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ON (α, β)-FUZZY IDEALS OF TERNARY SEMIGROUPS

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Abstract. In this paper, we introduce the concept of generalized fuzzy ideals in ternary semigroups, which is a generalization of the fuzzy ideals of semigroups. In this regard, we define (α, β)-fuzzy left (right, lateral) ideals, (α, β)-fuzzy quasi-ideals and (α, β)-fuzzy bi-ideals and investigate some related properties of ternary semigroups. Special concentration is paid to $(\in, \in \vee q)$-fuzzy left (right, lateral) ideals, $(\in, \in \vee q)$-fuzzy quasi-ideals and $(\in, \in \vee q)$-fuzzy bi-ideals. Finally, we characterize regular ternary semigroups in terms of these notions.

Key words and Phrases:Fuzzy subsets; $(\in, \in \vee \mathrm{q})$-fuzzy left (right, lateral) ideals; $(\in, \in \vee \mathrm{q})$-fuzzy quasi-ideals; $(\in, \in \vee \mathrm{q})$-fuzzy bi-ideals.

Abstrak. Dalam makalah ini, kami memperkenalkan konsep ideal fuzzy yang tergeneralisasi dalam semigroup ternary, yang merupakan perluasan dari ideal fuzzy pada semigroup. Dalam hal ini, ideal (α, β)-fuzzy kiri (kanan, lateral), kuasi ideal(α, β)-fuzzy dan bi-ideal (α, β)-fuzzy didefinisikan dan beberapa sifat dari semigroup ternary dibahas. Perhatian khusus ditujukan untuk ideal kiri (kanan, lateral $(\in, \in \vee q)$-fuzzy, kuasi ideal $(\in, \in \vee q)$-fuzzy dan bi-ideal$(\in, \in \vee q)$-fuzzy. Akhirnya, semigrup ternary regular dikarakterisasi berdasarkan sifat-sifat tersebut di atas.

Kata kunci: Subhimpunan fuzzy; ideal kiri $(\in, \in \vee \mathrm{q})$-fuzzy (kanan, lateral); kuasi ideal $(\in, \in \vee \mathrm{q})$-fuzzy; bi-ideal $(\in, \in \vee \mathrm{q})$-fuzzy.

1. INTRODUCTION

In 1965, Zadeh [27] defined fuzzy sets with a view to describe, study, and formulate mathematical situations which are imprecise and vaguely defined. Many

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researchers who are involved in applying, refining and teaching fuzzy sets have successfully applied this theory in many different fields such as weather forecasting, linguistic, psychology, economics, artificial intelligence, computer science, pattern recognition, mathematical programming, topological spaces, algebraic structures and so on. The idea of a quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [3], played a vital role to generate some different types of fuzzy subgroups. It is worth pointing out that Bhakat and Das [2, 4, 5] gave the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in view, Davvaz in [7, 8] introduced the concept of $(\in, \in \vee q)$-fuzzy sub-nearrings (R-subgroups, ideals) of a nearring and investigated some of their interesting properties. Jun and Song [14] discussed general forms of fuzzy interior ideals in semigroups. Kazanci and Yamak introduced the concept of a generalized fuzzy bi-ideal in semigroups [17] and gave some properties of fuzzy bi-ideals in terms of $(\in, \in \vee q)$-fuzzy bi-ideals. In [19], Khan et al. characterized ordered semigroups in terms of $(\in, \in \vee q)$-fuzzy interior ideals and gave some generalized forms of interior ideals in ordered semigroups. Many other researchers used the idea of generalized fuzzy sets and gave several characterizations results in different branches of algebra, for example see [1, 15, 16, 20, 22, 24, 25, 28]. For further reading on ternary semigroups, we refer the reader to [9, 10, 11, 12, 13, 18, 21, 26]. This paper is divided in the following sections:

In Section 2, we recall some definitions of (fuzzy) ternary semigroups, in Section 3, we define $(\in, \in \vee q)$-fuzzy left (right, lateral) ideals, $(\in, \in \vee q)$-fuzzy quasi-ideals and $(\in, \in \vee q)$-fuzzy bi-ideals and give some interesting results of ternary semigroups. In Section 4, we define the upper/lower parts of $(\in, \in \vee q)$-fuzzy left (right, lateral) ideals, (resp. $(\in, \in \vee q)$-fuzzy quasi-ideals and $(\in, \in \vee q)$-fuzzy bi-ideals) and characterize regular ternary semigroups.

2. PRELIMINARIES

Here we reproduce some results and definitions which are necessary for the subsequent sections. Throughout the paper, X is a ternary semigroup unless otherwise stated.

A *ternary semigroup* X is a non-empty set whose elements are closed under the ternary operation [ ] of multiplication and satisfy the associative law defined as follows:

$$[[abc]de] = [a[bcd]e] = [ab[cde]] \text{ for all } a, b, c, d, e \in X.$$

For simplicity we shall write $[abc]$ as abc. For non-empty subsets A, B and C of X, let $ABC := \{abc \mid a \in A, b \in B \text{ and } c \in C\}$.

An element a of a ternary semigroup X is called *regular* if there exist elements $x, y \in X$ such that $a = axaya$. A ternary semigroup X is regular if every element of X is regular.

A non-empty subset A of a ternary semigroup X is called *left* (resp. *right, lateral*) *ideal* of X if $X^2A \subseteq A$ (resp. $AX^2 \subseteq A$, $XAX \subseteq A$). A non-empty subset A of X is called an *ideal* of X if it is left, right and lateral ideal of X. A non-empty subset A of a ternary semigroup X is called *ternary subsemigroup* if $A^3 \subseteq A$. A ternary subsemigroup A of X is called a *bi-ideal* of X if $AXAXA \subseteq A$. A subset A of a ternary semigroup X is called a *quasi-ideal* of X if $X^2A \cap XAX \cap AX^2 \subseteq A$ (see [9]).

Now, we review some fuzzy logic concepts.

