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QUADRATIC INEQUALITY FOR OBTAINING FIXED POINT USING PROPERTY (E.A) IN MENGER SPACES

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Abstract. Employing the common property (E.A), we prove some fixed point theorems for occasionally weakly compatible via quadratic inequality for obtaining fixed point in Menger space. Our results extent generalize the results of Cho, Murthy and Stojakovic [2].

Key words : Menger space, compatible map, occasionally weakly compatible maps, property (E.A), common property (E.A), common fixed point.

1. INTRODUCTION

K. Menger[7] introduced the notion of probabilistic metric spaces (or statistical metric spaces), which is a generalization of metric spaces, and the study of these spaces was expanded rapidly with the pioneering works of Schweizer and Skler[9]. A contraction is one of the main tools to prove the existence and uniqueness results on fixed points in probabilistic analysis. A generalization of Banach contraction principle on a complete Menger space which is a milestone in developing fixed point theorems in Menger space was obtained by Sehgal and Bharucha[10] in 1972.

In 1982, Sessa[11] introduced weakly commuting mappings in metric space. Jungck[5] enlarged this concept to compatible mappings. The notion of compatible mappings in Menger space has been introduced by Mishra[8]. Cho, Murthy and Stojakovic [2] introduced the concept of compatible mappings of type (A) on Menger spaces, which is equivalent to the concept of compatible mappings under some conditions and prove some common fixed point theorems for compatible mappings of type (A) on Menger spaces and metric spaces. Infact, compatible maps and compatible mappings of type (A) are weak compatible but the reverse is always not true[2].

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Recently, Aamri and Moutawakil[1] defined the property (E. A) and the common property (E. A) and proved some common fixed point theorems in metric spaces. Kubiaczyk and Sharma[6] defined the property (E. A) in PM-spaces and used it to prove results on common fixed points. Doric et al.[3] have shown that the condition of occasionally weak compatibility reduces to weak compatibility in the presence of a unique point of coincidence of the given pair of mappings in Menger space.

In this paper, we prove some common fixed point theorems for two pairs of self-mappings by using the notion of occasionally weakly compatible/common property (E.A) using an inequality involving quadratic terms in Menger PM-space. Our results extend generalize and improve the results of Cho, Murthy and Stojakovic [2]. The quadratic inequality for obtaining fixed point using property (E.A) in Menger spaces is useful.

2. Preliminaries

Definition 2.1. [9] *A mapping $F : \mathbb{R} \to [0, 1]$ is called a distribution function if it satisfies the following conditions:*

(a) F is nondecreasing;

(b) F is left continuous, with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$.

We shall denote by $\mathbb{D}$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(x) = \begin{cases} 0 & \textit{if } x \leq 0 \\ 1 & \textit{if } x > 0. \end{cases}$$

Definition 2.2. [9] *A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a triangular norm (briefly t-norm) if for every $a, b, c \in [0, 1]$,*

(a) $\Delta(a, 1) = a$ for every $a \in [0, 1]$,

(b) $\Delta(0, 0) = 0$,

(c) $\Delta(a, b) = \Delta(b, a)$ for every $a, b \in [0, 1]$, and

(d) $c \geq a$ and $d \geq b$, then $\Delta(c, d) \geq \Delta(a, b)$ $(a, b, c, d \in [0, 1])$,

(e) $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

Definition 2.3. [9] *Let (X, F, Δ) be a PM-space. Then,*

(i): *a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, if for every $\epsilon > 0$ and $0 < \lambda < 1$ there exists a positive integer $\mathbb{Z}^+$ such that $F_{x_n,x(\epsilon)} > 1 - \lambda$ whenever $n \geq \mathbb{Z}^+$,*

(ii): *a sequence $\{x_n\}$ in X is called a cauchy sequence if for every $\epsilon > 0$ and $\lambda > 0$ we can find a positive integer $\mathbb{Z}^+$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ whenever $n, m \geq \mathbb{Z}^+$,*

(iii): *a Menger PM-space is said to be complete if every Cauchy sequence is convergent to a point in X.*

Definition 2.4. *Let A and S be two selfmaps on a nonempty set X, if $Ax = Sx = w(say)$, $w \in X$ for some $x \in X$, then x is called a coincidence point of A and S, and w is called a point of coincidence of A and S.*

In 1991, Mishra [8] introduced compatible Mappings in PM-Space setting.

