J. Indones. Math. Soc.
Vol. 29, No. 01 (2023), pp. 93–98.

S-PRIME IDEALS IN PRINCIPAL DOMAIN

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Abstract. Let R be a commutative ring and S be a multiplicative subset of R. The S-prime ideal is a generalization of the concept of prime ideal. In this paper, we completely determine all S-prime and S-maximal ideals of a principal domain. It is shown that the intersection of any descending chain of S-prime ideals in a principal domain is an S-prime ideal, also the S-radical is investigated.

Key words and Phrases: Principal domain, S-prime ideal, S-maximal ideal, S-radical.

1. Introduction

Throughout this paper all rings are commutative with identity $\neq 0$. Let R be a commutative ring and S be a multiplicative subset of R. Recently, Sevim et al. [11], studied the concept of S-prime ideal which is a generalization of prime ideal and used it to characterize integral domains, certain prime ideals, fields and S-Noetherian rings. An ideal P with $P \cap S = \emptyset$ is said to be S-prime ideal if there exists an element $s \in S$ such that, whenever $a, b \in R$, if $ab \in P$ then $sa \in P$ or $sb \in P$. Note that if S consist of units of R, then the notions of S-prime ideal and prime ideal coincide. Recall from [4] that an ideal P of R is said to be S-maximal ideal if $P \cap S = \emptyset$ and there exists $s \in S$ such that whenever $P \subseteq Q$ for some ideal Q of R, then either $sQ \subseteq P$ or $Q \cap S \neq \emptyset$. The S-radical of an ideal I is defined by $\sqrt[S]{I} = \{a \in R \;/ sa^n \in I \text{ for some } s \in S \text{ and } n \in \mathbb{N}\}$. In this paper we study the concept of S-prime ideal in a principal ideal domain, for instance, we completely determine all S-prime ideals of a principal ideal domain. In [4], the author showed that any S-maximal ideal is S-prime. If R is a principal ideal domain, we show that every non-zero S-prime ideal is S-maximal. Also the S-radical of an ideal is given.

Recall from [5] that a multiplicative subset S of R is said to be strongly multiplicative if for each family $(s_\alpha)_{\alpha\in\Lambda}$ we have $\cap_{\alpha\in\Lambda}(s_\alpha R) \cap S \neq \emptyset$. In [5], the

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2020 Mathematics Subject Classification: 13F10, 13A15, 13E15.
Received: 18-04-2022, accepted: 22-02-2023.

author showed that if S is a strongly multiplicative subset, then the intersection of any chain of S-prime ideals is an S-prime ideal, and in particular, any ideal disjoint with S is contained in a minimal S-prime ideal. Then the author asked the following question;

Question: Is the assumption "S strongly multiplicative subset" necessary for the theorem?

As part of our study, we give a negative answer to this question.

Her, we fix some notations that will be used throughout this paper. If R is a principal ideal domain. The set of all irreducible (prime) elements of R is denoted by $\mathbb{P}$. For a multiplicative subset S the set $\mathbb{P}_S$ is defined by $\mathbb{P}_S = \{p \in \mathbb{P} \ /(p) \cap S \neq \emptyset\}$, that is, $\mathbb{P}_S$ is the set of all irreducible elements of R that belong to some element of S. An irreducible element p is in $\mathbb{P}_S$ if and only if there exists $s \in S$ and $b \in R$ such that $s = bp$. Note that if $S = R \setminus \{0\}$, then $\mathbb{P}_S = \mathbb{P}$.

2. S-prime ideal in principal domain

We start this section by recalling the concept of S-prime ideals of a commutative ring R in order to give the form of all S-prime ideals in principal ideal domain.

Definition 2.1. *Let R be a commutative ring, S be a multiplicative subset of R and P be an ideal of R disjoint with S. Then P is said to be S-prime ideal if there exists an $s \in S$ such that for all $a, b \in R$ with $ab \in P$, we have $sa \in P$ or $sb \in P$.*

The following result will be frequently used and can be found in [5].

