J. Indones. Math. Soc.
Vol. 29, No. 03 (2023), pp. 328–337.

HOMOMORPHISMS OF COMPLEX KUMJIAN-PASK ALGEBRAS

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Abstract. Let Λ and Γ be row finite k-graphs without sources. We show that *-algebra homomorphisms $\phi : KP_{\mathbb{C}}(\Lambda) \to KP_{\mathbb{C}}(\Gamma)$ extend to *-algebra homomorphisms $\bar{\phi} : C^*(\Lambda) \to C^*(\Gamma)$. We also examine necessary and sufficient conditions for algebra homomorphisms between complex Kumjian-Pask algebras $KP_{\mathbb{C}}(\Lambda)$ and $KP_{\mathbb{C}}(\Gamma)$ which are *-preserving.

Key words and Phrases: Kumjian-Pask algebra, Leavitt path algebra, Homomorphism, k-graph

1. INTRODUCTION

For any commutative ring K with 1 and a row finite k-graph Λ without source, Aranda-Pino, Clark, Huef and Raeburn [13] introduced Kumjian-Pask algebra $KP_K(\Lambda)$. Since then many researchers sought to explore and extend this algebra, [16, 17, 18, 21, 7, 19, 20, 4, 3, 5, 9, 6, 8, 12] are among others.

Kumjian-Pask algebra is a purely algebraic analog of k-graph algebra $C^*(\Lambda)$ which was firstly introduced by Kumjian and Pask [10]. As another point of view, the Kumjian-Pask algebra is a generalization of the notion of Leavitt-path algebra of [2] for the row finite graph E to higher rank graph Λ. When we specify the ring K to be the complex field $\mathbb{C}$, we can investigate relation between k-graph algebras $C*(\Lambda)$ and the complex Kumjian-Pask algebras $KP_{\mathbb{C}}(\Lambda)$. In [16] it was shown that the algebraic structure of the complex Kumjian-Pask algebras is the same with the algebraic structure of the k-graph algebras. The relation can be seen through the injective algebra *-homomorphism $\iota_\Lambda : KP_{\mathbb{C}}(\Lambda) \to C^*(\Lambda)$ which maps generators of $KP_{\mathbb{C}}(\Lambda)$ to the generators of $C^*(\Lambda)$. Through ι_Λ we can view $KP_{\mathbb{C}}(\Lambda)$ as a dense *-subalgebra of $C^*(\Lambda)$. Furthermore when $|\Lambda^0| < \infty$, it was

2020 Mathematics Subject Classification: 16S10, 16W50, 46L05

Received: 03-08-2022, accepted: 03-10-2023.

proved in Proposition 4.3 of [16] that the $C^*(\Lambda)$ (as a *-algebra or as a *-ring) is isomorphic to the Kumjian-Pask algebra $KP_{\mathbb{C}}(\Lambda)$.

Kumjian-Pask algebras also can be seen as a generalization of Leavitt path algebras of directed graphs to higher rank graphs. Leavitt path algebras are purely algebraic analogs of graph algebras and results on Leavitt path algebras emerge as important as determining the C^*-algebra structure for the graph algebras. Similarly, investigating algebraic structure of Kumjian-Pask algebras is very important just as investigating the C^*-algebra structure of the k-graph algebras.

In [1], Abrams and Tomforde investigated homomorphisms between Leavitt path algebras. They proved that *-homomorphisms between complex Leavitt path algebras extend to homomorphisms between associated graph C^*-algebras. They also examined algebra homomorphisms between complex Leavitt path algebras and obtained necessary and sufficient conditions for an algebra homomorphism between complex Leavitt path algebras to be a *-algebra homomorphism. These results provide a more descriptive translation from Leavitt path algebras to graph C^*-algebras.

We aim to establish an anlogue of Abrams and Tomforde's results for Kumjian Pask algebras. To obtain this, we used results in [16] to generalize the idea of Abrams and Tomforde to a more general case, i.e to row finite k-graph without sources and we obtained an analogue of their results for higher rank graphs. We begin with some preliminaries in Section 2 to establish notations and basic facts on k-graphs, k-graph algebras and Kumjian-Pask algebras. In Result (Subsection 3.1), we demonstrate that every homomorphism between complex Kumjian-Pask algebras extends to $*$-homomorphism of k-graph algebras. This result is an extension of Abrams and Tomforde's result to higher rank graph. In Subsection 3.2, we employ the result in Subsection 3.1 to obtain necessary and sufficient conditions for an algebra homomorphism between complex Kumjian-Pask algebras which is *-preserving. This is a generalization of Proposition 5.3 of Abrams and Tomforde [1] to higher rank graphs.