A function f from a non-empty set X to a unit interval $[0, 1]$ of real numbers is called a *fuzzy subset* of X, that is $f : X \to [0, 1]$. For fuzzy subsets f and g of X, $f \leq g$ means that $f(x) \leq g(x)$ for all $x \in X$. The symbols $f \wedge g \wedge h$ and $f \vee g \vee h$ will mean the following fuzzy subsets of X :

$$(f \wedge g \wedge h)(x) = f(x) \wedge g(x) \wedge h(x) = \min\{f(x), g(x), h(x)\},$$

$$\text{and } (f \vee g \vee h)(x) = f(x) \vee g(x) \vee h(x) = \max\{f(x), g(x), h(x)\} \text{ for all } x \in X.$$

Let f be a fuzzy subset of X, define $U(f; t) = \{x \in X \mid f(x) \geq t\}$. We call $U(f; t)$ an *upper level set*.

A fuzzy subset f of a ternary semigroup X is called a *fuzzy ternary subsemigroup* of X if $f(abc) \geq f(a) \wedge f(b) \wedge f(b)$ for all $a, b, c \in X$.

A fuzzy subset f of a ternary semigroup X is a *fuzzy left* (resp. *lateral, right) ideal* of X if $f(abc) \geq f(c)$ (resp. $f(abc) \geq f(b), f(abc) \geq f(a)$) for all $a, b, c \in X$.

A fuzzy subset f of a ternary semigroup X is called a *fuzzy bi-ideal* of X if (i) $f(abc) \geq f(a) \wedge f(b) \wedge f(b)$ and (ii) $f(abcde) \geq f(a) \wedge f(c) \wedge f(e)$ for all $a, b, c, d, e \in X$.

We define the fuzzy subsets $\mathcal{X}$ and φ as follows:

$\mathcal{X} : X \longrightarrow [0, 1], x \longmapsto \mathcal{X}(x) = 1$ and $\varphi : X \longrightarrow [0, 1], x \longmapsto \varphi(x) = 0$ for all $x \in X$.

Let f, g and h be any three fuzzy subsets of X. We define the product $f \circ g \circ h$ of f, g and h as follows:

$$f \circ g \circ h(x) = \begin{cases} \bigvee\limits_{x=abc} \min\{f(a), g(b), h(c)\} & \text{if } \exists\, a, b, c \in X \text{ such that } x = abc \\ 0 & \text{if } x \neq abc. \end{cases}$$

A fuzzy subset f of a ternary semigroup X is called a *fuzzy quasi-ideal* of X if

$$f(a) \geq \min\{(f \circ \mathcal{X} \circ \mathcal{X})(a), (\mathcal{X} \circ f \circ \mathcal{X})(a), (\mathcal{X} \circ \mathcal{X} \circ f)(a)\}.$$

A fuzzy subset f of X of the form

$$f : X \longrightarrow [0, 1], y \longmapsto f(y) = \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{otherwise} \end{cases}$$

is called a *fuzzy point with support* x and value t and is denoted by $[x;t]$. Consider a fuzzy point $[x;t]$, a fuzzy subset f and $\alpha, \beta \in \{\in, \mathrm{q}, \in \vee\mathrm{q}, \in \wedge\mathrm{q}\}$ we define $[x;t]\alpha f$ as follows:

(1) $[x;t] \in f$ (resp. $[x;t]\mathrm{q}f$) means that $f(x) \geq t$ (resp. $f(x) + t > 1$) and in this case we say that $[x;t]$ belongs to (resp. quasi-coincident with) fuzzy subset f.

(2) $[x;t] \in \vee\mathrm{q}f$ (resp. $[x;t] \in \wedge\mathrm{q}f$) means that $[x;t] \in f$ or $[x;t]\mathrm{q}f$ (resp. $[x;t] \in f$ and $[x;t]\mathrm{q}f$).

By $[x;t]\overline{\alpha}f$, we mean that $[x;t]\alpha f$ does not hold.

Note that if f is a fuzzy subset X, defined by $f(x) \leq 0.5$ for all $x \in X$, then the set $\{[x;t] | [x;t] \in \wedge\mathrm{q}f\}$ is empty.

3. $(\in, \in \vee q)$-fuzzy ideals

In this section, we define $(\in, \in \vee q)$-fuzzy left (right, lateral) ideal, $(\in, \in \vee q)$-fuzzy quasi-ideals and $(\in, \in \vee q)$-fuzzy bi-ideal of a ternary semigroup X and study some of their basic properties.

Definition 3.1. *A fuzzy subset f of X is called $(\in, \in \vee q)$-fuzzy ternary subsemigroup of X, if for all $a, b, c \in X$ and $t, r, s \in (0, 1]$,*

$$[a;t] \in f, [b;r] \in f, [c;s] \in f \Longrightarrow [abc; \min\{t, r, s\}] \in \vee qf.$$

Definition 3.2. *A fuzzy subset f of X is called $(\in, \in \vee q)$-fuzzy left (resp. lateral, right) ideal of X, if for all $x, y, z \in X$ and $t \in (0, 1]$*

$$[z;t] \in f \ (resp.\ [y;t] \in f,\ [x;t] \in f) \Longrightarrow [xyz;t] \in \vee qf.$$

A fuzzy subset f is called an $(\in, \in \vee q)$-*fuzzy ideal* of X, if it is $(\in, \in \vee q)$-fuzzy left ideal, $(\in, \in \vee q)$-fuzzy lateral ideal and $(\in, \in \vee q)$-fuzzy right ideal of X.

Example 3.3. *Consider the set $Z_5^- = \{0, -1, -2, -3, -4\}$. Then, $(Z_5^-, \cdot)$ is a ternary semigroup where ternary multiplication "$\cdot$" is defined as*

$\cdot$	0	−1	−2	−3	−4
0	0	0	0	0	0
−1	0	1	2	3	4
−2	0	2	4	1	3
−3	0	3	1	4	2
−4	0	4	3	2	1

$\cdot$	0	−1	−2	−3	−4
0	0	0	0	0	0
1	0	−1	−2	−3	−4
2	0	−2	−4	−1	−3
3	0	−3	−1	−4	−2
4	0	−4	−3	−2	−1

Define a fuzzy subset f of Z_5^- as follows:

$$f(0) = t_0,\ f(-1) = f(-2) = f(-3) = f(-4) = t_1,$$

where $t_0, t_1 \in (0, 1]$ and $t_0 \geq t_1$ then it easy to calculate that f is an $(\in, \in \vee q)$-fuzzy ideal of Z_5^-.

