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# A NOTE ON THE EXISTENCE OF A UNIVERSAL POLYTOPE AMONG REGULAR 4-POLYTOPES

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**Abstract.** For a polytope $P$, the set of all of its vertices is denoted by $V(P)$. For polytopes $P$ and $Q$ of the same dimension, we write $P \subset Q$ if $V(P) \subset V(Q)$. An $n$-polytope ($n$-dimensional polytope) $Q$ is said to be universal for a family $\mathfrak{P}_n$ of all regular $n$-polytopes if $P \subset Q$ holds for every $P \in \mathfrak{P}_n$. The set $\mathfrak{P}_4$ consists of six regular 4-polytopes. It is stated implicitly in Coxeter [2] by applying finite discrete groups that a regular 120-cell is universal for $\mathfrak{P}_4$. Our purpose of this note is to give a simpler proof by using only metric properties. Furthermore, we show that the corresponding property does not hold in any other dimension but 4.

*Key words and Phrases*: Inclusion property.

**Abstrak.** Untuk suatu politop $P$, himpunan semua titik-titiknya dinotasikan dengan $V(P)$. Untuk politop $P$ dan $Q$ dengan dimensi sama, kita tulis $P \subset Q$ jika $V(P) \subset V(Q)$. Sebuah politop-$n$ (politop berdimensi-$n$) $Q$ dikatakan menjadi universal untuk suatu keluarga $\mathfrak{P}_n$ dari semua politop-$n$ regular jika $P \subset Q$ berlaku untuk setiap $P \in \mathfrak{P}_n$. Himpunan $\mathfrak{P}_4$ memuat 6 politop-$n$ regular. Telah dinyatakan secara implisit di Coxeter [2] dengan menerapkan grup diskrit hingga bahwa sebuah sel-120 regular adalah universal terhadap $\mathfrak{P}_4$. Pada paper ini kami akan memberi sebuah bukti yang lebih sederhana dengan hanya menggunakan sifat-sifat metrik. Lebih jauh kami menunjukkan bahwa sifat-sifat yang terkait tidak dipenuhi, kecuali pada dimensi 4.

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## 1. Introduction

For a polytope $P$, let us call the set of all of its vertices the *vertex set* of $P$ and denote it by $V(P)$. In this paper we investigate the problem of deciding whether a chosen proper subset of the vertex set of a given polytope is the vertex set of some other polytope or not.

**Definition 1.1.** *For polytopes $P$ and $Q$ of the same dimension, we say that $P$ is contained in $Q$ and write $P \subset Q$, if $V(P) \subset V(Q)$ holds.*

**Definition 1.2.** *We say that an $n$-dimensional polytope $Q$ is a universal polytope for a family $\mathfrak{P}$ of $n$-dimensional regular polytopes, if $P \subset Q$ holds for every $P \in \mathfrak{P}$.*

It is well known (see [2]) that there are 5 kinds of regular polytopes in dimension 3, 6 kinds in dimension 4 and 3 kinds in dimension $n \geq 5$. We investigate the question whether there exists a universal regular polytope or not in each dimension. We take up the case of dimension 3 in Section 2, of dimension 4 in Section 3 and of dimension $n \geq 5$ in Section 5, and obtain results on the inclusion relation among regular polytopes, and in particular, on the existence of a universal polytope in each dimension.

## 2. Inclusion Relation among 3-dimensional Regular Polyhedra and Non-existence of Universal Polyhedron in Dimension 3

There are 5 kinds of regular polyhedra in dimension 3: regular tetrahedra, cubes, regular octahedra, regular dodecahedra and regular icosahedra, and they have $4, 8, 6, 20, 12$ vertices, respectively. As shown in figure $1(a)$ below, if we choose 4 points (8 points) from the vertex set, consisting of 20 points, of a regular dodecahedron suitably, then we get the vertex set of a regular tetrahedron (a cube, respectively).

However, the situation is different for the case of regular octahedra and of regular icosahedra. Namely, it is well-known that no subset of the vertex set of a regular dodecahedron can be the vertex set of a regular octahedron or of a regular icosahedron. Since a regular dodecahedron has the most number of vertices among the 3-dimensional regular polyhedra, a universal polyhedron, if it exists in 3-dimension, must be a regular dodecahedron. Thus we conclude that there is no universal polyhedron among 3-dimensional polyhedra. (However, as indicated in figure $1(b)$ below, it is well-known that the vertex set of a regular octahedron or a regular icosahedron can be obtained from a cube or a dodecahedron by choosing the centroid from suitably chosen faces of a cube or a dodecahedron.)

