J. Indones. Math. Soc.
Vol. 29, No. 03 (2023), pp. 338–346.

MINIMUM ROMAN DOMINATING DISTANCE ENERGY OF A GRAPH

Lakshmanan, R.[1], and N. Annamalai[2]

[1]**PG and Research Department of Mathematics, Thiagarajar College, Madurai-625 009, Tamil Nadu, India, lakshmsc2004@yahoo.co.in**
[2]**Government Polytechnic College, Sankarapuram-606 401, Tamil Nadu, India, algebra.annamalai@gmail.com**

Abstract. In this correspondence, we introduced the concept of minimum roman dominating distance energy $E_{RDd}(G)$ of a graph G and computed minimum roman dominating distance energy of some standard graphs. Also, we discussed the properties of eigenvalues of a minimum roman dominating distance matrix $A_{RDd}(G)$. Finally, we derived the upper and lower bounds for $E_{RDd}(G)$.

Key words and Phrases: Distance Matrix, Energy of a graph, Roman dominating function, Roman domination.

1. Introduction

In 1978, I. Gutman[7] was introduced the concept of energy of a graph . The graph $G = (V, E)$ mean a simple connected graph with n vertices and m edges. The distance between two vertices u and v is the length shortest distance between u and v. The Wiener index $W(G)$ of G, is the sum of the lengths of the shortest paths between all pairs of vertices. Let $A = (a_{ij})$ be the adjacency matrix of a graph G. Then the energy $E(G)$ of a graph G is defined by the sum of absolute value of all eigenvalues of A. For more details about energy of a graph[1].

The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$. A *crown graph* S_k^0 is a bipartite graph with two sets of vertices $\{a_1, a_2, \ldots, a_k\}$ and $\{b_1, b_2, \ldots, b_k\}$ and with an edge from a_i to b_j whenever $i \neq j$. For a positive integer $n \geq 2$, a healthy spider is a star $K^*_{1,n-1}$ with all of its edges subdivided[3].

The distance matrix $A_d = (d_{ij})$ of G is a symmetric matrix of order n where d_{ij} is the distance between i^{th} and j^{th} vertices of a graph. The distance energy

2020 Mathematics Subject Classification: 05C50, 05C69.
Received: 06-04-2022, accepted: 04-11-2023.

$E_d(G)$ of the graph G is defined by the sum of absolute value of all eigenvalues of A_d. The distance matrix of an undirected graph has been widely studied in the literature, see [2, 4, 5, 6].

A set $S \subseteq V$ is a dominating set if every vertex of $V \setminus S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G, and a dominating set S of minimum cardinality is called a γ-set of G. E. J. Cockayne et al.[3] introduce the concept of roman domination in graphs. A *Roman dominating function* on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$.

Let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V \mid f(v) = i\}$ and $|V_i| = n_i$, for $i = 0, 1, 2$. Note that there exists a one-one correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus, we will write $f = (V_0, V_1, V_2)$. A function $f = (V_0, V_1, V_2)$ is a Roman dominating function (RDF) if $V_2 \succ V_0$, where $\succ$ means that the set V_2 dominates the set V_0. The weight of f is $f(V) = \sum\limits_{v \in V} f(v) = 2n_2 + n_1$.

The Roman domination number, denoted $\gamma_R(G)$, equals the minimum weight of an RDF of G, and we say that a function $f = (V_0, V_1, V_2)$ is a γ_R-function if it is an RDF and $f(V) = \gamma_R(G)$.