Definition 3.4. *A fuzzy subset f of X is called $(\in, \in \vee q)$-fuzzy bi-ideal of X, if for all $a, b, c, d, e \in X$ and $t, r, s \in (0, 1]$*

(1) $[a;t] \in f, [b;r] \in f$ *and* $[c;s] \in f \Longrightarrow [abc; \min\{t, r, s\}] \in \vee qf$*;*

(2) $[a;t] \in f, [c;r] \in f, [e;s] \in f$ *and* $b, d \in X \Longrightarrow [abcde; \min\{t, r, s\}] \in \vee qf$.

Example 3.5. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a semigroup with respect to "$*$" and $abc = (a * b) * c$ for all $a, b, c \in X$, where "$*$" is defined by the following Cayley table:*

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

Then, $(X, *)$ is a ternary semigroup and $\{0\}, \{0, 1\}, \{0, 1, 2\}, \{0, 1, 2, 4\}$ and X are bi-ideals of X. Define a fuzzy subset $f : X \longrightarrow [0, 1]$ of X as follows:

$$f(0) = 0.8,\ f(1) = 0.7,\ f(2) = 0.6,\ f(4) = 0.5,\ f(3) = f(5) = 0.4.$$

Then,

$$U(f; t) = \begin{cases} X & \text{if} \quad 0 < t \leq 0.4 \\ \{0, 1, 2, 4\} & \text{if} \quad 0.4 < t \leq 0.5 \\ \{0, 1, 2\} & \text{if} \quad 0.5 < t \leq 0.6 \\ \{0, 1\} & \text{if} \quad 0.6 < t \leq 0.7 \\ \{0\} & \text{if} \quad 0.7 < t \leq 0.8. \end{cases}$$

Then, f is an $(\in, \in \vee q)$-fuzzy bi-ideal of X.

Proposition 3.6. *Let A be a ternary subsemigroup of a ternary semigroup X and a fuzzy subset f of X is defined by*

$$f(x) = \begin{cases} \geq \frac{1}{2} & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then,

(1) *f is a $(q, \in \vee q)$-fuzzy ternary subsemigroup of X.*
(2) *f is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of X.*

Proof. (1) Let $x, y, z \in X$ and $t, r, s \in (0, 1]$ be such that $[x; t]qf, [y; r]qf, [z; s]qf$. Then, $f(x)+t > 1, f(y)+r > 1$ and $f(z)+s > 1$ which shows that $f(x), f(y), f(z) > 0$. Thus, $x, y, z \in A$. Since A is a ternary subsemigroup, so $xyz \in A \Longrightarrow f(xyz) \geq 0.5$.

If $t \wedge r \wedge s \leq 0.5$, then

$$f(xyz) \geq t \wedge r \wedge s \text{ and so } [xyz; \min\{t, r, s\}] \in f.$$

If $t \wedge r \wedge s > 0.5$ then

$$f(xyz) + t \wedge r \wedge s > 0.5 + 0.5 = 1 \text{ and so } [xyz; \min\{t, r, s\}]q\lambda.$$

Hence, $[xyz; \min\{t, r, s\}] \in \vee q\lambda$.

(2) Let $x, y, z \in X$ and $t, r, s \in (0, 1]$ be such that $[x; t], [y; r], [z; s] \in f$. Then, $f(x) \geq t > 0, f(y) \geq r > 0$ and $f(z) \geq s > 0$. Thus $x, y, z \in A$. Since A is a ternary subsemigroup, so $xyz \in A \Longrightarrow f(xyz) \geq 0.5$.

If $t \wedge r \wedge s \leq 0.5$, then

$$f_1(xyz) \geq t \wedge r \wedge s \text{ and so } [xyz; \min\{t, r, s\}] \in f.$$

If $t \wedge r \wedge s > 0.5$, then

$$f_1(xyz) + t \wedge r \wedge s > 0.5 + 0.5 = 1 \text{ and so } [xyz; \min\{t, r, s\}]qf.$$

Hence, $[xyz; \min\{t, r, s\}] \in \vee qf$. □

Theorem 3.7. *Let f be a fuzzy subset of X. Then, f is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of X if and only if*

$$f(xyz) \geq \min\{f(x), f(y), f(z), 0.5\} \text{ for all } x, y, z \in X.$$

Proof. Let f be $(\in, \in \vee q)$-fuzzy ternary subsemigroup of X. Let us suppose on the contrary that there exist $x, y, z \in X$ such that

$$f(xyz) < \min\{f(x), f(y), f(z), 0.5\}.$$

Choose $t \in (0, 0.5]$ such that

$$f(xyz) < t \leq \min\{f(x), f(y), f(z), 0.5\}.$$

Then, $[x; t], [y; t], [z; t] \in f$ but $f(xyz) < t \Longrightarrow [xyz; t]\overline{\in} f$. Also

$$f(xyz) + t < 0.5 + 0.5 = 1 \Longrightarrow [xyz; \min\{t, t, t\}] = [xyz; t]\overline{q}f.$$

Thus, $[xyz; t]\overline{\in \vee q}$, a contradiction. Hence,

$$f(xyz) \geq \min\{f(x), f(y), f(z), 0.5\} \text{ for all } x, y, z \in X.$$

Conversely, assume that $f(xyz) \geq \min\{f(x), f(y), f(z), 0.5\}$ for all $x, y, z \in X$. Let $[x; t], [y; r], [z; s] \in f$ for some $t, r, s \in (0, 1]$ then $f(x) \geq t, f(y) \geq r$ and $f(z) \geq s$, so

$$f(xyz) \geq \min\{f(x), f(y), f(z), 0.5\} \geq \min\{t, r, s, 0.5\}$$

If $t \wedge r \wedge s \leq 0.5$, then

$$f(xyz) \geq t \wedge r \wedge s \text{ so that } [xyz; \min\{t, r, s\}] \in f.$$