2. Preliminaries of Kumjian-Pask Algebras

We recall some terminologies which will be used in the sequel. Kumjian and Pask [10] introduced k-graphs also commonly referred to as higher rank graphs. Higher rank graphs are higher-dimensional analogues of directed graphs. To deal with higher rank graphs, we view the pointwise additive semigroup $\mathbb{N}^k := \{0, 1, 2, ...\}^k$ as the morphisms in a category with one object, and the composition map in the category is given by the addition in $\mathbb{N}^k$. We denote this category by $\mathbb{N}^k$. In [10, Definition 1], a k-graph is defined as a countable category $\Lambda = (\Lambda^0, \Lambda, r, s)$ together with a functor $d : \Lambda \to \mathbb{N}^k$ that satisfies the factorisation property: for each (morphism) $\lambda \in \Lambda$ and each decomposition $d(\lambda) = m + n$ with $m, n \in \mathbb{N}^k$, there exists unique μ and ν such that $d(\mu) = m,\ d(\nu) = n$ with $\lambda = \mu\nu$.

Given any k-graph Λ and $n \in \mathbb{N}^k$, the symbol Λ^n denotes the set of paths of degree n and the symbol $\Lambda^{\neq 0}$ denotes the set of paths of nonzero degree. The symbol $v\Lambda^n$ denotes the set of paths degree $n \in \mathbb{N}^k$ with range $v \in \Lambda^0$.

Definition 2.1. *Suppose Λ is a k-graph. If for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, the set $v\Lambda^n$ is finite, Λ is called row-finite. If $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, Λ is called has no sources.*

In [13], Aranda-Pino, Clark, Huef and Raeburn introduced the concept of the Kumjian-Pask algebra of k-graph without sources which gives a purely algebraic model for the k-graph algebras. This algebra is a generalization of Leavitt path algebra $L_K(E)$ for row finite graph E and a field K to higher-rank graph. When K is the field of complex numbers, then $L_{\mathbb{C}}(E)$ describes the algebraic structure of the graph algebra $C^*(E)$ of [11].

Let Λ be a k-graph. For any $\lambda \in \Lambda$ define a ghost path λ^*, as a new path with the degree, source and range defined by

$$d(\lambda^*) = -d(\lambda),\ r(\lambda^*) = s(\lambda),\ s(\lambda^*) = r(\lambda).$$

The set of ghost paths is denoted by $G(\Lambda)$, or $G(\Lambda^{\neq 0})$ if the vertices are excluded. For $\lambda, \mu \in \Lambda^{\neq 0}$ with $r(\mu^*) = s(\lambda^*)$, we set $\lambda^*\mu^* = (\mu\lambda)^*$ to obtain the composition on $G(\Lambda)$.

Definition 2.2. [13, Definition 3.1] *Let R be a commutative ring with 1 and Λ be a row finite k-graph without sources. Functions $P : \Lambda^0 \to A$ and $S : \Lambda^{\neq 0} \to A$ satisfying Kumjian-Pask relations:*

(KP1) $\{P_v : v \in \Lambda^0\}$ *is a family of mutually orthogonal idempotents,*

(KP2) for all $\lambda, \mu \in \Lambda^{\neq 0}$ *with* $r(\mu) = s(\lambda)$, *we have*

$$S_\lambda S_\mu = S_{\lambda\mu},\ S_{\mu^*}S_{\lambda^*} = S_{(\lambda\mu)^*},\ P_{r(\lambda)}S_\lambda = S_\lambda = S_\lambda P_{s(\lambda)},$$
$$P_{s(\lambda)}S_{\lambda^*} = S_{\lambda^*} = S_{\lambda^*}P_{r(\lambda)},$$

(KP3) for all $\lambda, \mu \in \Lambda^{\neq 0}$ *with* $d(\lambda) = d(\mu)$ *we have*

$$S_{\lambda^*}S\mu = \delta_{\lambda,\mu}P_{s(\lambda)},$$

(KP4) for all $v \in \Lambda^0$ *and all* $n \in \mathbb{N}^k \setminus \{0\}$ *we have*

$$P_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S_{\lambda^*},$$

is called a Kumjian-Pask Λ-family (P, S) in an R-algebra A.