Theorem 1.1. [3] *For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.*

Kanna et al. [8, 10] studied the minimum covering distance energy of a graph and also they were studied Laplacian minimum dominating energy of a graph. Kanna et al. [9] introduced the concept of the minimum dominating distance energy of a graph. Let G be the graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Let D be a minimum dominating set of a graph G. The minimum dominating distance matrix of G is the square matrix $A_{Dd}(G) := (d'_{ij})$ where

$$d'_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } v_i \in D \\ d(v_i, v_j) & \text{otherwise.} \end{cases}$$

Let $\delta_1, \delta_2, \ldots, \delta_n$ be the eigenvalues of $A_{Dd}(G)$. Then the minimum dominating energy $E_{Dd}(G)$ of G is

$$E_{Dd}(G) = \sum_{j=1}^{n} |\delta_j|.$$

In this paper, we introduce the concept of minimum roman dominating distance energy of a graph in section 2. In Section 3, we find the minimum roman dominating distance energy of some standard graphs. In section 4, we discussed the properties of eigenvalues of a minimum roman dominating distance matrix $A_{RDd}(G)$. We derived the upper and lower bounds for $E_{RDd}(G)$ in Section 5.

2. THE MINIMUM ROMAN DOMINATING ENERGY

In this section, we introduce the concept of minimum roman dominating energy of a graph.

Let $f = (V_0, V_1, V_2)$ be a γ_R-function of a graph G. The minimum roman dominating distance matrix $A_{RDd}(G)$ of G is defined as $A_{RDd}(G) := (\bar{d}_{ij})$ where

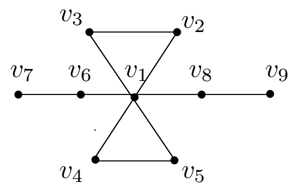
$$\bar{d}_{ij} = \begin{cases} 2 & \text{if } i = j \text{ and } v_i \in V_2 \\ 1 & \text{if } i = j \text{ and } v_i \in V_1 \\ d(v_i, v_j) & \text{otherwise.} \end{cases}$$

Let $\rho_1, \rho_2, \ldots, \rho_n$ be the eigenvalues of $A_{RDd}(G)$. Then the minimum roman dominating distance energy $E_{RDd}(G)$ of G is defined as

$$E_{RDd}(G) = \sum_{k=1}^{n} |\rho_k|.$$

Note that $tr(A_{RDd}(G)) = \gamma_R(G)$.

Example 2.1. *The minimum roman dominating function of the following graph G*



is $f = (V_0, V_1, V_2)$ where $V_2 = \{v_1\}, V_1 = \{v_7, v_9\}$ and $V_0 = \{v_2, v_3, v_4, v_5, v_6, v_8\}$. Then the minimum roman dominating distance matrix is

$$A_{RDd}(G) = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 2 & 2 & 2 & 3 & 2 & 3 \\ 1 & 1 & 0 & 2 & 2 & 2 & 3 & 2 & 3 \\ 1 & 2 & 2 & 0 & 1 & 2 & 3 & 2 & 3 \\ 1 & 2 & 2 & 1 & 0 & 2 & 3 & 2 & 3 \\ 1 & 2 & 2 & 2 & 2 & 0 & 1 & 2 & 3 \\ 2 & 3 & 3 & 3 & 3 & 1 & 1 & 3 & 4 \\ 1 & 2 & 2 & 2 & 2 & 2 & 3 & 0 & 1 \\ 2 & 3 & 3 & 3 & 3 & 3 & 4 & 1 & 1 \end{pmatrix}_{9\times 9}.$$

Then the characteristic equation of $A_{RDd}(G)$ is

$$\rho^9 - 4\rho^8 - 171\rho^7 - 1034\rho^6 - 2339\rho^5 - 1284\rho^4 + 2659\rho^3 + 4438\rho^2 + 2410\rho + 444 = 0$$

and the eigenvalues are $\rho_1 = -3, \rho_2 = -1, \rho_3 = -1, \rho_4 \approx -4.5615, \rho_5 \approx -0.4384, \rho_6 \approx -3.9721, \rho_7 \approx -0.8397, \rho_8 \approx 1.2642,$ and $\rho_9 \approx 17.5476$. Hence the minimum roman dominating energy of G is $E_{RDd}(G) \approx 33.6237$.