If $t \wedge r \wedge s > 0.5$, then

$$f(xyz) + t \wedge r \wedge s > 0.5 + 0.5 = 1 \text{ so we get } [xyz; \min\{t, r, s\}]qf.$$

Hence, $[xyz; \min\{t, r, s\}] \in \vee qf$. □

Theorem 3.8. *A non-empty subset f of X is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of X if and only if $U(f; t)(\neq \emptyset)$ is a ternary subsemigroup of X for all $t \in (0, 0.5]$.*

Proof. Let f be an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of X and $x, y, z \in U(f; t)$ for some $t \in (0, 0.5]$.Then, $f(x) \geq t$, $f(y) \geq t$ and $f(z) \geq t$. Since f is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup, so

$$f(xyz) \geq \min\{f(x), f(y), f(z), 0.5\} \geq \min\{t, 0.5\} = t$$

and so $xyz \in U(f; t)$. Consequently, $U(f; t)$ is a ternary subsemigroup of X.

Conversely, let $U(f;t)$ be a ternary subsemigroup of X for all $t \in (0, 0.5]$. Suppose that there exist $x, y, z \in X$ such that

$$f(xyz) < \min\{f(x), f(y), f(z), 0.5\}.$$

Choosing $t \in (0, 0.5]$ such that

$$f(xyz) < t \leq \min\{f(x), f(y), f(z), 0.5\}.$$

Then, $x, y, z \in U(f;t)$, but $xyz \notin U(f;t)$, which contradicts our supposition. Hence,

$$f(xyz) \geq \min\{f(x), f(y), f(z), 0.5\}$$

and so f is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup. □

Theorem 3.9. *Let A be a left (resp. lateral, right) ideal of X and f be a fuzzy subset defined as*

$$f(x) = \begin{cases} \geq \frac{1}{2} & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then,

(1) *f is $(q, \in \vee q)-$fuzzy left (resp. lateral, right) ideal of X.*
(2) *f is $(\in, \in \vee q)$-fuzzy left (resp. lateral, right) ideal of X.*

Proof. The proof is similar to the proof of Proposition 3.6. □

Theorem 3.10. *A fuzzy subset f of X is an $(\in, \in \vee q)$-fuzzy left (resp. lateral, right) ideal of X if and only if*

$$f(xyz) \geq \min\{f(z), 0.5\} \,(\textit{resp. } f(xyz) \geq \min\{f(y), 0.5\},\ f(xyz) \geq \min\{f(x), 0.5\}).$$

for all $x, y, z \in X$.

Proof. The proof is similar to the proof of Theorem 3.7. □

By Theorem [11], we have the following corollary.

Corollary 3.11. *A fuzzy subset f of X is an $(\in, \in \vee q)$-fuzzy ideal of X if and only if it satisfies the following*

(1) $f(xyz) \geq \min\{f(z), 0.5\}$,
(2) $f(xyz) \geq \min\{f(y), 0.5\}$,
(3) $f(xyz) \geq \min\{f(x), 0.5\}$ *for all $x, y, z \in X$.*

Theorem 3.12. *A non-empty subset f of X is an $(\in, \in \vee q)$-fuzzy left (resp. lateral, right) ideal of X if and only if $U(f;t)(\neq \emptyset)$ is a left (resp. lateral, right) ideal of X for all $t \in (0, 0.5]$.*

Proof. The proof is similar to the proof of Theorem 3.8. □

Theorem 3.13. *Let A be a bi-ideal of X and f be a fuzzy subset defined as*

$$f(x) = \begin{cases} \geq \frac{1}{2} & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then,

(1) f *is a* $(q, \in \vee q)-$*fuzzy bi-ideal of* X.
(2) f *is an* $(\in, \in \vee q)$-*fuzzy bi-ideal of* X.

Proof. The proof is similar to the proof of Proposition 3.6. □

Theorem 3.14. *A fuzzy subset* f *of* X *is an* $(\in, \in \vee q)$-*fuzzy bi-ideal of* X *if and only if it satisfies the following*

(1) $f(abc) \geq \min\{f(a), f(b), f(c), 0.5\}$ *for all* $a, b, c \in X$.
(2) $f(abcde) \geq \min\{f(a), f(c), f(e), 0.5\}$ *for all* $a, b, c, d, e \in X$.

Proof. The proof is similar to the proof of Theorem 3.10. □

Theorem 3.15. *A non-empty subset* f *of* X *is an* $(\in, \in \vee q)$-*fuzzy bi-ideal of* X *if and only if* $U(f;t)(\neq \emptyset)$ *is a bi-ideal of* X *for all* $t \in (0, 0.5]$.

Proof. The proof is similar to the proof of Theorem 3.8. □

Example 3.16. *Let* $Z^- = X$ *be the set of all negative integers. Then,* Z^- *is a ternary semigroup. If* $B = 5X$, *then*

$$BXBXB = 5XX5XX5X = 125X \subseteq 5X.$$

Hence, B *is a bi-ideal of* X. *Define* $f : X \longrightarrow [0,1]$ *by*

$$f(x) = \begin{cases} \geq t & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}$$

for any $t \in (0, 0.5)$ *then* $U(f;t) = \{x \in X | f(x) \geq t\} = \{5X\}$. *Since* $5X$ *is a bi-ideal of* X *so by Theorem 3.15,* f *is* $(\in, \in \vee q)$-*fuzzy bi-ideal of* X.

Definition 3.17. *A fuzzy subset* f *of* X *is called* $(\in, \in \vee q)$-*fuzzy quasi ideal of* X, *if it satisfies*

$$f(x) \geq \min\{(f \circ \mathcal{X} \circ \mathcal{X})(x), (\mathcal{X} \circ f \circ \mathcal{X})(x), (\mathcal{X} \circ \mathcal{X} \circ f)(x)\ , 0.5\}.$$

Theorem 3.18. *Let* f *be an* $(\in, \in \vee q)$-*fuzzy quasi ideal of* X. *Then, the set* $f_0 = \{x \in X | f(x) > 0\}$ *is a quasi-ideal of* X.