Proposition 2.2. *Let R be a commutative ring and S be a multiplicative subset of R. Let P be an ideal of R. The following statements are equivalent*

(1) P is an S-prime ideal of R.
(2) There exists $s \in S$ such that $(P : s)$ is a prime ideal of R.

The S-prime ideals of a principal ideal domain are completely determined in the following result.

Theorem 2.3. *Let R be a principal ideal domain and S be a multiplicative subset of R and let I be an ideal of R. The following statements are equivalent:*

(1) I is an S-prime ideal of R,
(2) $I = (0)$ or $I = (vp)$ for some $p \in \mathbb{P} - \mathbb{P}_S$ and $v \in R$ such that $(v) \cap S \neq \emptyset$.

Proof. (2) $\Rightarrow$ (1). If $I = (0)$, then I is an S-prime ideal since it is a prime ideal. Now, let $I = (vp)$ where $p \in \mathbb{P} - \mathbb{P}_S$ and $(v) \cap S \neq \emptyset$. There exists an $s_0 \in S$ and $v' \in R$ such that $s_0 = vv'$. Let $x \in (I : s_0)$, then $xs_0 \in I$ so $xs_0 = \alpha vp$ for some $\alpha \in R$, therefore $s_0 x \in (p)$, hence $x \in (p)$ since $s_0 \notin (p)$. It follows that $(I : s_0) \subseteq (p)$. On the other hand, we have $ps_0 = v'vp \in I$, that is $p \in (I : s_0)$, so that $(I : s_0) = (p)$ is a prime ideal of R. Thus I is an S-prime ideal of R.
(1) $\Rightarrow$ (2). Let $I = (a)$ be a non-zero S-prime ideal of R. Let $s_0 \in S$ such that $(I : s_0)$ is a prime ideal of R. Since $(0) \neq I \subseteq (I : s_0)$ there exists an irreducible

element p of R such that $(I : s_0) = (p)$. As $ps_0 \in I$, we have $ps_0 = a'a$ for some $a' \in R$, in particular $a'a \in (p)$ so $a' \in (p)$ or $a \in (p)$. If $a' \in (p)$, then $a' = a''p$ where $a'' \in R$, that is $ps_0 = a'a = aa''p$, so that $s_0 = a''a \in (a) \cap S$, a contradiction. Thus $a \in (p)$, hence $a = vp$ where $v \in R$. Now $ps_0 = a'a = a'vp$, so $s_0 = a'v$, that is $(v) \cap S \neq \emptyset$. It follows that $I = (vp)$ and $(v) \cap S \neq \emptyset$ and $p \in \mathbb{P} - \mathbb{P}_S$; in fact if $p \in \mathbb{P}_S$ then $(p) \cap S \neq \emptyset$, so there is an element $s \in S$ such that $s = cp$ where $c \in R$. Then clearly $ss_0 = cs_0p \in I$, which is not compatible with the fact that $I \cap S = \emptyset$. □

Remark 2.4. *(1) If S is a multiplicative subset of a commutative ring R, then there exists a saturated multiplicative subset S' of R such that $\mathrm{Spec}_S R = \mathrm{Spec}_{S'} R$ (see the appendix).*

(2) If S is a saturated multiplicative subset of a principal ideal domain R. Then an ideal P is S-prime if and only if P is the zero ideal or $P = (sp)$ where $s \in S$ and $p \in \mathbb{P} - \mathbb{P}_S$.