Remark 2.3. The notation for the Kumjian-Pask family follows the convention in [15]: where lowercase letters will be used when the Kumjian-Pask family has a universal property.

The Kumjian-Pask algebra, denoted as $KP_R(\Lambda)$, is defined for a row finite k-graph Λ without sources and commutative ring R with 1. It is an R-algebra generated by the Kumjian-Pask Λ-family (p, s) [13]. To show the existence of such algebra, Aranda-Pino and his collaborators define a free algebra $\mathbb{F}_R(w(X))$ on $X := \Lambda^0 \cup \Lambda^{\neq 0} \cup G(\Lambda^{\neq 0})$. Let I be the ideal generated by the union of the following sets:

- $\{vw - \delta_{v,w}v : v, w \in \Lambda^0\}$;
- $\{\lambda - \mu\nu, \lambda^* - \nu^*\mu^* : \lambda, \mu, \nu \in \Lambda^{\neq 0}$ and $\lambda = \mu\nu\}$
$\cup\{r(\lambda)\lambda - \lambda, \lambda - \lambda s(\lambda), s(\lambda)\lambda^* - \lambda^*, \lambda^* - \lambda^* r(\lambda) : \lambda \in \Lambda^{\neq 0}\}$;
- $\{\lambda^*\mu - \delta_{\lambda,\mu}s(\lambda) : \lambda, \mu \in \Lambda^{\neq 0}$ such that $d(\lambda) = d(\mu)\}$;
- $\{v - \sum_{\lambda \in v\Lambda^n} \lambda\lambda^* : v \in \Lambda^0,\ n \in \mathbb{N}^k \setminus \{0\}\}$.

The quotient $\mathbb{F}_R(w(X))/I$ is then defined as the Kumjian-Pask algebra $KP_R(\Lambda)$. In [13, Theorem 3.4] it was demonstrated that this algebra is universal in the following sense: whenever (Q, T) is a Kumjian-Pask Λ-family in an R-algebra A, there exists a unique R-algebra homomorphism $\pi_{Q,T} : KP_R(\Lambda) \to A$ such that

$$\pi_{Q,T}(p_v) = Q_v,\ \pi_{Q,T}(s_\lambda) = T_\lambda,\ \pi_{Q,T}(s_{\mu^*}) = T_{\mu^*}$$

for $v \in \Lambda^0$ and $\lambda, \mu \in \Lambda^{\neq 0}$.

Remark 2.4. The Kumjian-Pask relations (KP1) through (KP4), along with the convention that $S_v := P_v$ and $S_{v^*} = P_v$ for $v \in \Lambda^0$ implies that the Kumjian-Pask algebra $KP_R(\Lambda)$ generated by a Kumjian-Pask family (P, S) is

$$\operatorname{span}\{S_\lambda S_{\mu^*} : \lambda, \mu \in \Lambda \text{ with } s(\lambda) = s(\mu)\}. \tag{1}$$

When we consider the coefficient ring R is the field of complex number $\mathbb{C}$, the structure of $KP_{\mathbb{C}}(\Lambda)$ is a *-algebra. This is possible because we can define an involution in $KP_{\mathbb{C}}(\Lambda)$ through the mapping

$$cs_\lambda s_{\mu^*} \mapsto \bar{c}s_\mu s_{\lambda^*}.$$

Suppose Λ is a row finite k-graph without sources, Kumjian and Pask [10] define a Cuntz-Krieger Λ-family in $C^*(\Lambda)$, i.e. a family of partial isometries $\{T_\lambda : \lambda \in \Lambda\}$ which satisfying the Cuntz-Krieger relations:

(CK1) $\{T_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections,
(CK2) $T_{\lambda\mu} = T_\lambda T_\mu$ for all $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$,
(CK3) $T_\lambda^* T_\lambda = T_{s(\lambda)}$,
(CK4) $T_v = \sum_{\lambda \in v\Lambda^m} T_\lambda T_\lambda^*$ for $v \in \Lambda^0$ and $m \in \mathbb{N}^k$.