Definition 2.5. [8] *Let (X, F, Δ) be a Menger space such that the t-norm Δ is continuous and the mappings A and S are selfmaps on X. Then, the pair (A, S) is said to be compatible if*

$$\lim_{n\to\infty} F_{ASx_n, SAx_n}(t) = 1,$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

In 1992, Cho, Murthy and Stojakovic [2] introduced the following.

Definition 2.6. [2] *Let (X, F, Δ) be a Menger space such that T-norm t is continuous and A, S be mapping X into itself compatible maps of type(A) if*

$$\lim_{n\to\infty} F_{SAx_n, AAx_n}(t) = 1, \text{ and } \lim_{n\to\infty} F_{ASx_n, SSx_n}(t) = 1,$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \quad \text{for some } z \in X.$$

Remark (a) If the selfmappings A and S are both continuous, then definition 2.5 iff definition 2.6 [2].

(b) If the selfmappings A and S are not continuous, then the definition 2.5 and definition 2.6 are independent [2].

Definition 2.7. [4] *Let (X, F, Δ) be a Menger space such that the t-norm Δ is continuous and A, S be mapping X into itself. Then, S and T are said to be weakly compatible if they commute at their coincidence points.*

Definition 2.8. [4] *Let (X, F, Δ) be a Menger space such that the t-norm Δ is continuous and A, S be mapping X into itself. Then, S and T are said to be occasionally weakly compatible(owc) if and if there is a point x in X which is coincidence point A and S at which A and S commute.*

In 2008, Kubiaczyk and Sharma [6] defined the notion of property (E.A) in Menger spaces as follow:

Definition 2.9. [6] *Let (X, F, Δ) be a Menger space such that the t-norm Δ is continuous and A, S be mapping X into itself. Then, A and S are said to be satisfy the property (E.A), if there exists a sequence $\{x_n\}$ such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \quad \textit{for some } z \in X.$$

Definition 2.10. [6] *Let (X, F, Δ) be a Menger space such that the t-norm Δ is continuous and A, B, S and T be four selfmappings on X. Then the pairs (A, S) and (B, T) are said to be satisfy the common property (E.A), if there exists two sequence $\{x_n\}$,$\{y_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z, \quad \textit{for some } z \in X.$$

Lemma 2.11. [8] *Let (X, F, Δ) be a Menger space such that the t-norm Δ is continuous and A, B, S and T be self mapping of a Menger space. If there exists a constant $k \in (0, 1)$ such that*

$$F_{x,y}(kt) \geq F_{x,y}(t),$$

for all $x, y \in X$ and $t > 0$. Then $x = y$.

In 1992, Cho, Murthy and Stojakovic proved the following theorem.

Theorem 2.12. [2] *Let (X, F, t) be a complete Menger space with $t(x, y) = min\{x, y\}$ for all $x, y \in [0, 1]$ and A, B, S, T be mappings from X into itself such that (i) $A(X) \subset T(X)$ and $B(X) \subset S(X)$, (ii) the pairs (A, S) and (B, T) are compatible of type (A), (iii) one of A, B, S and T is continuous, (iv) there exists a constant $k \in (0, 1)$ such that*

$$(F_{Ax,By}(kt))^2 \quad \geq \quad min \begin{Bmatrix} (F_{Sx,Ty}(t))^2, F_{Sx,Ax}(t)F_{Ty,By}(t), F_{Sx,By}(2t)F_{Ty,Ax}(t), \\ F_{Sx,By}(t)F_{Ty,Ax}(t), F_{Sx,By}(2t)F_{Ty,By}(t) \end{Bmatrix}.$$

for all $x, y \in X$ and $t \geq 0$. Then A, B, S and T have a unique common fixed point in X.

In the rest of this paper, X will stand for the Menger PM space. $C(A, S)$ and $C(B, T)$ will denotes the set of common coincidence points of the respective pairs (A, S) and (B, T).