Proof. In order to prove that f_0 is a quasi-ideal of X, we need to show that $X^2 f_0 \cap X f_0 X \cap f_0 X^2 \subseteq f_0$. Let $a \in X^2 f_0 \cap X f_0 X \cap f_0 X^2$. This implies that $a \in X^2 f_0$, $a \in X f_0 X$ and $a \in f_0 X^2$. Thus, there exist $x_1, y_1, x_2, y_2, x_3, y_3$ in X and a_1, a_2, a_3 in f_0 such that $a = x_1 y_1 a_1$, $a = x_2 a_2 y_2$ and $a = a_3 x_3 y_3$ thus $f(a_1) > 0, f(a_2)$ and $f(a_3) > 0$. We have

$$\begin{aligned} (f \circ \mathcal{X} \circ \mathcal{X})(\mathsf{a}) &= \bigvee_{a = p p_1 q_1} \{f(p) \wedge \mathcal{X}(p_1) \wedge \mathcal{X}(q_1)\} \\ &\geq f(a_3) \wedge \mathcal{X}(x_3) \wedge \mathcal{X}(y_3) \\ &= f(a_3). \end{aligned}$$

Similarly, $(\mathcal{X} \circ f \circ \mathcal{X})(x) \geq f(a_2)$ and $(\mathcal{X} \circ \mathcal{X} \circ f)(x) \geq f(a_1)$. Thus

$$\begin{aligned} f(a) &\geq \min\{(\mathcal{X} \circ \mathcal{X} \circ f)(a), (\mathcal{X} \circ f \circ \mathcal{X})(a), (f \circ \mathcal{X} \circ \mathcal{X})(a), 0.5\} \\ &\geq \min\{f(a_1), f(a_2), f(a_3), 0.5\} \\ &> 0 \text{ because } f(a_3) > 0, f(a_2) > 0 \text{ and } f(a_1) > 0 \end{aligned}$$

Thus, $a \in f_0$. Hence, f_0 is a quasi-ideal of X. □

In the following lemma, we characterize $(\in, \in \vee q)$-fuzzy quasi-ideals in terms of their characteristic functions.

Lemma 3.19. *A non-empty subset A of X is a quasi-ideal of X if and only if C_A is an $(\in, \in \vee q)$-fuzzy quasi-ideal of X.*

Proof. Suppose that A is a quasi-ideal of X and C_A is the characteristic function of A. If $x \notin A$, then $x \notin X^2A$ or $x \notin XAX$ or $x \notin AX^2$. Thus, $(\mathcal{X} \circ \mathcal{X} \circ C_A)(x) = 0$ or $(\mathcal{X} \circ C_A \circ \mathcal{X})(x) = 0$ or $(C_A \circ \mathcal{X} \circ \mathcal{X})(x) = 0$ and so

$$\min\{(\mathcal{X} \circ \mathcal{X} \circ C_A)(x), (\mathcal{X} \circ C_A \circ \mathcal{X})(x), (C_A \circ \mathcal{X} \circ \mathcal{X})(x), 0.5\} = 0 = C_A(x).$$

If $x \in A$, then

$$C_A(x) = 1 \geq \min\{(\mathcal{X} \circ \mathcal{X} \circ C_A)(x), (\mathcal{X} \circ C_A \circ \mathcal{X})(x), (C_A \circ \mathcal{X} \circ \mathcal{X})(x), 0.5\}.$$

Hence, C_A is $(\in, \in \vee q)$-fuzzy quasi-ideal of X.

Conversely, assume that C_A is an $(\in, \in \vee q)$-fuzzy quasi-ideal of X. Hence, A is quasi-ideal of X. □

Lemma 3.20. *Every $(\in, \in \vee q)$-fuzzy left (resp. lateral, right) ideal of X is an $(\in, \in \vee q)$-fuzzy quasi-ideal of X.*

Proof. Let $a \in X$ and let f be $(\in, \in \vee q)$-fuzzy left ideal then

$$(\mathcal{X} \circ \mathcal{X} \circ f)(a) = \bigvee_{a=p_1p_2p_3} \{\mathcal{X}(p_1), \mathcal{X}(p_2), f(p_3)\} = \bigvee_{a=p_1p_2p_3} f_1(p_3)$$

This implies that

$$\begin{aligned} (\mathcal{X} \circ \mathcal{X} \circ f)(a) \wedge 0.5 &= \left(\bigvee_{a=p_1p_2p_3} f(p_3)\right) \wedge 0.5 \\ &= \bigvee_{a=p_1p_2p_3} (f(p_3) \wedge 0.5) \\ &\leq \bigvee_{a=p_1p_2p_3} f(p_1p_2p_3) \text{ because } f \text{ is } (\in, \in \vee q)\text{-fuzzy left ideal of } X. \\ &= f(a) \end{aligned}$$

Thus $(\mathcal{X} \circ \mathcal{X} \circ f)(a) \wedge 0.5 \leq f(a)$. Hence,

$$f(a) \geq (\mathcal{X} \circ \mathcal{X} \circ f)(a) \wedge 0.5 \geq \min\{(f \circ \mathcal{X} \circ \mathcal{X})(x), (\mathcal{X} \circ f \circ \mathcal{X})(x), (\mathcal{X} \circ \mathcal{X} \circ f)(x), 0.5\}.$$

Thus, f is an $(\in, \in \vee q)$-fuzzy quasi-ideal of X. □

Theorem 3.21. *Every $(\in, \in \vee q)$-fuzzy quasi-ideal of X is an $(\in, \in \vee q)$-fuzzy bi-ideal of X.*