Note that this graph has unique minimum roman dominating function.

3. Minimum Roman Dominating Distance Energy of Some Standard Graphs

In this section, we studied the minimum roman dominating distance energy of complete, complete bipartite, crown, star and healthy spider graphs.

Denote J_n is an $n \times n$ all ones matrix, I_n is an $n \times n$ identity matrix, D_k is a diagonal matrix whose k^{th} diagonal entry is zero and other diagonal entries are two, $\mathbf{a_n}$ is a $1 \times n$ row vector $[a, a, \ldots, a]$ and $\mathbf{a'_n}$ is the transpose of $\mathbf{a_n}$.

Theorem 3.1. *For any integer $n \geq 3$, $E_{RDd}(K_n) = 2n - 2$.*

Proof. For a complete graph K_n, the minimum roman dominating function is $f = (V_0, V_1, V_2)$ where $V_2 = \{v_i\}$ for any i, $V_1 = \emptyset$ and $V_0 = V \setminus V_2$. Then the minimum roman dominating distance matrix $A_{RDd}(K_n) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \text{ and } v_i \in V_2 \\ 0 & \text{if } i = j \text{ and } v_i \in V_0 \\ 1 & \text{otherwise} \end{cases}$$

One can easily show that the characteristic polynomial of $A_{RDd}(K_n)$ is $(\rho+1)^{n-2}(\rho^2 - n\rho + n - 3)$. Hence the eigenvalues are -1 with multiplicity $n-2$ and $\frac{n \pm \sqrt{n^2-4n+12}}{2}$. Since for $n \geq 3$, $n \geq \sqrt{n^2 - 4n + 12}$. Therefore, the eigenvalues $\frac{n+\sqrt{n^2-4n+12}}{2}$ and $\frac{n-\sqrt{n^2-4n+12}}{2}$ are positive. Hence the sum of absolute values of all eigenvalues is $2n - 2$. That is, $E_{RDd}(K_n) = 2n - 2$.

Corollary 3.2. *For any integer $n \geq 3$, $E_{RDd}(K_n) = E_d(K_n) = E(K_n)$.*

Proof. Let $n \geq 2$ be an integer. The adjacency matrix of a complete graph K_n is $J_n - I_n$ and the eigenvalues are $n - 1$ and -1 with multiplicity $n - 1$, the energy of K_n is $2n - 2$.

Theorem 3.3. *For any integer $r \geq 2$,*

$$E_{RDd}(K_{r,r}) = 2(2r - 4) + \sqrt{(r-2)^2 + 8} + \sqrt{(3r-2)^2 + 24}.$$

Proof. Let $X = \{v_1, v_2, \ldots, v_r\}$ and $Y = \{w_1, w_2, \ldots, w_r\}$ be a partition of the vertex set of a complete bipartite graph $K_{r,r}$. Then the minimum roman dominating function is $f = (V_0, V_1, V_2)$ where $V_2 = \{v_i, w_j\}$ for any $1 \leq i, j \leq r$, $V_1 = \emptyset$ and $V_0 = V \setminus V_2$. Then the minimum roman dominating distance matrix is

$$A_{RDd}(K_{r,r}) = \left[\begin{array}{c|c} 2J_r - D_i & J_r \\ \hline J_r & 2J_r - D_j \end{array} \right]_{2r \times 2r}.$$

One can easily show that the characteristic equation of $A_{RDd}(K_{r,r})$ is $(\rho+2)^{2r-4}(\rho^2 - (r-2)\rho - 2)(\rho^2 - (3r-2)\rho - 6) = 0$. Then the eigenvalues are -2 with multiplicity $2r - 4$, $\frac{(r-2) \pm \sqrt{(r-2)^2+8}}{2}$ and $\frac{(3r-2) \pm \sqrt{(3r-2)^2+24}}{2}$. Therefore, the eigenvalues $\frac{(r-2)-\sqrt{(r-2)^2+8}}{2}$, $\frac{(3r-2)-\sqrt{(3r-2)^2+24}}{2}$ are negative and the eigenvalues $\frac{(r-2)+\sqrt{(r-2)^2+8}}{2}$, $\frac{(3r-2)+\sqrt{(3r-2)^2+24}}{2}$ are positive.