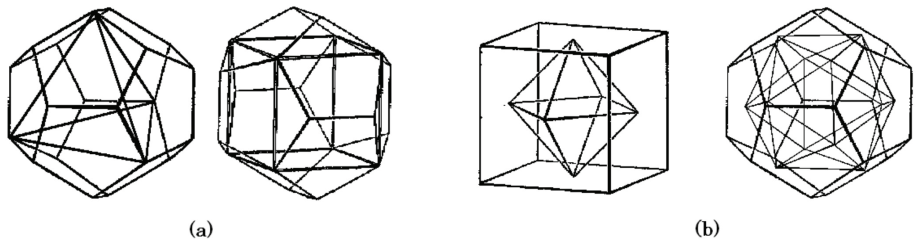

Figure 1. Inclusion relation among regular polyhedra

## 3. Inclusion Relation among 4-dimensional Regular Polytopes and Existence of Universal Polytopes in Dimension 4

There are 6 types of 4-dimensional regular polytopes. They are regular 5-cell (denoted by $C_5$, in the sequel), regular 8-cell ($C_8$), regular 16-cell ($C_{16}$), regular 24-cell ($C_{24}$), regular 120-cell ($C_{120}$) and regular 600-cell ($C_{600}$). They have the vertex sets consisting of $5, 16, 8, 24, 600, 120$ vertices, respectively. The following theorem describes the inclusion relationship among these 6 types.

**Theorem 3.1.** *The regular* 120*-cell is a universal polytope for* 4*-dimensional regular polytopes. More precisely, the following inclusion relations hold:*

*(i)* $C_{16} \subset C_8 \subset C_{24} \subset C_{600} \subset C_{120}$

*(ii)* $C_5 \subset C_{120}$

Proof. The book by Coxeter [2] lists in pages $156 \sim 158$ the coordinates of all the vertices for each of the 6 kinds of 4-dimensional regular polytopes. However, Coxeter uses different coordinate systems for describing coordinates for polytopes in classes $C_{24}$ and $C_{120}$, and for those in classes $C_5, C_{16}, C_8, C_{600}$. Let us call the former $\alpha$-system and the latter $\beta$-system. In order to establish the inclusion relation we seek, let us transform $\alpha$-system to $\beta$-system.

For this purpose, let us denote by $\mathfrak{P}$ the set of 24 points consisting of all possible permutations of the 4 points $(\pm 2, \pm 2, 0, 0)$, (here and below, all possible combinations of the signs are allowed) chosen from the vertex set of $C_{120}$. Let us also denote by $\mathfrak{Q}$ the set of 24 points, 16 of which are $(\pm 2, \pm 2, \pm 2, \pm 2)$ obtained by doubling the coordinates in $\beta$-system of the vertices of $C_8$ and, 8 others are all possible permutations of $(\pm 4, 0, 0, 0)$, which are obtained by doubling the coordinates in $\beta$-system of the vertices of $C_{16}$. It is then enough to find a $4 \times 4$ matrix $R$ which gives a linear transformation mapping 4 pairwise orthogonal points

$P_1(2,2,0,0)$, $P_2(2,-2,0,0)$, $P_3(0,0,2,2)$, $P_4(0,0,2,-2)$ in $\mathfrak{P}$ onto 4 pairwise orthogonal points $Q_1(4,0,0,0)$, $Q_2(0,4,0,0)$, $Q_3(0,0,4,0)$, $Q_4(0,0,0,4)$ in $\mathfrak{Q}$, respectively. For example,

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

gives such a linear transformation. This is an orthogonal transformation followed by multiplication by $\sqrt{2}$.

According to Coxeter's book, the coordinates in $\alpha$-system of the 600 vertices of $C_{120}$ are given as follows (we denote by $\tau$ the golden ratio $\frac{1+\sqrt{5}}{2}$):
All possible permutations of $(\pm 2, \pm 2, 0, 0)$, $(\pm\sqrt{5}, \pm 1, \pm 1, \pm 1)$, $(\pm\tau, \pm\tau, \pm\tau, \pm\frac{1}{\tau^2})$, $(\pm\tau^2, \pm\frac{1}{\tau}, \pm\frac{1}{\tau}, \pm\frac{1}{\tau})$, and all possible even permutations of $(\pm\tau^2, \frac{1}{\tau^2}, \pm 1, 0)$, $(\pm\sqrt{5}, \pm\frac{1}{\tau}, \pm\tau, 0)$, $(\pm\tau, \pm 1, \pm\tau, \pm\frac{1}{\tau})$. If we transform these points by the linear transformation given by $R$, we get the following disjoint sets of points (we denote below by $\sigma$ the number $\frac{3\sqrt{5}+1}{2}$ and by $\sigma'$ the number $\frac{3\sqrt{5}-1}{2}$):