For $v \in \Lambda^0$, we write $Q_v := T_v$, and for $\mu \in \Lambda^{\neq 0}$ we denote $T_{\mu^*} := T_\mu^*$. A straightforward computation shows that the family $(Q, T) := \{T_\lambda : \lambda \in \Lambda\}$ satisfies (KP1) through (KP4), and hence is a Kumjian-Pask Λ-family in $C^*(\Lambda)$. For example, let $\mu, \lambda \in \Lambda^{\neq 0}$ with $d(\lambda) = d(\mu)$. If $v = r(\lambda) = r(\mu)$, it is easy to see that $T_{\lambda^*}T_\mu = Q_{s(\lambda)}$ by (CK3). If $v = r(\lambda) \neq r(\mu)$, then $\lambda \neq \mu$ which implies

$$\begin{aligned} T_{\lambda^*}T_\mu &= T_{\lambda^*}T_\lambda T_{\lambda^*}T_\mu T_{\mu^*}T_\mu \\ &= T_{\lambda^*}(T_\lambda T_{\lambda^*})(T_\mu T_{\mu^*})T_\mu \\ &= 0 \end{aligned}$$

because $T_\lambda T_{\lambda^*}$ and $T_\mu T_{\mu^*}$ are orthogonal. Then (KP3) is satisfied.

The existence of a unique ($\mathbb{C}$-algebra) homomorphism $\pi_{Q,T} : KP_{\mathbb{C}}(\Lambda) \to C^*(\Lambda)$ such that

$$\pi_{Q,T}(P_v) = Q_v,\ \pi_{Q,T}(S_\lambda) = T_\lambda,\ \pi_{Q,T}(S_{\mu^*}) = T_{\mu^*} := T_\mu^*, \tag{2}$$

is guaranted by Theorem 3.4 of [13]. Hence from (1), $\pi_{Q,T}$ maps $KP_{\mathbb{C}}(\Lambda)$ onto $A = \operatorname{span}\{T_\lambda T_\mu^* : \lambda, \mu \in \Lambda\}$, and $\pi_{Q,T}$ is injective by Theorem 4.1 of [13]. By [13, Lemma 7.4], the algebra A forms a dense *-subalgebra of $C^*(\Lambda)$. Since it is $\mathbb{Z}^k$-graded, $KP_{\mathbb{C}}(\Lambda)$ can be viewed as a dense *-subalgebra of the k-graph algebra $C^*(\Lambda)$. In [16] it was discussed some condition which implies equivalence between $KP_{\mathbb{C}}(\Lambda)$ and $C^*(\Lambda)$ as *-algebras or as *-rings.

Remark 2.5. When a specific Kumjian-Pask family is not explicitly specified, Rosjanuardi in [16] write ι_Λ for the injection $KP_{\mathbb{C}}(\Lambda) \to C^*(\Lambda)$ instead of particular $\pi_{Q,T}$, and we will follow this notation.

3. MAIN RESULTS

3.1. **Extension of *-homomorphism of Kumjian-Pask Algebras.**

For any graphs E and F, in [1] Abrams and Tomforde showed that $*$-homomorphisms between $L_{\mathbb{C}}(E)$ and $L_{\mathbb{C}}(F)$ extends to *-homomorphisms between $C^*(E)$ and $C^*(F)$. Hence, if $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ (as *-algebras), it implies that $C^*(E) \cong C^*(F)$. We prove that a similar result for higher-rank graph can be obtained.

Proposition 3.1. *Let Λ and Γ be row finite k-graphs without sources. If $\iota_\Lambda, \iota_\Gamma$ are injections as in Remark 2.5, and $\phi : KP_{\mathbb{C}}(\Lambda) \to KP_{\mathbb{C}}(\Gamma)$ is a *-algebra homomorphism, then there is a unique *-algebra homomorphism $\bar{\phi} : C^*(\Lambda) \to C^*(\Gamma)$ such that the following diagram*

$$\begin{array}{ccc} C^*(\Lambda) & \xrightarrow{\bar{\phi}} & C^*(\Gamma) \\ {\scriptstyle \iota_\Lambda}\uparrow & & \uparrow{\scriptstyle \iota_\Gamma} \\ KP_{\mathbb{C}}(\Lambda) & \xrightarrow{\phi} & KP_{\mathbb{C}}(\Gamma) \end{array}$$

commutes. Moreover, if ϕ is a *-algebra isomorphism, then $\bar{\phi}$ is a *-algebra isomorphism.