Hence the sum of absolute values of all eigenvalues is $2(2r-4)+\sqrt{(r-2)^2+8}+\sqrt{(3r-2)^2+24}$. That is, $E_{RDd}(K_{r,r}) = 2(2r-4)+\sqrt{(r-2)^2+8}+\sqrt{(3r-2)^2+24}$.

Theorem 3.4. *For any $n \geq 3$, $E_{RDd}(K_{1,n-1}) = 4n-6$.*

PROOF. Consider the star graph $K_{1,n-1}$ with vertex set $V = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$, where $deg(v_0) = n-1$. The minimum roman dominating distance function is $f = (V_0, V_1, V_2)$ where $V_2 = \{v_0\}$, $V_1 = \emptyset$ and $V_0 = V \setminus V_2 = \{v_1, v_2, \ldots, v_{n-1}\}$. Then the minimum roman dominating distance matrix is

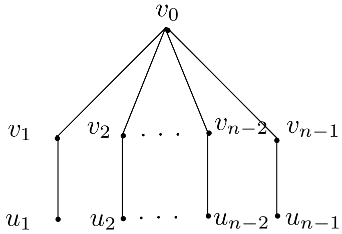
$$A_{RDd}(K_{1,n-1}) = \left[\begin{array}{c|c} 2 & \mathbf{1_{n-1}} \\ \hline \mathbf{1'_{n-1}} & 2J_{n-1} - 2I_{n-1} \end{array}\right]_{n\times n}.$$

The characteristic equation is $(\rho+2)^{n-2}(\rho^2 - (2n-2)\rho + 3n - 7) = 0$. Then the eigenvalues are -2 with multiplicity $n-2$, $(n-1) \pm \sqrt{n^2-5n+8}$. Hence the sum of absolute values of all eigenvalues is $4n-6$. That is, $E_{RDd}(K_{1,n-1}) = 4n-6$.

Theorem 3.5. *For an odd integer $n \geq 4$,*

$$11n - 19 \leq E_{RDd}(K^*_{1,n-1}) \leq 6n^2 - 4n - 16.$$

PROOF. Consider the healthy spider graph $K^*_{1,n-1}$ with vertex set $V = \{v_0, v_1, v_2, \ldots, v_{n-1}, u_1, u_2, \ldots, u_{n-1}\}$. The vertex v_0 is adjacent with $v_1, v_2, \ldots, v_{n-1}$ and for $1 \leq i \leq n-1$, u_i is adjacent with v_i. Then the minimum roman dominating function is $f = (V_0, V_1, V_2)$ where $V_2 = \{v_0\}$, $V_1 = \{u_1, u_2, \ldots, u_{n-1}\}$ and $V_0 = \{v_1, v_2, \ldots, v_{n-1}\}$.



Then the minimum roman dominating distance matrix is

$$A_{RDd}(K^*_{1,n-1}) = \left[\begin{array}{c|c|c} 2 & \mathbf{1_{n-1}} & \mathbf{2_{n-1}} \\ \hline \mathbf{1'_{n-1}} & 2J_{n-1} - 2I_{n-1} & 3J_{n-1} - 2I_{n-1} \\ \hline \mathbf{2'_{n-1}} & 3J_{n-1} - 2I_{n-1} & 4J_{n-1} - 3I_{n-1} \end{array}\right]_{2n-1\times 2n-1}.$$

The characteristic equation of $A_{RDd}(K^*_{1,n-1})$ is

$$(\rho^2 + 5\rho + 2)^{n-2}[\rho^3 - (6n-9)\rho^2 - (n^2 - 7n + 14)\rho + 2n^2 - 3n - 3] = 0.$$