Proof. Suppose that f is $(\in, \in \vee q)$-fuzzy quasi-ideal of X. Now,

$$\begin{aligned} f(xyz) &\geq (f \circ \mathcal{X} \circ \mathcal{X})(xyz) \wedge (\mathcal{X} \circ f \circ \mathcal{X})(xyz) \wedge (\mathcal{X} \circ \mathcal{X} \circ f)(xyz) \wedge 0.5 \\ &= \left[\bigvee_{xyz=abc} f(a) \wedge \mathcal{X}(b) \wedge \mathcal{X}(c) \right] \wedge \left[\bigvee_{xyz=pqr} \mathcal{X}(p) \wedge f(q) \wedge \mathcal{X}(r) \right] \\ &\quad \wedge \left[\bigvee_{xyz=p_1q_1r_1} \mathcal{X}(p_1) \wedge \mathcal{X}(q_1) \wedge f(r_1) \right] \wedge 0.5 \\ &\geq [f(x) \wedge \mathcal{X}(y) \wedge \mathcal{X}(z)] \wedge [\mathcal{X}(x) \wedge f(y) \wedge \mathcal{X}(z)] \wedge [\mathcal{X}(x) \wedge \mathcal{X}(y) \wedge f(z)] \wedge 0.5 \\ &= f(x) \wedge f(y) \wedge f(z) \wedge 0.5, \end{aligned}$$

so $f(xyz) \geq \min\{f_1(x), f_1(y), f_1(z), 0.5\}$. In the same way, we can prove that $f(abcde) \geq \min\{f(a), f(c), f(e)0.5\}$. Thus f is an $(\in, \in \vee q)$-fuzzy bi-ideal of X. □

Theorem 3.22. *Every $(\in, \in \vee q)$-fuzzy left (resp. lateral, right) ideal of X is an $(\in, \in \vee q)$-fuzzy bi-ideal of X.*

Proof. Let f an be $(\in, \in \vee q)$-fuzzy left (resp. lateral, right) ideal of X. Then,

$$f(abc) \geq \min\{f(c), 0.5\} \text{ for all } a, b, c \in X.$$

Suppose that there exist $x, y, z \in X$ such that

$$f(xyz) < \min\{f(x), f_1(y), f_1(z), 0.5\}.$$

Choose $t \in (0, 0.5]$ such that

$$f(xyz) < t \leq \min\{f(x), f(y), f(z), 0.5\}.$$

Then, $[z; t] \in f$ but $[xyz; t] \overline{\in} f$. Also, we have

$$f(xyz) + t < 0.5 + 0.5 = 1,$$

which implies that $[xyz; t] \overline{q} f$. Thus, $[xyz; t] \overline{\in \vee q} f$, which contradicts the hypothesis. Thus,

$$f(xyz) \geq \min\{f(x), f(y), f(z), 0.5\}.$$

Similarly, suppose that there exist $a, b, c, d, e \in X$ such that

$$f(abcde) < \min\{f(a), f(c), f(e), 0.5\}.$$

Choose $t \in (0, 0.5]$ such that

$$f(a(bcd)e) < t \leq \min\{f(a), f(c), f(e), 0.5\}.$$

Then, $[e; t] \in f$ but $[a(bcd)e; t] \overline{\in} f$. Also,

$$f(abcde) + t < 0.5 + 0.5 = 1$$

for all $a, b, c, d, e \in X$. Thus, $[abcde; t] \overline{q} f$. Hence, $[abcde; t] \overline{\in \vee q} f$, a contradiction. Therefore,

$$f(abcde) \geq \min\{f(a), f(c), f(e), 0.5\}.$$

Consequently, f is an $(\in, \in \vee q)$-fuzzy bi-ideal of X. □

4. REGULAR TERNARY SEMIGROUPS

In this section, we characterize regular ternary semigroups by the properties of their $(\in, \in \vee q)$-fuzzy ideals and $(\in, \in \vee q)$-fuzzy bi-ideals.

Definition 4.1. *Let f_1, f_2 and f_3 be fuzzy subsets of X. We define $f_1^-, (f_1 \wedge f_2 \wedge f_3)^-, (f_1 \vee f_2 \vee f_3)^-, (f_1 \circ f_2 \circ f_3)^-$ as follows.*

$f_1^-(x) = f_1(x) \wedge 0.5,$
$(f_1 \wedge f_2 \wedge f_3)^-(x) = (f_1 \wedge f_2 \wedge f_3)(x) \wedge 0.5,$
$(f_1 \vee f_2 \vee f_3)^-(x) = (f_1 \vee f_2 \vee f_3)(x) \wedge 0.5,$
$(f_1 \circ f_2 \circ f_3)^-(x) = (f_1 \circ f_2 \circ f_3)(x) \wedge 0.5$ *for all* $x \in X$.

Lemma 4.2. [24] *Let f_1 and f_2 be fuzzy subsets of a ternary semigroup X. Then, the following conditions hold:*

(1) $(f_1 \wedge f_2)^- = f_1^- \wedge f_2^-,$
(2) $(f_1 \vee f_2)^- = f_1^- \vee f_2^-,$
(3) $(f_1 \circ f_2)^- = f_1^- \circ f_2^-.$

Lemma 4.3. *Let f_1, f_2 and f_3 be fuzzy subsets of X. Then, the following conditions hold:*

(1) $(f_1 \wedge f_2 \wedge f_3)^- = f_1^- \wedge f_2^- \wedge f_3^-,$
(2) $(f_1 \vee f_2 \vee f_3)^- = f_1^- \vee f_2^- \vee f_3^-,$
(3) $(f_1 \circ f_2 \circ f_3)^- = f_1^- \circ f_2^- \circ f_3^-.$

Proof. The proof is a consequence of the previous lemma. □

Lemma 4.4. [24] *Let A, B be non-empty subsets of a semigroup X. Then, the following conditions hold:*

(1) $(C_A \wedge C_B)^- = C_{A\cap B}^-,$
(2) $(C_A \vee C_B)^- = C_{A\cup B}^-,$
(3) $(C_A \circ C_B)^- = C_{AB}^-.$

Lemma 4.5. *Let A, B and D be non-empty subsets of a semigroup X. Then, the following conditions hold:*

(1) $(C_A \wedge C_B \wedge C_D)^- = C_{A\cap B\cap D}^-,$
(2) $(C_A \vee C_B \vee C_D)^- = C_{A\cup B\cup D}^-,$
(3) $(C_A \circ C_B \circ C_D)^- = C_{ABD}^-.$

Proof. The proof is a consequence of the previous lemma. □

Lemma 4.6. *Let A be a non-empty subset of a ternary semigroup X. Then, A is a left (resp. lateral, right) ideal of X if and only if C_A^- is an $(\in, \in \vee q)$-fuzzy left (resp. lateral, right) ideal of X.*

Proof. Let A be a left ideal of a ternary semigroup X. Then, by Theorem 3.9, C_A^- is an $(\in, \in \vee q)$-fuzzy left ideal of X.