Proof. Let $\{s_\lambda : \lambda \in \Lambda\}$ be a generating Cuntz-Krieger Λ-family in $C^*(\Lambda)$ and $\{t_\lambda : \lambda \in \Gamma\}$ be a generating Cuntz-Krieger Γ-family in $C^*(\Gamma)$. Let $KP_{\mathbb{C}}(\Lambda)$ and $KP_{\mathbb{C}}(\Gamma)$ respectively be generated by Kumjian-Pask families (s', p') and (t', q').

Given a *-algebra homomorphism $\phi : KP_{\mathbb{C}}(\Lambda) \to KP_{\mathbb{C}}(\Gamma)$. Since $\iota_\Gamma \circ \phi$ is a *-algebra homomorphism, $\{\iota_\Gamma(\phi(s'_\lambda)) : \lambda \in \Lambda\}$ forms a Cuntz-Krieger Λ-family in $C^*(\Gamma)$. The universal property of $C^*(\Lambda)$ then implies the existence of a unique *-algebra homomorphism $\bar{\phi} : C^*(\Lambda) \to C^*(\Gamma)$ such that $\bar{\phi}(s_\lambda) = \iota_\Gamma(\phi(s'_\lambda))$, $\forall \lambda \in \Lambda$. A routine calculation on the generators of $C^*(\Lambda)$ and $KP_{\mathbb{C}}(\Lambda)$ shows that $\bar{\phi} \circ \iota_\Lambda = \iota_\Gamma \circ \phi$, i.e. the diagram commutes.

If ϕ is an isomorphism, then so is ϕ^{-1}. As in the previous paragraph, $\{\iota_\Lambda(\phi^{-1}(t'_\lambda)) : \lambda \in \Gamma\}$ is a Cuntz-Krieger Γ-family in $C^*(\Lambda)$. Hence the universal property of $C^*(\Gamma)$ gives a unique *-algebra homomorphism $\rho : C^*(\Gamma) \to C^*(\Lambda)$ such that $\rho(t_\lambda) = \iota_\Lambda(\phi^{-1}(t'_\lambda)),\ \forall \lambda \in \Gamma$.

Computations on the generators show that

$$\bar{\phi} \circ \rho(t_\lambda t_\mu^*) = \bar{\phi}(\iota_\Lambda \circ \phi^{-1}(t'_\lambda t'^*_\mu)) = \iota_\Gamma \circ \phi(\phi^{-1}(t'_\lambda t'^*_\mu)) = \iota_\Gamma(t'_\lambda t'^*_\mu) = t_\lambda t_\mu^*,\ \forall \lambda\mu \in \Gamma$$

and

$$\begin{aligned}\rho \circ \bar{\phi}(s_\lambda s_\mu^*) = \rho(\iota_\Gamma(\phi(s'_\lambda s'^*_\mu)) = \rho(\iota_\Gamma(t'_\lambda t'^*_\mu)) &= \rho(t_\lambda t_\mu^*)) \\ &= \iota_\Lambda(\phi^{-1}(t'_\lambda t'^*_\mu)) \\ &= \iota_\Lambda(s'_\lambda s'^*_\mu) = s_\lambda s_\mu^*,\ \forall \lambda, \mu \in \Lambda.\end{aligned}$$

Hence $\bar{\phi} \circ \rho = Id_{C^*(\Gamma)}$ and $\rho \circ \bar{\phi} = Id_{C^*(\Lambda)}$. Therefore $\bar{\phi}$ is a *-algebra isomorphism. □

As a consequence of above proposition, we obtain a generalization of Corollary 4.5 of [1] to Kumjian-Pask algebras.