The sum of absolute values of the roots of $(\rho^2+5\rho+2)^2 = 0$ is $5n-10$ and the sum of absolute values of roots of the $\rho^3 - (6n-9)\rho^2 - (n^2-7n+14)\rho + 2n^2 - 3n - 3 = 0$ is greater than or equal to $6n-9$. Therefore,

$$E_{RDd}(K^*_{1,n-1}) \geq 11n - 19.$$

By Cauchy's bound for roots of a polynomial, all the roots of $\rho^3 - (6n-9)\rho^2 - (n^2 - 7n + 14)\rho + 2n^2 - 3n - 3 = 0$ lies in the closed interval $[-(M+1), M+1]$

where $M = 2n^2 - 3n - 3$. Therefore, the sum of absolute values of these roots is bounded above by $3(2n^2 - 3n - 2) = 6n^2 - 9n - 6$. Thus,

$$E_{RDd}(K^*_{1,n-1}) \leq 5n - 10 + 6n^2 - 9n - 6 = 6n^2 - 4n - 16.$$

Theorem 3.6. *For any integer $k \geq 2$, $E_{RDd}(S^0_k) = 7k - 6 + \sqrt{k^2 - 4k + 12}$.*

Proof. Consider the crown graph S^0_k with vertex set $V = X \cup Y$ where $X = \{v_1, v_2, \ldots, v_k\}$ and $Y = \{w_1, w_2, \ldots, w_k\}$. The minimum roman dominating distance function is $f = (V_0, V_1, V_2)$ where $V_2 = \{v_i, w_i\}$ for any $1 \leq i \leq k$, $V_1 = \emptyset$ and $V_0 = V \setminus V_2$. Then the minimum roman dominating distance matrix is

$$A_{RDd}(S^0_k) = \left[\begin{array}{c|c} 2J_k - D_i & J_k + 2I_k \\ \hline J_k + 2I_k & 2J_k - D_i \end{array}\right]_{2k \times 2k}.$$

The characteristic equation of $A_{RDd}(S^0_k)$ is

$$\rho^{2k-2}(\rho + 4)^{2k-2}[\rho^2 - (3k + 2)\rho + 6(k - 1)][(\rho^2 + (6 - k)\rho - 2k + 6] = 0.$$

Then the eigenvalues are -4 with multiplicity $k - 2$, 0 with multiplicity $k - 2$, $\frac{(-6+k)\pm\sqrt{k^2-4k+12}}{2}$ and $\frac{(3k+2)\pm\sqrt{9k^2-12k+28}}{2}$. Hence the sum of absolute values of all eigenvalues is $7k-6+\sqrt{k^2 - 4k + 12}$. That is, $E_{RDd}(S^0_k) = 7k-6+\sqrt{k^2 - 4k + 12}$.

Theorem 3.7. *Let G and H be two disjoint graphs. Then $E_{RDd}(G \cup H) = E_{RDd}(G) + E_{RDd}(H)$.*

Proof. Let A and B be the minimum roman dominating distance matrix of G and H, respectively. Then the minimum roman dominating distance matrix of $G \cup H$ is

$$A_{RDd}(G \cup H) = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right].$$

The characteristic polynomial of $A_{RDd}(G \cup H)$ is the product of characteristic polynomial of A and B. Therefore, $E_{RDd}(G \cup H) = E_{RDd}(G) + E_{RDd}(H)$.

4. Properties of Eigenvalues of Minimum Roman Dominating Distance Matrix $A_{RDd}(G)$

In this section, we discussed the relation between the eigenvalues of the minimum roman dominating distance matrix $A_{RDD}(G)$ and the minimum roman dominating energy γ_R of G.