Conversely, suppose that C_A^- is an $(\in, \in \vee q)$-fuzzy left ideal of X. If $z \in A$ then $C_A^-(z) = 0.5$. So $z_{0.5} \in C_A^-$. Since C_A^- is $(\in, \in \vee q)$-fuzzy left ideal of X,

$$C_A^-(xyz) \geq 0.5 \text{ or } C_A^-(xyz) + 0.5 > 1 \text{ for all } x, y \in X.$$

If $C_A^-(xyz) + 0.5 > 1$, then $C_A^-(xyz) > 0.5$. Thus, $C_A^-(xyz) \geq 0.5$ which gives $C_A^-(xyz) = 0.5$ and hence $xyz \in A$ for all $x, y \in X$ and $z \in A$. Consequently, A is a left ideal of X. □

Lemma 4.7. *Let B be a non-empty subset of a ternary semigroup X. Then, B is a bi-ideal of X if and only if C_B^- is an $(\in, \in \vee q)$-fuzzy bi-ideal of X.*

Proof. The proof is similar to the proof of Lemma 4.6. □

Proposition 4.8. *Let f be an $(\in, \in \vee q)$-fuzzy left (resp. lateral, right) ideal of X. Then, f^- is a fuzzy left (resp. lateral, right) ideal of X.*

Proof. It is straightforward. □

Theorem 4.9. [9] *A ternary semigroup X is regular if and only if for every left ideal L, lateral ideal T and right ideal R of X we have $R \cap T \cap L = RTL$.*

Theorem 4.10. *For a ternary semigroup X the following conditions are equivalent:*

(1) *X is regular.*
(2) *$(f_1 \wedge f_2 \wedge f_3)^- = (f_1 \circ f_2 \circ f_3)^-$ for every $(\in, \in \vee q)$-fuzzy right ideal f_1, $(\in, \in \vee q)$-fuzzy lateral ideal f_2 and $(\in, \in \vee q)$-fuzzy left ideal f_3 of X.*

Proof. (1) $\Longrightarrow$ (2) : Let X be a regular ternary semigroup and let f_1 be an $(\in, \in \vee q)$-fuzzy right ideal, f_2 be an $(\in, \in \vee q)$-fuzzy lateral ideal and f_3 be an $(\in, \in \vee q)$-fuzzy left ideal of X. Now, we obtain

$$\begin{aligned}
(f_1 \circ f_2 \circ f_3)^-(a) &= (f_1 \circ f_2 \circ f_3)(a) \wedge 0.5 \\
&= \left(\bigvee_{a=bcd} f_1(b) \wedge f_2(c) \wedge f_3(d) \right) \wedge 0.5 \\
&= \bigvee_{a=bcd} \{f_1(b) \wedge f_2(c) \wedge f_3(d) \wedge 0.5\} \\
&= \bigvee_{a=bcd} \{(f_1(b) \wedge 0.5) \wedge (f_2(c) \wedge 0.5) \wedge (f_3(d) \wedge 0.5) \wedge 0.5\} \\
&\leq \bigvee_{a=bcd} \{f_1(bcd) \wedge f_2(bcd) \wedge f_3(bcd) \wedge 0.5\} \\
&= f_1(a) \wedge f_2(a) \wedge f_3(a) \wedge 0.5 \\
&= (f_1^- \wedge f_2^- \wedge f_3^-)(a).
\end{aligned}$$

Hence, $(f_1 \circ f_2 \circ f_3)^- \leq (f_1 \wedge f_2 \wedge f_3)^-$.

On the other hand, since X is regular so for $a \in X$ there exist $x, y \in X$ such that $a = axaya = axayaxaya$. Thus,

$$\begin{aligned}
(f_1 \circ f_2 \circ f_3)^-(a) &= (f_1 \circ f_2 \circ f_3)(a) \wedge 0.5 \\
&= \left(\bigvee_{a=bcd} f_1(b) \wedge f_2(c) \wedge f_3(d) \right) \wedge 0.5 \\
&= \bigvee_{a=bcd} \{f_1(b) \wedge f_2(c) \wedge f_3(d) \wedge 0.5\} \\
&\geq f_1(axa) \wedge f_2(yax) \wedge f_3(aya) \wedge 0.5 \\
&\geq (f_1(a) \wedge 0.5) \wedge (f_2(a) \wedge 0.5) \wedge (f_3(a) \wedge 0.5) \wedge 0.5 \\
&= f_1(a) \wedge 0.5 \wedge f_2(a) \wedge 0.5 \wedge f_3(a) \wedge 0.5 \wedge 0.5 \\
&= f_1(a) \wedge f_2(a) \wedge f_3(a) \wedge 0.5 \\
&= (f_1^- \wedge f_2^- \wedge f_3^-)(a).
\end{aligned}$$

Hence, $(f_1 \circ f_2 \circ f_3)^-(a) = (f_1 \wedge f_2 \wedge f_3)^-(a)$.