Example 2.5. *Let $R = \mathbb{Z}$ and $S = \{2^k \ / k \in \mathbb{N}\}$. Note that $\mathbb{P}_S = \{2\}$. Let I be a non-zero S-prime ideal of $\mathbb{Z}$. Then $I = (vp)$ where p is a prime integer and $v \in \mathbb{Z}$ such that $p \neq 2$ and $(v) \cap S \neq \emptyset$ that is $mv = 2^k$ for some $m \in \mathbb{Z}$ and $k \in \mathbb{N}$. Thus $v = \pm 2^l$ for some $l \in \mathbb{N}$. It follows that the S-prime ideals of $\mathbb{Z}$ are the zero ideal and the ideals of the form $(2^l p)$ where $p \neq 2$ is a prime integer and $l \in \mathbb{N}$.*

Lemma 2.6. *Let R be a principal ideal domain. Let $(I_n)_{n \in \mathbb{N}}$ be a descending chain of ideals of R. Then I_n stabilize or $\cap_n I_n = (0)$.*

Proof. Let $I = (a) = \cap_n I_n$ and assume that $a \neq 0$. If a is invertible, then the chain stabilize. If a is not invertible, consider the commutative ring $R' = R/(a)$. Then R' is Noetherian and $\dim R' = 0$, so R' is an Artinian ring. Thus $\overline{I_n}$ stabilize (in R'). There exists N such that for all $n \geq N$, $\overline{I_n} = \overline{I_N}$, so $I_n = I_N$. □

Proposition 2.7. *Let R be a principal ideal domain. If $(Q_n)_n$ is a descending chain of S-prime ideals of R, then $\cap_n Q_n$ is an S-prime ideal of R.*

Proof. This follows from the previous lemma. □

Theorem 2.8. *Let R be a principal ideal domain. Then every ideal which is disjoint with S is contained in a minimal S-prime ideal.*

Proof. Let I be an ideal of R with $I \cap S = \emptyset$. Let

$$\Gamma = \{Q \ / Q \text{ is an } S\text{-prime ideal and } I \subseteq Q\}$$

Note that Γ is not empty since $I \subseteq P$ for some prime ideal P of R with $P \cap S \neq \emptyset$, which is an S-prime ideal of R. If $(Q_n)_n$ is a descending chain of S-prime ideals of R containing I, then by the previous Proposition, $Q = \cap_n Q_n$ is an S-prime ideal containing I. By applying the Zorn's lemma, we get the desired results. □

Proposition 2.9. *Let R be a principal ideal domain and S be a multiplicative subset of R. The following statements are equivalent.*

(1) S is a strongly multiplicative subset of R.

(2) $S \subseteq U(R)$, where $U(R)$ is the set of invertible elements of R.

Proof. If $S \subseteq U(R)$, then S is clearly a strongly multiplicative subset since for any $s \in S$ we have $sR = R$. Now, assume that $S \nsubseteq U(R)$. Then there exists a nonzero element $s \in S$ which is not invertible. Let $p \in \mathbb{P}$ such that $(s) \subset (p)$, then for any $n \in \mathbb{N}$, $(s^n) \subseteq (p^n)$, thus $\cap_{n\in\mathbb{N}}(s^n) \subseteq \cap_{n\in\mathbb{N}}(p^n) = (0)$, in particular $\cap_{n\in\mathbb{N}}(s^n) \cap S = \emptyset$. Thus S is not a strongly multiplicative subset of R. □

Example 2.10. *Let p be an irreducible element of a principal ideal domain R and $S = \{p^n \ / \ n \in \mathbb{N}\}$. Then S is not a strongly multiplicative subset since $\cap_{n\in\mathbb{N}}(p^n R) \cap S = \emptyset$. But the intersection of a chain of S-prime ideals pf R is an S-prime ideal of R.*

3. S-maximal ideal in principal domain

Definition 3.1. *Let R be a commutative ring and S be a multiplicative subset. Let P be an ideal of R with $P \cap S = \emptyset$. Then P is said to be an S-maximal ideal of R if there exists $s \in S$ such that whenever $P \subseteq Q$ for some ideal Q of R then either $sQ \subseteq P$ or $Q \cap S \neq \emptyset$.*

Remark 3.2. *Every S-maximal ideal of R is an S-prime ideal of R (see* [4]*).*

Lemma 3.3. *Let R be a principal ideal domain and S be a multiplicative subset of R. Then (0) is an S-maximal ideal of R if and only if $\mathbb{P}_S = \mathbb{P}$*