3. MAIN RESULTS

We start with noting down following lemmas which will be crucial to our main results.

Proposition 3.1. *Let A, B, S and T be self mappings of a Menger space (X, F, Δ), where Δ is a continuous t-norm, and satisfy the following inequality*

$$(F_{Ax,By}(kt))^2 \quad \geq \quad min \begin{Bmatrix} (F_{Sx,Ty}(t))^2, \ F_{Sx,Ax}(t)F_{Ty,Ax}(t), \ F_{Sx,Ax}(t)F_{Ty,By}(t), \\ F_{Sx,By}(t)F_{Ty,Ax}(t), F_{Sx,By}(t)F_{Ty,By}(t) \end{Bmatrix}, \tag{1}$$

for all $x, y \in X$ and $t > 0$ and suppose that $B(X) \subset S(X)$, the pair (B, T) satisfies property (E.A) and $T(X)$ is a closed subspace of X .
Then, the followings
(i) the pairs (A, S) and (B, T) shares common property of (E.A.),
(ii) $C(A, S) \neq \emptyset$ and $C(B, T) \neq \emptyset$ holds.

Proof. Suppose the pair (B, T) satisfy the property $(E.A)$, then there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z, \quad \text{for some } z \in X. \tag{2}$$

Since $B(X) \subset S(X)$, there exists a sequence $\{x_n\}$ in X such that $By_n = Sx_n$. Therefore, from (2) we obtain that

$$\lim_{n\to\infty} Sx_n = z. \tag{3}$$

On taking $x = x_n$ and $y = y_n$ in (1), we get

$$(F_{Ax_n,By_n}(kt))^2 \quad \geq \quad min\begin{Bmatrix} (F_{Sx_n,Ty_n}(t))^2, F_{Sx_n,Ax_n}(t)F_{Ty_n,Ax_n}(t), F_{Sx_n,Ax_n}(t)F_{Ty_n,By_n}(t), \\ F_{Sx_n,By_n}(t)F_{Ty_n,Ax_n}(t), F_{Sx_n,By_n}(t)F_{Ty_n,By_n}(t) \end{Bmatrix}.$$

Taking limits as $n \to \infty$ and using (2) and (3), we get By Lemma 2.11, we obtain that

$$\begin{aligned}
(F_{\lim_{n\to\infty} Ax_n,z}(kt))^2 \qquad &\geq min\begin{Bmatrix} (F_{z,z}(t))^2,\ F_{z,\lim\limits_{n\to\infty} Ax_n}(t)F_{z,\lim\limits_{n\to\infty} Ax_n}(t), \\ F_{z,\lim\limits_{n\to\infty} Ax_n}(t)F_{z,z}(t), F_{z,z}(t)F_{z,\lim\limits_{n\to\infty} Ax_n}(t), \\ F_{z,z}(t)F_{z,z}(t) \end{Bmatrix}, \\
&= min\begin{Bmatrix} 1,\ F_{z,\lim\limits_{n\to\infty} Ax_n}(t)F_{z,\lim\limits_{n\to\infty} Ax_n}(t), \\ F_{\lim\limits_{n\to\infty} Ax_n,z}(t)\cdot 1, 1\cdot F_{\lim\limits_{n\to\infty} Ax_n,z}(t),\ 1 \end{Bmatrix} \\
&\geq min\left\{F_{z,\lim\limits_{n\to\infty} Ax_n}(t)F_{z,\lim\limits_{n\to\infty} Ax_n}(t),\ F_{\lim\limits_{n\to\infty} Ax_n,z}(t)\right\}, \\
&= (F_{z,\lim\limits_{n\to\infty} Ax_n}(t))^2.
\end{aligned}$$

$$\lim_{n\to\infty} Ax_n = z. \tag{4}$$

From (2)-(4), we can conclude that condition (i) holds.
Since $T(X)$ is a closed subspace of X, we have $Tv = z$ for some $v \in X$. Now, we have to show that $Bv = z$. On taking $x = x_n$ and $y = v$ in inequality (1), we get

$$(F_{Ax_n,Bv}(kt))^2 \quad \geq \quad min\begin{Bmatrix} (F_{Sx_n,Tv}(t))^2, F_{Sx_n,Ax_n}(t)F_{Tv,Ax_n}(t), F_{Sx_n,Ax_n}(t)F_{Tv,Bv}(t), \\ F_{Sx_n,Bv}(t)F_{Tv,Ax_n}(t), F_{Sx_n,Bv}(t)F_{Tv,Bv}(t) \end{Bmatrix}.$$