- A : The set of 16 points consisting of $(\pm 2, \pm 2, \pm 2, \pm 2)$
- B : The set of 8 points consisting of all possible permutations of $(\pm 4, 0, 0, 0)$
- C : The set of 192 points consisting of all possible permutations of $(\pm 2\tau, \pm 2, \pm\frac{2}{\tau}, 0)$
- D : The set of 256 points obtained by putting an even number of minus signs to coordinates of each point in the set of all permutations of the numbers $(\sqrt{5}, \sqrt{5}, \sqrt{5}, 1)$, $(\tau^2, \tau^2, \frac{\sqrt{5}}{\tau}, \frac{1}{\tau})$, $(\sigma, \frac{1}{\tau}, \frac{1}{\tau}, \frac{1}{\tau})$, $(\sqrt{5}\tau, \tau, \frac{1}{\tau^2}, \frac{1}{\tau^2})$
- E : The set of 128 points obtained by putting an odd number of minus signs to coordinates of each point in the set of all permutations of the numbers $(\sigma', \tau, \tau, \tau)$, $(3, \sqrt{5}, 1, 1)$

Now, using $\beta$-system of coordinates, we can compute distances between pairs of points, dihedral angles and dichoral angles, and can obtain the following results.

- (1) : $V(C_5)$ is the set of 5 points consisting of all possible permutations of the point $(-\sigma', \tau, \tau, \tau)$ belonging to the set $E$ and the point $(-2, -2, -2, -2)$ belonging to the set $A$
- (2) : $V(C_8) = A$
- (3) : $V(C_{16}) = B$
- (4) : $V(C_{24}) = A \cup B$
- (5) : $V(C_{120}) = A \cup B \cup C \cup D \cup E$
- (6) : $V(C_{600}) = A \cup B \cup C'$, where $C'$ is a subset of $C$ consisting of 96 points obtained by applying all possible even permutations to $(\pm 2\tau, \pm 2, \pm\frac{2}{\tau}, 0)$.

From (1) $\sim$ (6) we see that the vertex sets of $C_5, C_8, C_{16}, C_{24}, C_{600}$ are all proper subsets of the vertex set of $C_{120}$, and therefore, we conclude that $C_{120}$ is a universal polytope for 4-dimensional regular polytopes. □

Although Theorem 3.1 above shows that $C_{120}$ is a universal polytope for 4-dimensional polytopes, we note that the inclusion relation splits in two branches. You might think that, by splitting 120 vertices of $C_{600}$ suitably into 24 groups of 5 vertices each, it may be possible to obtain 24 concentric 5-cells. However, we can show that such a procedure is impossible. Let us first quote the following theorem (see [1]) which we need for giving a proof for our Theorem 3.2.

For a given polytope $\Pi$ with $v$ vertices $P_1, P_2, \cdots P_v$, we define the *diagonal weight* of $\Pi$ as the sum of the squares of the lengths of all diagonals and sides of $\Pi$, and denote it by $\alpha(\Pi)$. Namely,

$$\alpha(\Pi) = \sum_{P_i, P_j} (d(P_i, P_j))^2,$$

where $d(P_i, P_j)$ is the distance between $P_i$ and $P_j$, and the sum is taken over all possible pairs of $P_i$ and $P_j$.

Then we have

**Theorem A.** Let $R$ be a regular $n$-dimensional polytope with $v$ vertices $P_1, P_2, \cdots P_v$ which is inscribed in a unit $n$-sphere. Then the diagonal weight $\alpha(R)$ is $v^2$ for every dimension $n \geq 2$.

Using this theorem we obtain the following:

**Theorem 3.2.** *The regular* 5*-cell* $C_5$ *is not contained in the regular* 600*-cell* $C_{600}$*; namely,* $C_5 \not\subset C_{600}$.

PROOF.Let us compute the length of the side of $C_5$. 5 vertices of $C_5$ lie on its circum-sphere of radius 4. $C_5$ also has 10 sides, and their length $d = d_i (1 \leq i \leq 10)$ are all equal. Therefore, by Theorem A, $\sum_i \left(\frac{d_i}{4}\right)^2 = \sum \left(\frac{d}{4}\right)^2 = 5^2$. Consequently, each side has the length $d = 2\sqrt{10}$. On the other hand, the lengths of the diagonals of the $C_{600}$ which is inscribed in the same sphere are

$$2(\sqrt{5}-1),\ 4,\ 2\sqrt{10-2\sqrt{5}},\ 4\sqrt{2},\ 2(\sqrt{5}+1),\ 4\sqrt{3},\ 2\sqrt{10+2\sqrt{5}},\ 8$$

listed in increasing order. Since these numbers are all different from $2\sqrt{10}$, we conclude that $C_5 \not\subset C_{600}$. □

## 4. Inclusion Relation among $n$-dimensional Polytopes for $n \geq 5$ and Non-existence of Universal Polytopes in Dimensions $n \geq 5$

There are 3 kinds of regular polytopes in dimension $n \geq 5$. They are $n$-**simplexes** (denoted in the sequel by $\alpha_n$), $n$**-orthoplexes** ($\beta_n$) and $n$**-cubes** ($\gamma_n$), and they have $n+1$, $2n$, $2^n$ vertices, respectively.