(2) $\Longrightarrow$ (1) : Suppose that R, T and L are right, lateral and left ideals of X, respectively. Then, by Lemma 4.6, C_R^-, C_T^- and C_L^- are $(\in, \in \vee q)$-fuzzy right ideal, $(\in, \in \vee q)$-fuzzy lateral ideal and $(\in, \in \vee q)$-fuzzy left ideal of X respectively. Thus, we have

$$\begin{aligned}
C_{RTL}^- &= C_R^- \circ C_T^- \circ C_L^- \text{ by Lemma 4.5} \\
&= (C_R \wedge C_T \wedge C_L)^- \text{ by Theorem 4.10 } (ii) \\
&= C_{R\cap T\cap L}^- \text{ by Lemma 4.5.}
\end{aligned}$$

Thus, $RTL = R \cap T \cap L$, and so X is regular. □

Theorem 4.11. *The following assertions for a ternary semigroup X are equivalent.*

(1) *X is regular.*
(2) $f_1^- = (f_1 \circ \mathcal{X} \circ f_1 \circ \mathcal{X} \circ f_1)^-$ *for every $(\in, \in \vee q)$-fuzzy bi-ideal f_1 of X.*

Proof. (1) $\Longrightarrow$ (2) : Let X be regular ternary semigroup and f_1 be an $(\in, \in \vee q)$-fuzzy bi-ideal of X. Since X is regular, so for $a \in X$ there exist $x, y \in X$ such that

$a = axaya$. Now, we obtain

$$\begin{aligned}
& (f_1 \circ \mathcal{X} \circ f_1 \circ \mathcal{X} \circ f_1)^-(a) \\
= & \left(\bigvee_{a=bcd} (f_1 \circ \mathcal{X} \circ f_1)^- (b) \wedge \mathcal{X}(c) \wedge f_1(d) \right) \wedge 0.5 \\
= & \bigvee_{a=bcd} \Big((f_1 \circ \mathcal{X} \circ f_1)^- (b) \wedge \mathcal{X}(c) \wedge f_1(d) \wedge 0.5 \Big) \\
= & \bigvee_{a=bcd} \left\{ \left(\bigvee_{b=xyz} f_1(x) \wedge \mathcal{X}(y) \wedge f_1(z) \wedge 0.5 \right) \wedge \mathcal{X}(c) \wedge f_1(d) \wedge 0.5 \right\} \\
= & \bigvee_{a=(xyz)cd} f_1(x) \wedge \mathcal{X}(y) \wedge f_1(z) \wedge \mathcal{X}(c) \wedge f_1(d) \wedge 0.5 \\
\geq & f_1(a) \wedge \mathcal{X}(x) \wedge f_1(a) \wedge \mathcal{X}(y) \wedge f_1(a) \wedge 0.5 \\
= & f_1(a) \wedge 1 \wedge f_1(a) \wedge 1 \wedge f_1(a) \wedge 0.5 \\
= & f_1(a) \wedge 0.5 \\
= & f_1^-(a).
\end{aligned}$$

Therefore, $(f_1 \circ \mathcal{X} \circ f_1 \circ \mathcal{X} \circ f_1)^- \geq f_1^-$. On the other hand, we have

$$\begin{aligned}
& (f_1 \circ \mathcal{X} \circ f_1 \circ \mathcal{X} \circ f_1)^-(a) \\
= & \left(\bigvee_{a=bcd} (f_1 \circ \mathcal{X} \circ f_1)^- (b) \wedge \mathcal{X}(c) \wedge f_1(d) \right) \wedge 0.5 \\
= & \bigvee_{a=bcd} \Big((f_1 \circ \mathcal{X} \circ f_1)^- (b) \wedge \mathcal{X}(c) \wedge f_1(d) \wedge 0.5 \Big) \\
= & \bigvee_{a=bcd} \left\{ \left(\bigvee_{b=xyz} f_1(x) \wedge \mathcal{X}(y) \wedge f_1(z) \wedge 0.5 \right) \wedge \mathcal{X}(c) \wedge f_1(d) \wedge 0.5 \right\} \\
= & \bigvee_{a=(xyz)cd} (f_1(x) \wedge \mathcal{X}(y) \wedge f_1(z) \wedge \mathcal{X}(c) \wedge f_1(d) \wedge 0.5) \\
= & \bigvee_{a=(xyz)cd} (f_1(x) \wedge f_1(z) \wedge f_1(d) \wedge 0.5) \\
\leq & \bigvee_{a=(xyz)cd} f_1(xyzcd) \wedge 0.5 \text{ as } f_1 \text{ is } (\in, \in \vee q) - \text{fuzzy bi-ideal of } X \\
= & f_1(a) \wedge 0.5 \\
= & f_1^-(a).
\end{aligned}$$

Hence, we get $(f_1 \circ C_{\mathcal{X}} \circ f_1 \circ \mathcal{X} \circ f_1)^- \leq f_1^-$. Thus,

$$(f_1 \circ \mathcal{X} \circ f_1 \circ \mathcal{X} \circ f_1)^- = f_1^-.$$

(2) $\Longrightarrow$ (1) : Let B be a bi-ideal of X. Then, by Lemma 4.7, C_B^- is an $(\in,\in\vee q)$-fuzzy bi-ideal of X. If $a\in B$ then $C_B^-(a)=0.5$. Now, by (2), we obtain

$$\begin{aligned} C_B^-(a) &= (C_B\circ\mathcal{X}\circ C_B\circ\mathcal{X}\circ C_B)^-(a)\\ &= \bigvee_{a=bfg}(C_B\circ\mathcal{X}\circ C_B)^-(b)\wedge\mathcal{X}(f)\wedge C_B(g)\wedge 0.5\\ &= \bigvee_{a=bfg}\left\{\bigvee_{b=cde}C_B(c)\wedge\mathcal{X}(d)\wedge C_B(e)\wedge 0.5\right\}\wedge\mathcal{X}(f)\wedge C_B(g)\wedge 0.5\\ &= \bigvee_{a=(cde)fg}C_B(c)\wedge\mathcal{X}(d)\wedge C_B(e)\wedge\mathcal{X}(f)\wedge C_B(g)\wedge 0.5\\ &= \bigvee_{a=(cde)fg}C_B(c)\wedge C_B(e)\wedge C_B(g)\wedge 0.5\\ &= \bigvee_{a=(cde)fg}(C_B(c)\wedge 0.5)\wedge(C_B(e)\wedge 0.5)\wedge(C_B(g)\wedge 0.5)\\ &= \bigvee_{a=(cde)fg}C_B^-(c)\wedge C_B^-(e)\wedge C_B^-(g)\\ &= 0.5 \text{ as } C_B^-(a)=0.5, \end{aligned}$$

which is only possible if $c,e,g\in B$. Thus, $cdefg\in BXBXB$ and hence $a\in BXBXB$ so $B\subseteq BXBXB$ but $BXBXB\subseteq B$ it follows that X is regular. □

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