On taking limits as $n \to \infty$, using (3) and (4), we get

$$\begin{aligned}
(F_{z,Bv}(kt))^2 \quad &\geq \quad min\begin{Bmatrix} (F_{z,z}(t))^2, F_{z,z}(t)F_{z,z}(t), F_{z,z}(t)F_{z,Bv}(t), \\ F_{z,Bv}(t)F_{z,z}(t), F_{z,Bv}(t)F_{z,Bv}(t) \end{Bmatrix}, \\
&= \quad F_{z,Bv}(t)F_{z,Bv}(t).
\end{aligned}$$

From Lemma 2.11, we get $Bv = z$. Therefore,

$$Bv = Tv = z. \tag{5}$$

Then $v \in C(B,T)$,i.e., $C(B,T) \neq \emptyset$. Since $B(X) \subset S(X)$, there exists a point $u \in X$ such that $Su = z$. Now, finally we have to show that $Au = z$.
On taking $x = u$ and $y = v$ in inequality (1), we get

$$(F_{Au,Bv}(kt))^2 \geq min\left\{\begin{array}{c}((F_{Su,Tv}(t))^2, F_{Su,Au}(t)F_{Tv,Au}(t), F_{Su,Au}(t)F_{Tv,Bv}(t), \\ F_{Su,Bv}(t)F_{Tv,Au}(t), F_{Su,Bv}(t)F_{Tv,Bv}(t)\end{array}\right\},$$

$from$ (5), $we\ get$

$$(F_{Au,z}(kt))^2 \geq min\left\{\begin{array}{c}((F_{z,z}(t))^2, F_{z,Au}(t)F_{z,Au}(t), F_{z,Au}(t)F_{z,z}(t), \\ F_{z,z}(t)F_{z,Au}(t), F_{z,z}(t)F_{z,z}(t)\end{array}\right\},$$

From Lemma 2.11, we get $Au = z$, therefore,

$$Su = Au = z, \tag{6}$$

and hence $u \in C(A,S)$ (i.e., $C(A,S) \neq \emptyset$), condition (ii) holds.
From (5) and (6), we get

$$Su = Au = Bv = Tv = z. \tag{7}$$

Thus, z is a common point of coincidence of the pairs (A,S) and (B,T). □

Proposition 3.2. *Let A, B, S and T be self mapping of a Menger space (X, F, t) and where t is a continuous t-norm, and satisfy the inequality* (1)*, and suppose that $A(X) \subset T(X)$, the pair (A,S) satisfies property (E.A.) and $S(X)$ is a closed subspace of X. Then the condition* (*i*) *and* (*ii*) *of Proposition 3.1 holds.*

Proof. Suppose the pair (A,S) satisfy the property $(E.A)$, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \quad \text{for some } z \in X. \tag{8}$$

Since $A(X) \subset T(X)$, there exists a sequence $\{y_n\}$ in X such that $Ax_n = Ty_n$. Therefore, from (2) we obtain that

$$\lim_{n\to\infty} Ty_n = z. \tag{9}$$

On taking $x = x_n$ and $y = y_n$ in (1), we get

$$(F_{Ax_n,By_n}(kt))^2 \geq min\left\{\begin{array}{c}(F_{Sx_n,Ty_n}(t))^2, F_{Sx_n,Ax_n}(t)F_{Ty_n,Ax_n}(t), F_{Sx_n,Ax_n}(t)F_{Ty_n,By_n}(t), \\ F_{Sx_n,By_n}(t)F_{Ty_n,Ax_n}(t), F_{Sx_n,By_n}(t)F_{Ty_n,By_n}(t)\end{array}\right\}.$$

