$

Proof. Replacing $n + m$ by n in Eq. (25), we can write

$$H_{n+m+1} = R^{n+m+1} H_0 = R^{n+1} H_m.$$

The bottom right entry of the matrix H_{n+m+1} is equal to the bottom right entry of the product matrix, which gives the first result. □

Note that in Theorem 4.1, symmetric exchanges of n with m in each equation are possible.

Theorem 4.2. *For any integers n and k, we can write*

$$h_{n-k} = q^{1-k}\left(u_k h_{n-1} - u_{k-1} h_n\right) = q^{-k}\left(h_n u_{k+1} - h_{n+1} u_k\right) \tag{27}$$

and

$$w_{n-k} = q^{1-k}\left(u_k w_{n-1} - u_{k-1} w_n\right) = q^{-k}\left(w_n u_{k+1} - w_{n+1} u_k\right). \tag{28}$$

Proof. By Eq. (25), we obtain

$$H_{n-k} = R^{n-k} H_0 = R^{-k} R^n H_0 = \left(R^k\right)^{-1} H_n.$$

By Eq. (18) after computing the inverse of R^k, we can write

$$H_{n-k} = \frac{1}{q^k}\begin{bmatrix} q\left(u_k h_n - u_{k-1} h_{n+1}\right) & q\left(u_k h_{n-1} - u_{k-1} h_n\right) \\ u_{k+1} h_n - u_k h_{n+1} & u_{k+1} h_{n-1} - u_k h_n \end{bmatrix}.$$

This completes the proof of Eq. (27). Eq. (28) can be proved similarly. □

From Theorems 4.1 and 4.2, we obtain the following conclusion:

Theorem 4.3 (Melham identity)**.** *Let n and k be any integers. Then,*

$$h_{n+k+1}{}^2 - q^{2k+1} h_{n-k}{}^2 = d^2 u_{2k+1}\left[\left(b^2 - a^2 q\right) u_{2n+1} - aq\left(2b - ap\right) u_{2n}\right]. \tag{29}$$

Proof. Considering Eqs. (26) and (27), we can write

$$\begin{aligned} h_{n+k+1}{}^2 - q^{2k+1} h_{n-k}{}^2 &= u_{k+1}{}^2 h_{n+1}{}^2 + q^2 u_k{}^2 h_n{}^2 - 2q u_{k+1} u_k h_{n+1} h_n \\ &\quad - q^{2k+1} q^{-2k}\left\{h_n{}^2 u_{k+1}{}^2 + h_{n+1}{}^2 u_k{}^2 - 2u_{k+1} u_k h_{n+1} h_n\right\} \\ &= \left(u_{k+1}{}^2 - q u_k{}^2\right)\left(h_{n+1}{}^2 - q h_n{}^2\right) \end{aligned}$$

Applying Eq. (14) to the last equation, we obtain the claimed result. □

Theorem 4.4 (General bilinear formula)**.** *Let a, b, c, d, and r be any integers satisfying $a+b=c+d$. Then, we have*

$$u_a h_b - u_c h_d = q^r\left(u_{a-r} h_{b-r} - u_{c-r} h_{d-r}\right) \tag{30}$$

and

$$u_a w_b - u_c w_d = q^r\left(u_{a-r} w_{b-r} - u_{c-r} w_{d-r}\right). \tag{31}$$

Proof. Employing the matrix equations in (23) and (24), we obtain $R^a H_b = R^c H_d$. Considering the bottom left entry of the result, we can write

$$u_a h_b - u_c h_d = q\left(u_{a-1} h_{b-1} - u_{c-1} h_{d-1}\right).$$

Repeating the same operations r times yields Eq. (30). The other result can be proved in a similar way. □

5. Conclusion

The present work describes a new class of a second-order integer sequence that are closely related to Horadam's generalized one. In this scope, the main definition and the related Binet-type formula are given. To be clear, it is called the Horadam-Lucas sequence. We give some special identities for this generalized sequence, including De Moivre's Formula, Pythagorean formula, Vajda's identity, and Gelin-Cesáro identity. A matrix treatment using the determinantal and multiplicative properties is developed to achieve additional results such as the Melham identity and General bilinear formula.

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