Theorem 4.1. *Let $G = (V, E)$ be a graph and let $f = (V_0, V_1, V_2)$ be a γ_R-function. If $\rho_1, \rho_2, \ldots, \rho_n$ are the eigenvalues of minimum roman dominating distance matrix $A_{RDd}(G)$, then*

(i) $\sum_{i=1}^{n} \rho_i = \gamma_R(G)$

(ii) $\sum_{i=1}^{n} \rho_i^2 = \gamma_R(G) + 2m + 2M$ *where* $M = \sum_{i<j,\, d(v_i,v_j)\neq 1} d(v_i, v_j)^2$ *and* $m = |E|$.

PROOF. (i) The sum of the eigenvalues of $A_{RDd}(G)$ is the trace of $A_{RDd}(G)$. Therefore,

$$\sum_{i=1}^{n}\rho_i = \sum_{i=1}^{n} d(v_i, v_i) = 2|V_2| + |V_1| = \gamma_R(G).$$

(ii) The sum of squares of the eigenvalues of $A_{RDd}(G)$ is trace of $(A_{RDd}(G))^2$. Therefore,

$$\begin{aligned}\sum_{i=1}^{n}\rho_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} d(v_i, v_j)d(v_j, v_i)\\ &= \sum_{i=1}^{n} d(v_i, v_i)^2 + \sum_{i\neq j} d(v_i, v_j)d(v_j, v_i)\\ &= \sum_{i=1}^{n} d(v_i, v_i)^2 + \sum_{i<j} d(v_i, v_j)^2\\ &= \gamma_R(G) + 2\sum_{i<j} d(v_i, v_j)^2\\ &= \gamma_R(G) + 2m + 2M.\end{aligned}$$

Corollary 4.2. *Let G be a graph with diameter 2 and let $f = (V_0, V_1, V_2)$ be a γ_R-function. If $\rho_1, \rho_2, \ldots, \rho_n$ are eigenvalues of minimum roman dominating distance matrix $A_{RDd}(G)$, then*

$$\sum_{i=1}^{n}\rho_i^2 = \gamma_R(G) + 2(2n^2 - 2n - 3m).$$

PROOF. We know that in $A_{RDd}(G)$ there are $2m$ elements with 1 and $n(n-1)-2m$ elements with 2 and hence corollary follows from the above theorem.

5. BOUNDS FOR MINIMUM ROMAN DOMINATING ENERGY

In this section, we discussed the bounds for minimum roman dominating energy.

The proofs of the following Theorems are similar to the proofs in [9].

Theorem 5.1. *Let G be a graph. If $f = (V_0, V_1, V_2)$ is a γ_R-function and $P = |det(A_{RDd}(G)|$, then*

$$\sqrt{(2m + 2M + \gamma_R) + n(n-1)P^{\frac{n}{2}}} \leq E_{RDd}(G) \leq \sqrt{n(2m + 2M + \gamma_R(G))}$$

where γ_R is a roman domination number.

By Theorem 1.1,we have the following corollary.

Corollary 5.2. *Let G be a graph. If $f = (V_0, V_1, V_2)$ is a γ_R-function and $P = |det(A_{RDd}(G))|$ then*

$$\sqrt{(2m + 2M + \gamma) + n(n-1)P^{\frac{n}{2}}} \leq E_{RDd}(G) \leq \sqrt{n(2m + 2M + 2\gamma(G))},$$

where γ is a minimum domination number of G.

Remark 5.3. *In Theorem 3.5, for the healthy spider graph $K^*_{1,n-1}$, $m = 2n - 2, M = (n-1)(19n-6)$ and $\gamma_R(K^*_{1,n-1}) = n+1$. Hence $\sqrt{n(2m + 2M + \gamma_R(G))} = \sqrt{(2n-1)(38n^2 + 31n - 3)} > 6n^2 - 4n - 16$.*

Theorem 5.4. *If $\rho_1(G)$ is the largest eigenvalue of a minimum roman dominating distance matrix $A_{RDd}(G)$, then*

$$\rho_1(G) \geq \frac{2W(G) + \gamma_R(G)}{n},$$

where $W(G)$ is the Wiener index of G.