Taking limits as $n \to \infty$ on both sides and using (8) and (9), we get

$$
\begin{aligned}
(F_{z,\lim_{n\to\infty} By_n}(kt))^2 &\geq min\left\{\begin{matrix}(F_{z,z}(t))^2,\ F_{z,z}(t)F_{z,z}(t), F_{z,z}(t)F_{z,\lim_{n\to\infty} By_n}(t), \\ F_{z,\lim_{n\to\infty} By_n}(t)F_{z,z}(t), F_{z,\lim_{n\to\infty} By_n}(t)F_{z,\lim_{n\to\infty} By_n}(t)\end{matrix}\right\}, \\
&= min\left\{\begin{matrix}1,\ 1\cdot 1,\ 1\cdot F_{z,\lim_{n\to\infty} By_n}(t), F_{z,\lim_{n\to\infty} By_n}(t)\cdot 1, \\ F_{z,\lim_{n\to\infty} By_n}(t)F_{z,\lim_{n\to\infty} By_n}(t)\end{matrix}\right\}, \\
&\geq min\left\{F_{z,\lim_{n\to\infty} By_n}(t), F_{z,\lim_{n\to\infty} By_n}(t)\ F_{\lim_{n\to\infty} By_n, z}(t)\right\}, \\
&= (F_{z,\lim_{n\to\infty} By_n}(t))^2.
\end{aligned}
$$

By Lemma 2.11, we obtain that

$$
\lim_{n\to\infty} By_n = z. \tag{10}
$$

From (8)-(10), we can conclude that condition (i) holds.
Since $T(X)$ is a closed subspace of X, we have $Tv = z$ for some $v \in X$. Now, we have to show that $Bv = z$.

$$
(F_{Ax_n,Bv}(kt))^2 \geq min\left\{\begin{matrix}(F_{Sx_n,Tv}(t))^2, F_{Sx_n,Ax_n}(t)F_{Tv,Ax_n}(t), F_{Sx_n,Ax_n}(t)F_{Tv,Bv}(t), \\ F_{Sx_n,Bv}(t)F_{Tv,Ax_n}(t), F_{Sx_n,Bv}(t)F_{Tv,Bv}(t)\end{matrix}\right\}.
$$

On taking limits as $n \to \infty$, using (3) and (4), we get

$$
\begin{aligned}
(F_{z,Bv}(kt))^2 &\geq min\left\{\begin{matrix}(F_{z,z}(t))^2, F_{z,z}(t)F_{z,z}(t), F_{z,z}(t)F_{z,Bv}(t), \\ F_{z,Bv}(t)F_{z,z}(t), F_{z,Bv}(t)F_{z,Bv}(t)\end{matrix}\right\}, \\
&= (F_{z,Bv}(t)F_{z,Bv}(t).
\end{aligned}
$$

From Lemma 2.11, we get $Bv = z$. Therefore,

$$
Bv = Tv = z. \tag{11}
$$

Then $v \in C(B,T)$,i.e., $C(B,T) \neq \emptyset$. Since $B(X) \subset S(X)$, there exists a point $u \in X$ such that $Su = z$. Now, finally we have to show that $Au = z$. On taking $x = u$ and $y = v$ in inequality (1), we get

$$
(F_{Au,Bv}(kt))^2 \geq min\left\{\begin{matrix}((F_{Su,Tv}(t))^2, F_{Su,Au}(t)F_{Tv,Au}(t), F_{Su,Au}(t)F_{Tv,Bv}(t), \\ F_{Su,Bv}(t)F_{Tv,Au}(t), F_{Su,Bv}(t)F_{Tv,Bv}(t)\end{matrix}\right\},
$$

$$
(F_{Au,z}(kt))^2 \geq min\left\{\begin{matrix}((F_{z,z}(t))^2, F_{z,Au}(t)F_{z,Au}(t), F_{z,Au}(t)F_{z,z}(t), \\ F_{z,z}(t)F_{z,Au}(t), F_{z,z}(t)F_{z,z}(t)\end{matrix}\right\},
$$