Corollary 3.2. *Let Λ and Γ be any row finite k-graphs without sources such that $KP_{\mathbb{C}}(\Lambda) \cong KP_{\mathbb{C}}(\Gamma)$ (as *-algebras). Then $C^*(\Lambda) \cong C^*(\Gamma)$ as *-algebras.*

3.2. **Necessary and sufficient conditions for *-preserving homomorphisms.** An algebra homomorphism ϕ between *-algebras A and B is said to be *-preserving, if it preserves the adjoint, i.e. $\phi(a^*) = \phi(a)^*$ for every $a \in A$. Notably, not all algebra homomorphisms between algebras are *-preserving. Abrams and Tomforde [1, Example 4.1] give an example of algebra homomorphism between algebras of continuous functions which is not $*$-preserving. In [14] Power gives an example of algebra homomorphism between digraph algebras that is not *-preserving.

In this section we characterise homomorphisms between complex Kumjian-Pask algebras which are *-preserving. Our result generalizes that of Abrams and Tomforde [1, Proposition 5.3] which is a necessary and sufficient condition of *-preserving algebra homomorphism between Leavitt-path algebras.

Lemma 3.3. *Let Λ and Γ be row finite k-graphs without sources. If*

$$\psi : KP_{\mathbb{C}}(\Lambda) \to KP_{\mathbb{C}}(\Gamma)$$

is an algebra homomorphism between complex Kumjian-Pask algebras and $\psi(s_\lambda s_\lambda^*)$ is a projection, then

$$\psi(s_\lambda^*)^* \psi(s_\lambda)^* = \psi(s_\lambda s_\lambda^*).$$

Proof. Since $\psi(s_\lambda)\psi(s_\lambda^*) = \psi(s_\lambda s_\lambda^*)$, by taking adjoint of both sides we get

$$\psi(s_\lambda^*)^* \psi(s_\lambda)^* = (\psi(s_\lambda)\psi(s_\lambda^*))^* = (\psi(s_\lambda s_\lambda^*))^* = \psi(s_\lambda s_\lambda^*),$$

because $\psi(s_\lambda s_\lambda^*)$ is a projection. □

Lemma 3.4. *Let Λ and Γ be row finite k-graphs without sources, and*

$$\psi : KP_{\mathbb{C}}(\Lambda) \to KP_{\mathbb{C}}(\Gamma)$$

is an algebra homomorphism between complex Kumjian-Pask algebras. If $\psi(s_\lambda)$ and $\psi(s_\lambda^*)$ are contractions, then both

$$\psi(s_\lambda^*) \text{ and } \psi(s_\lambda)^*$$

are partial isometries, and

$$\psi(s_\lambda^*)\psi(s_\lambda s_\lambda^*) = \psi(s_\lambda^*),\ \psi(s_\lambda)^*\psi(s_\lambda s_\lambda^*) = \psi(s_\lambda)^*.$$

Proof. From (KP3) and (KP2), we obtain

$$\psi(s_\lambda s_\lambda^*)\psi(s_\lambda s_\lambda^*) = \psi(s_\lambda s_\lambda^* s_\lambda s_\lambda^*) = \psi(s_\lambda p_{s(\lambda)} s_\lambda^*) = \psi(s_\lambda s_\lambda^*),$$

hence $\psi(s_\lambda s_\lambda^*)$ is an idempotent. As $\|\psi(s_\lambda s_\lambda^*)\| \leq \|\psi(s_\lambda)\|\|\psi(s_\lambda^*)\|$, $\psi(s_\lambda s_\lambda^*)$ is a contractive idempotent, therefore it is a projection. Since $\psi(s_\lambda)\psi(s_\lambda^*) = \psi(s_\lambda s_\lambda^*)$, by applying [1, Lemma 5.2] to the contractions $\psi(s_\lambda)$ and $\psi(s_\lambda^*)$, yields that $\psi(s_\lambda^*)\psi(s_\lambda s_\lambda^*)$ is a partial isometry. But

$$\psi(s_\lambda^*)\psi(s_\lambda s_\lambda^*) = \psi(s_\lambda^* s_\lambda s_\lambda^*) = \psi(p_{s(\lambda)} s_\lambda^*) = \psi(s_\lambda^*),$$

therefore $\psi(s_\lambda^*)$ is a partial isometry.