**Theorem 4.1.** *There exists no universal polytope for n-dimensional regular polytopes for any* $n \geq 5$.

PROOF. Let us determine the lengths and the number of sides and diagonals for each of the 3 kinds of regular $n$-dimensional polytopes.

$(A)$ For $\alpha_n$:

Let the coordinates of $n$ among the $n+1$ vertices of the $n$-simplex be given by the all permutations of $(1,0,0,\cdots,0)$. By symmetry, we can write $(x,x,\cdots,x)$. The distances between any pair of the vertices are all equal, and their value is $\sqrt{2}$. Hence, we have $(x-1)^2+(n-1)x^2=2$, from which we conclude that $x=\frac{1\pm\sqrt{1+n}}{n}$. We choose here $x=\frac{1-\sqrt{1+n}}{n}$. Then, we see that the radius of the circum-sphere of our simplex must equal $\sqrt{\frac{n}{n+1}}$. Hence, for the $n$-simplex whose circum-sphere has radius 1, the length between any pair of vertices and the number of such distinct pairs (i.e., its sides) are $L_1=\sqrt{\frac{2(n+1)}{n}}$ and $n_1=\frac{(n+1)n}{2}$, respectively.

$(B)$ For $\beta_n$:

Let the coordinates of the $2n$ vertices of an $n$-orthoplex be given by all the permutations of $(\pm1,0,0,\cdots,0)$. Then the radius of the circum-sphere for the $n$-orthoplex is 1, and the length of a side of this orthoplex is $L_1=\sqrt{2}$ and the number of sides is $n_1=\frac{n(n-1)}{2}$, and the length of a diagonal is $L_2=2$ and the number of diagonals is $n_2=n$.

$(C)$ For $\gamma_n$:

Let the coordinates of the $2^n$ vertices of an $n$-cube be given by all the permutations of $(\pm1,\pm1,\cdots,\pm1)$. Then the radius of the circum-sphere of this $n$-cube is $\sqrt{n}$. Hence for the $n$-cube whose circum-sphere has radius 1, the lengths of its sides and diagonals and their numbers are given by $L_i=2\sqrt{\frac{i}{n}}$ and $n_i=\frac{n!}{i!(n-i)!}$ for $1\le i\le n$.

Now in order to complete the proof, let us suppose that there exists a universal polytope in $n$-dimension $(n\ge5)$. Then it has to be an $n$-cube, since $n$-cubes have the largest number of vertices among regular $n$-polytopes. But then from $(A)\sim(C)$ we conclude that there must exist positive integers $k$ and $\ell$ for which $\sqrt{\frac{4k}{n}}=\sqrt{\frac{2(n+1)}{n}}$ and $\sqrt{\frac{4\ell}{n}}=\sqrt{2}=\sqrt{\frac{2n}{n}}$ must hold. However, from the former identity we get $4k=2(n+1)$ and hence $n=2k-1$, implying that $n$ must be odd, while from the latter identity we get $4\ell=2n$ and hence $n=2\ell$, implying that $n$ must be even. Thus we get a contradiction, and therefore, we conclude that there is no universal polytope in dimension $n\ge5$. □

## 5. CONCLUDING REMARKS

We conclude from Theorem 1, Theorem 2 and Theorem 3 that only in 4-dimension, universal polytopes exist. In this sense, 4-dimensional space exhibits a very different characteristic from other dimensions.

The referees pointed out that an old theorem by Hess says that every regular star-polytope of dimension $n$ has the same vertices as a regular convex polytope of dimension $n$ (see Theorem 7D6 in [3]). When applied with $n = 4$, Theorem 3.1 can be stated in the stronger form: The 120-cell is universal among all regular 4-polytopes, convex or starry.

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## References

[1] Akiyama, J., and Sato, I., "On the Diagonal Weights of Inscribed Polytopes", *to appear in Elemente der Mathematik*.
[2] Coxeter, H.S.M., *Regular Polytopes*, Dover Publications, 1973.
[3] McMullen, P., and Schulte E., *Abstract Regular Polytopes*, Cambridge Univ. Press, 2002.