From Lemma 2.11, we get $Au = z$, therefore,

$$
Su = Au = z, \tag{12}
$$

and hence $u \in C(A,S)$ (i.e., $C(A,S) \neq \Phi$), condition (ii) holds.
From (5) and (6), we get

$$
Su = Au = Bv = Tv = z. \tag{13}
$$

Thus, z is a common point of coincidence of the pairs (A,S) and (B,T). □

Theorem 3.3. *In addition to the hypothesis of Proposition 3.1* ($Proposition$ 3.2)*, if both the pairs* (A, S) *and* (B, T) *are satisfies occasionally weakly compatible condition. Then* A, B, S *and* T *have a unique common fixed point in* X*.*

Proof. By Proposition 3.1 ($Proposition$ 3.2), we have

$$Av = Sv = Bu = Tu = z(say), \tag{14}$$

where $v \in C(A, S)$ and $u \in C(B, T)$. Since the pair (A, S) is owc, we get

$$Av = Sv(= z) \Rightarrow Az = ASv = SAv = Sz. \tag{15}$$

Again the pair (B, T) is owc, we have

$$Bu = Tu(= z) \Rightarrow Bz = BTu = TBu = Tz. \tag{16}$$

On taking $x = z$ and $y = u$ in inequality (1), we get

$$(F_{Az,Bu}(kt))^2 \geq min \begin{Bmatrix} (F_{Sz,Tu}(t))^2, F_{Sz,Az}(t)F_{Tu,Az}(t), F_{Sz,Az}(t)F_{Tu,Bu}(t), \\ F_{Sz,Bu}(t)F_{Tu,Az}(t), F_{Sz,Bu}(t)F_{Tu,Bu}(t) \end{Bmatrix},$$

By using (8)-(10), we obtain that

$$\begin{aligned} (F_{Az,z}(kt))^2 &\geq min \begin{Bmatrix} (F_{Az,z}(t))^2, F_{Az,Az}(t)F_{z,Az}(t), F_{Az,Az}(t)F_{z,z}(t), \\ F_{Az,z}(t)F_{z,Az}(t), F_{Az,z}(t)F_{z,z}(t) \end{Bmatrix}, \\ &= F_{Az,z}(t)F_{z,Az}(t). \end{aligned}$$

From Lemma 2.11 ,we get

$$Az = z, \tag{17}$$

from (15), we have

$$Az = Sz = z. \tag{18}$$

Now, we have to show that z is a common fixed point of B and T.
For this purpose, substitute $x = v$ and $y = z$ in our inequality (1), we get

$$(F_{Av,Bz}(kt))^2 \geq min \begin{Bmatrix} (F_{Sv,Tz}(t))^2, F_{Sv,Av}(t)F_{Tz,Av}(t), F_{Sv,Av}(t)F_{Tz,Bz}(t), \\ F_{Sv,Bz}(t)F_{Tz,Av}(t), F_{Sv,Bz}(t)F_{Tz,Bz}(t) \end{Bmatrix}.$$

By using (9) and (10), we get

$$\begin{aligned} (F_{z,Bz}(kt))^2 &\geq min \begin{Bmatrix} (F_{z,Bz}(t))^2, F_{z,z}(t)F_{Bz,z}(t), F_{z,z}(t)F_{Bz,Bz}(t), \\ F_{z,Bz}(t)F_{Bz,z}(t), F_{z,Bz}(t)F_{Bz,Bz}(t) \end{Bmatrix}, \\ &= F_{z,Bz}(t)F_{z,Bz}(t). \end{aligned}$$

From Lemma 2.11, we get

$$Bz = z, \tag{19}$$

from (10) and (13), we have

$$Bz = Tz = z. \tag{20}$$

We conclude from (12) and (14), we get z is common fixed point of A, B, S and T in X.