Proof. If (0) is an S-maximal ideal of R and $p \in \mathbb{P}$ then $(p) \cap S \neq \emptyset$ since $s(p) \nsubseteq (0)$, so $p \in \mathbb{P}_S$. Conversely, assume that $\mathbb{P}_S = \mathbb{P}$. Let $Q = (a)$ be an ideal of R. If $a = 0$ then $1(Q) = (0) \subseteq (0)$. If $a \neq 0$. Then either $Q = R$, in this case $Q \cap S \neq \emptyset$, or $Q \neq R$, in this case $a = p_1^{n_1} \cdots p_m^{n_m}$ where $p_1, \cdots, p_m \in \mathbb{P}$ and $n_1, \cdots, n_m$ are positive integers. Since $p_i \in \mathbb{P} = \mathbb{P}_S$ there exists $\alpha_i \in R$ such that $\alpha_i p_i \in S$, so $\prod_{i=1}^{m}(\alpha_i p_i)^{n_i} \in (a) \cap S$. Thus $Q \cap S \neq \emptyset$. □

Classically, in a principal ideal domain every non-zero prime ideal is a maximal ideal, it's S-version is the following result.

Theorem 3.4. *Let R be a principal ideal domain. Then every non-zero S-prime ideal is an S-maximal ideal.*

Proof. Let P be a non-zero S-prime ideal of R. Then $P = (vp)$ for some $p \in \mathbb{P} - \mathbb{P}_S$ and $v \in R$ with $(v) \cap S \neq \emptyset$. Let $Q = (a)$ be an ideal of R with $P \subseteq Q$. Since $vp \in (a)$, $vp = ba$ for some $b \in R$. In particular $ab \in (p)$, so $a \in (p)$ or $b \in (p)$.
First case, if $a \in (p)$, then $a = a'p$ for some $a' \in R$, so that $vp = ba'p$, thus $v = ba'$. As $(v) \cap S \neq \emptyset$, there exists $t \in R$ such that $s = tv = tba' \in S$. Therefore $sa = tva'p \in (vp)$. It follows that $sQ \subseteq P$.
Second case, if $a \notin (p)$, then $b \in (p)$. So $b = b'p$ for some $b' \in R$. Hence $v = b'a$ since $vp = ba = b'ap$. Thus $\emptyset \neq (v) \cap S \subseteq (a) \cap S$. It follows that $Q \cap S \neq \emptyset$.

□

4. S-radical in principal domain

Definition 4.1. *Let R be a commutative ring and S be a multiplicative subset of R. The S-radical of an ideal I is defined by*

$$\sqrt[S]{I} = \{a \in R \ / sa^n \in I \text{ for some } s \in S \text{ and } n \in \mathbb{N}\}$$

Theorem 4.2. *Let R be a principal ideal domain and S be a multiplicative subset of R. Let $I = (a)$ be a proper ideal of R write $a = \prod_{j=1}^{m} q_j^{m_j} \prod_{i=1}^{d} p_i^{n_i}$ where $q_j \in \mathbb{P}_S$ and $p_i \in \mathbb{P} - \mathbb{P}_S$. Then $\sqrt[S]{I} = (\prod_{i=1}^{d} p_i)$.*

Proof. Since $q_j \in \mathbb{P}_S$, there exists $\alpha_j \in R$ such that $\alpha_j q_j \in S$. Let $n = max(n_i)$, then $\prod_{j=1}^{m}(\alpha_j q)^{m_i}(\prod_{i=1}^{d} p_i)^n \in I$, thus $\prod_{i=1}^{d} p_i \in \sqrt[S]{I}$, that is $(\prod_{i=1}^{d} p_i) \subseteq \sqrt[S]{I}$. Conversely, let $x \in \sqrt[S]{I}$, then $sx^n \in I$ for some $s \in S$ and $n \in \mathbb{N}$. Let $b \in R$ such that $sx^n = b\prod_{j=1}^{m} q^{m_i} \prod_{i=1}^{d} p_i^{n_i}$. Then for each $1 \le i \le d$, $sx^n \in (p_i)$, since $(p_i) \cap S = \emptyset$ and (p_i) is a prime ideal of R, we have $x^n \in (p_i)$, so $x \in (p_i)$. Thus $x \in \cap_{i=1}^{d}(p_i) = (\prod_{i=1}^{d} p_i)$. It follows that $\sqrt[S]{I} = (\prod_{i=1}^{d} p_i)$. □