Since $\psi(s_\lambda s_\lambda^*)$ is a projection, (KP3) and (KP2) implies

$$\begin{aligned}\psi(s_\lambda)^*\psi(s_\lambda s_\lambda^*) &= \psi(s_\lambda)^*(\psi(s_\lambda s_\lambda^*))^* = \psi(s_\lambda)^*\psi(s_\lambda^*)^*\psi(s_\lambda)^* \\ &= (\psi(s_\lambda)\psi(s_\lambda^*)\psi(s_\lambda))^* = (\psi(s_\lambda s_\lambda^* s_\lambda))^* \\ &= (\psi(s_\lambda p_{s(\lambda)}))^* = (\psi(s_\lambda))^*.\end{aligned}$$

From the hypotesis, $\psi(s_\lambda)$ is a contraction, hence $\psi(s_\lambda)^*$ is so. Now apply [1, Lemma 5.2] to the contractions $\psi(s_\lambda)^*$ and $\psi(s_\lambda^*)$. Since

$$\psi(s_\lambda^*)^*\psi(s_\lambda)^* = (\psi(s_\lambda)\psi(s_\lambda^*))^* = (\psi(s_\lambda s_\lambda^*))^* = \psi(s_\lambda s_\lambda^*),$$

$\psi(s_\lambda)^*\psi(s_\lambda s_\lambda^*)$ is a partial isometry. Lemma 3.3 together with (KP3) and (KP2) imply that

$$\begin{aligned}\psi(s_\lambda)^*\psi(s_\lambda s_\lambda^*) &= \psi(s_\lambda)^*\psi(s_\lambda^*)^*\psi(s_\lambda)^* = (\psi(s_\lambda s_\lambda^* s_\lambda))^* \\ &= (\psi(s_\lambda p_{s(\lambda)}))^* = (\psi(s_\lambda))^*.\end{aligned}$$

Therefore $\psi(s_\lambda)^*$ is a partial isometry. □

If E, F are graphs, and $\psi : L_{\mathbb{C}}(E) \to L_{\mathbb{C}}(F)$ is an algebra homomorphism between Leavitt path algebras, Abrams and Tomforde [1] gave a necessary and sufficient condition so that ψ is a *-homomorphism. The technique described in [1] can be applied to obtain a similar result for complex Kumjian-Pask algebras.

Proposition 3.5. *Let Λ and Γ be row finite k-graphs without sources. If $\psi : KP_{\mathbb{C}}(\Lambda) \to KP_{\mathbb{C}}(\Gamma)$ is an algebra homomorphism between Kumjian-Pask algebras, then ψ is an algebra *-homomorphism if and only if: $\|\psi(s_\mu)\| \leq 1$, $\|\psi(s_\mu^*)\| \leq 1$, $\forall \mu \in \Lambda^1$, and $\|\psi(p_v)\| \leq 1$ $\forall v \in \Lambda^0$ that is a sink.*

Proof. Suppose $\psi : KP_{\mathbb{C}}(\Lambda) \to KP_{\mathbb{C}}(\Gamma)$ is an algebra*-homomorphism. Then ψ must be contractive because Proposition 3.1 implies that ψ extends to a *-homomorphism of $C^*(\Lambda)$ to $C^*(\Gamma)$, and hence, we obtain the necessary condition.

To prove the sufficient condition, suppose $v \in \Lambda^0$. We want to show that $\psi(p_v)$ is an idempotent. We first assume that v is not a sink. Then for every $e_i \in \mathbb{N}^k$, the set $\Lambda_v^{e_i}$ is not empty. Suppose $\lambda \in \Lambda_v^{e_{i0}}$ for some $e_{i0} \in \mathbb{N}^k$. Since p_v is an idempotent, then $\psi(p_v)$ is also an idempotent. When v is a sink, $\|\psi(p_v)\| \leq 1$ by the hypothesis. But

$$\|\psi(p_v)\| = \|\psi(p_v)^2\| \leq \|\psi(p_v)\|^2,$$

which implies $\|\psi(p_v)\| \geq 1$. Hence $\psi(p_v)$ is an idempotent with $\|\psi(p_v)\| = 1$. Hence for every $v \in \Lambda^0$, $\psi(p_v)$ is a projection. Therefore for every $v \in \Lambda^0$,