Proof. Let X be any nonzero vector. Then, we have

$$\begin{aligned}\rho_1(G) &= \max_{X \neq 0}\left\{\frac{X'A_{RDd}X}{X'X}\right\} \\ &\geq \frac{J'A_{RDd}J}{J'J} \quad \text{where } J = [1, 1 \cdots, 1] \\ &= \frac{2\sum\limits_{i<j} d(v_i, v_j) + \gamma_R(G)}{n} \\ &= \frac{2W(G) + \gamma_R(G)}{n}.\end{aligned}$$

Theorem 5.5. *Let G be a graph of diameter 2 and $\rho_1(G)$ is the largest eigenvalue of a minimum roman dominating distance matrix $A_{RDd}(G)$, then*

$$\rho_1(G) \geq \frac{2n^2 - 2m - 2n + \gamma_R(G)}{n}.$$

Proof. Let G be a connected graph of diameter 2 and let $d(v_i) = d_i$. Then i-th row of A_{RDd} consists of d_i one's and $n - d_i - 1$ two's except in the i^{th} column, also $tr(A_{RDd}) = \gamma_R(G)$. By using Raleigh's principle, for $J = [1, 1, \cdots, 1]$, we have

$$\rho_1(G) \geq \frac{J'A_{RDd}J}{J'J} = \frac{\sum\limits_{i=1}^{n}[d_i + 2(n - d_i - 1)] + tr(A_{RDd})}{n} = \frac{2n^2 - 2m - 2n + \gamma_R(G)}{n}.$$

CONCLUSION

In this paper, we introduced the concept of minimum roman dominating distance energy $E_{RDd}(G)$ of a graph G and computed minimum roman dominating distance energy of complete, complete bipartite, crown, star and healthy spider graphs. Also, we discussed the properties of eigenvalues of a minimum roman dominating distance matrix $A_{RDd}(G)$. Finally, we derived the upper and lower bounds for $E_{RDd}(G)$.

REFERENCES

[1] Bapat, R. B., *Graphs and Matrices*, Hindustan Book Agency, 2011.

[2] Bapat, R., Kirkland, S. J. and Neumann, M., On distance matrices and Laplacians, *Linear Algebra and its Applications*, **401** (2005), 193-209.

[3] Cockayne, E.J., Paul A Dreyer Jr., Sandra M Hedetniemi and Hedetniemi, S.T., Roman domination in graphs, *Discrete Mathematics*, **278** (2004), 11-22.

[4] Graham, R. L., Hoffman, A. J. and Hosoya, H., On the distance matrix of a directed graph, *Journal of Graph Theory*, **1(1)** (1977), 85-88.

[5] Graham, R. L. and Lovász, L., Distance matrix polynomials of trees, *Advances in Mathematics*, **29(1)** (1978), 60-88.

[6] Graham, R. L. and Pollak, H. O., On the addressing problem for loop switching, *The Bell System Technical Journal*, **50** (1971), 2495-2519.

[7] Gutman, I., The energy of a graph, *Ber. Math-Statist. Sekt. Forschungsz. Graz*, **103** (1978), 1-22.

[8] Rajesh Kanna, M.R., Dharmendra, B. N. and Pradeep Kumar, R., Minimum covering distance energy of a graph, *Applied Mathematical Sciences*, **7(111)** (2013), 5525-5536.

[9] Rajesh Kanna, M.R., Dharmendra, B. N. and Sridhara, G., Minimum dominating distance energy of a graph, *Journal of Indonesian Mathematical Society*, **20(1)** (2014), 19-29.

[10] Rajesh Kanna, M. R., Dharmendra, B. N. and Sridhara, G., Laplacian minimum dominating energy of a graph, *International Journal of Pure and Applied Mathematics*, **89(4)** (2013), 565-581.