5. Appendix

Here we show, to studying the concept of S-prime ideal, we can always assume that the multiplicative subset S is saturated. So, for a multiplicative subset S of a commutative ring R, denote S' the set defined by $S' = \{a \in R \ /(a) \cap S \neq \emptyset\}$.

Proposition 5.1. *With the previous notations, we have*

(1) $S \subseteq S'$ and S' is a saturated multiplicative subset.
(2) If I is an ideal of R, then $I \cap S = \emptyset$ if and only if $I \cap S' = \emptyset$.
(3) If P is an ideal of R, then P is S-prime if and only if P is S'-prime.
(4) If P is an ideal of R, then P is S-maximal if and only if P is S'-maximal.
(5) If I is an ideal of R, then $\sqrt[S]{I} = \sqrt[S']{I}$.

Proof. (1) Clearly $S \subseteq S'$, $0 \notin S'$ and $1 \in S'$. If $a, b \in S'$, then $aa' \in S$ and $bb' \in S$ for some $a', b' \in R$, so $(a'b')(ab) \in S$, that is $ab \in S'$. If $ab \in S'$, then $abt \in S$ for some $t \in R$, so $a, b \in S'$.

(2) Clearly, if $I \cap S \neq \emptyset$, then $I \cap S' \neq \emptyset$. If $I \cap S' \neq \emptyset$, then there exists $i \in I$ such that $(i) \cap S \neq \emptyset$, so $ia \in S$ for some $a \in R$. Thus $ia \in I \cap S$.

(3) If P is an S-prime ideal of R, then it is easy to see that P is also an S'-prime ideal of R. Conversely, assume that P is an S'-prime ideal of R. Then $(P : s')$ is a prime ideal for some $s' \in S'$. We have $ts' \in S$ for some $t \in R$. Now; we show that $(P : ts') = (P : s')$. If $x \in (P : ts')$, then $xts' \in P$, so $xt \in (P : s')$. Since $t \notin (P : s')$, we have $x \in (P : s')$, hence $(P : ts') \subseteq (P : s')$. If $x \in (P : s')$, then $xs' \in P$, hence $xts' \in P$, thus $x \in (P : ts')$. It follows that $(P : ts')$ is a prime ideal of R, therefore P is an S-prime ideal of R.

(4) If P is an S-maximal ideal. We fix an element $s \in S$ as in the definition, in particular $s \in S'$. If $P \subseteq Q$ and $Q \cap S' = \emptyset$, then $Q \cap S = \emptyset$, so $sQ \subseteq P$. It follows that P is an S'-maximal ideal of R. Now, assume that Q is an

S'-maximal ideal and fix $s' \in S'$ as in the definition. There exits $t \in R$ such that $ts' \in S$. If $P \subseteq Q$ with $Q \cap S = \emptyset$, then $Q \cap S' = \emptyset$, so $s'Q \subseteq P$, thus $st'Q \subseteq tP \subseteq P$.

(5) From the definition we have $\sqrt[S]{I} \subseteq \sqrt[S']{I}$. Let $x \in \sqrt[S']{I}$, then $s'x^n \in I$ for some $s' \in S'$ and $n \in \mathbb{N}$. There exists $t \in R$ such that $ts' \in S$, so $ts'x^n \in I$, thus $x \in \sqrt[S]{I}$.

□

Acknowledgement. The author would like to thank the referee for his/her great efforts in proofreading the manuscript.

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