$$\psi(p_v^*) = \psi(p_v) = \psi(p_v)^*.$$

Suppose $\lambda \in \Lambda^{\neq 0}$, from (KP3) and (KP2), we see that $s_\lambda s_\lambda^*$ is an idempotent, which implies that $\psi(s_\lambda s_\lambda^*)$ is also an idempotent. From the hypothesis, we get

$$\|\psi(s_\lambda s_\lambda^*)\| \leq \|\psi(s_\lambda)\|\|\psi(s_\lambda^*)\| \leq 1,$$

hence $\psi(s_\lambda s_\lambda^*)$ is contractive idempotent, therefore it is a projection. Lemma 3.4 implies that

$$\psi(s_\lambda)\psi(s_\lambda)^* = \psi(s_\lambda)\psi(s_\lambda)^*\psi(s_\lambda s_\lambda^*) = \psi(s_\lambda)\psi(s_\lambda^*) = \psi(s_\lambda s_\lambda^*).$$

From (KP3), (KP2) and Lemma 3.4 we get

$$\begin{aligned}
\psi(s_\lambda)^* &= \psi(s_\lambda p_{s(\lambda)})^* = \psi(p_{s(\lambda)})^*\psi(s_\lambda)^* = \psi(p_{s(\lambda)})\psi(s_\lambda)^* \\
&= \psi(s_\lambda^* s_\lambda)\psi(s_\lambda)^* = \psi(s_\lambda^*)\psi(s_\lambda)\psi(s_\lambda)^*\psi(s_\lambda s_\lambda^*) \\
&= \psi(s_\lambda^*)\psi(s_\lambda)\psi(s_\lambda)^*\psi(s_\lambda)\psi(s_\lambda^*) \\
&= \psi(s_\lambda^*)\psi(s_\lambda)\psi(s_\lambda^*) = \psi(s_\lambda^* s_\lambda s_\lambda^*) \\
&= \psi(p_{s(\lambda)} s_\lambda^*) = \psi(s_\lambda^*).
\end{aligned}$$

Hence $\psi(s_\lambda^*) = \psi(s_\lambda)^*$ for all $\lambda \in \Lambda^{\neq 0}$, then (1) and (2) imply that $\psi(x^*) = \psi(x)^*$ for all $x \in KP_{\mathbb{C}}(\Lambda)$. Therefore ψ is an algebra *-homomorphism.

□

Corollary 3.6. *Let Λ and Γ be row finite k-graphs without sources such that $|\Lambda^0| < \infty$ and $|\Gamma^0| < \infty$. If $\psi : C^*(\Lambda) \to C^*(\Gamma)$ is an algebra homomorphism between k-graph algebras, then ψ is an algebra *-homomorphism if and only if: $\|\psi(s_\mu)\| \leq 1$, $\|\psi(s_\mu^*)\| \leq 1$, $\forall \mu \in \Lambda^1$, and $\|\psi(p_v)\| \leq 1$ $\forall v \in \Lambda^0$ that is a sink.*

Proof. Proposition 4.3 of [16] implies that the inclusions $\iota_\Lambda : KP_{\mathbb{C}}(\Lambda) \to C^*(\Lambda)$ and $\iota_\Gamma : KP_{\mathbb{C}}(\Gamma) \to C^*(\Gamma)$ are surjective, hence $KP_{\mathbb{C}}(\Lambda) \cong C^*(\Lambda)$ and $KP_{\mathbb{C}}(\Gamma) \cong C^*(\Gamma)$. Proposition 4.3 then gives the result. □

4. CONCLUDING REMARKS

We have obtained an analogue of Abrams and Tomforde's results for Kumjian-Pask algebras by leveraging the fundamental properties of Kumjian-Pask algebras. We extended Abrams and Tomforde's concepts to row finite k-graph without sources and we obtained an analog of their results for higher rank graphs.

Our results provide descriptive properties that can be translated from Kumjian-Pask algebras to higher rank graph C*-algebras. We believe that our findings can be further generalized to describe properties of Kumjian-Pask algebras of more general k-graph.

Acknowledgement. This research was funded by Hibah Kompetensi and UPI Research Grant (Penelitian Bidang Keilmuan-Sekolah Pasca Sarjana).

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