Uniqueness: Assume that z' is another fixed point of A, B, S and T, which is distinct from z.
On taking $x = z$ and $y = z'$ in (1), we get

$$\begin{aligned}(F_{Az,Bz'}(kt))^2 &\geq min\left\{\begin{array}{c}(F_{Sz,Tz'}(t))^2, F_{Sz,Az}(t)F_{Tz',Az}(t), F_{Sz,Az}(t)F_{Tz',Bz'}(t), \\ F_{Sz,Bz'}(t)F_{Tz',Az}(t), F_{Sz,Bz'}(t)F_{Tz',Bz'}(t)\end{array}\right\}, \\ &= (F_{z,z'}(t))^2.\end{aligned}$$

From Lemma 2.11, we get $z = z'$. □

In the following, we prove the existence of common fixed points of A, B, S and T by imposing the condition common property(E.A)and relaxing the two containments $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

Proposition 3.4. *Let A, B, S and T be self mapping of a Menger space (X, F, Δ),satisfying the inequality (1) and suppose that (i) the pairs (A, S) and (B, T) satisfying a common property (E.A),(ii) $S(X)$ and $T(X)$ are closed subspaces of X. Then $C(B, T) \neq \emptyset$ and $C(A, S) \neq \emptyset$.*

Proof. As the pairs (A, S) and (B, T) satisfying a common property $(E.A)$, then there exists two sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z, \tag{21}$$

for some $z \in X$. Since $S(X)$ and $T(X)$ are closed subspaces of X. Then,

$$z = Su = Tv, \tag{22}$$

for some $u, v \in X$.
Now, we show that $Bv = z$, taking $x = x_n$ and $y = v$ in (1), we have

$$(F_{Ax_n,Bv}(kt))^2 \geq min\left\{\begin{array}{c}(F_{Sx_n,Tv}(t))^2, F_{Sx_n,Ax_n}(t)F_{Tv,Ax_n}(t), F_{Sx_n,Ax_n}(t)F_{Tv,Bv}(t), \\ F_{Sx_n,Bv}(t)F_{Tv,Ax_n}(t), F_{Sx_n,Bv}(t)F_{Tv,Bv}(t)\end{array}\right\}.$$

On taking limits as $n \to \infty$, using (15) and (16), we obtain that, we get

$$\begin{aligned}(F_{z,Bv}(kt))^2 &\geq min\left\{\begin{array}{c}(F_{z,z}(t))^2, F_{z,z}(t)F_{z,z}(t), F_{z,z}(t)F_{z,Bv}(t), \\ F_{z,Bv}(t)F_{z,z}(t), F_{z,Bv}(t)F_{z,Bv}(t)\end{array}\right\}, \\ &= F_{z,Bv}(t)F_{z,Bv}(t).\end{aligned}$$

From Lemma 2.11 , we get

$$Bv = z, \tag{23}$$

from (16) and (17),we have

$$Bv = Tv = z. \tag{24}$$

Hence $C(B,T) \neq \emptyset$.
Next,we claim that $Au = z$. On taking $x = u$ and $y = v$ in inequality (1),we get

$$\begin{aligned}(F_{Au,Bv}(kt))^2 &\geq min\begin{Bmatrix}(F_{Su,Tv}(t))^2, F_{Su,Au}(t)F_{Tv,Au}(t), F_{Su,Au}(t)F_{Tv,Bv}(t), \\ F_{Su,Bv}(t)F_{Tv,Au}(t), F_{Su,Bv}(t)F_{Tv,Bv}(t)\end{Bmatrix}, \\ &\geq min\begin{Bmatrix}(F_{z,z}(t))^2, F_{z,Au}(t)(F_{z,Au}(t), F_{z,Au}(t)F_{z,z}(t), \\ F_{z,z}(t)F_{z,Au}(t), F_{z,z}(t)F_{z,z}(t)\end{Bmatrix}, \\ &= (F_{Au,z}(t))^2.\end{aligned}$$

From Lemma 2.11, we get

$$Au = z. \tag{25}$$

from (16) and (19), we have

$$Su = Au = z. \tag{26}$$

Therefore $C(A,S) \neq \emptyset$. □

Proposition 3.5. *Let A, B, S and T be self mapping of a Menger space (X, F, Δ), satisfying the inequality* (1) *and suppose that* (i) *the pairs* (A,S) *and* (B,T) *satisfying a common property (E.A),*(ii) $A(X)$ *and* $B(X)$ *are closed subspaces of X. Then $C(B,T) \neq \emptyset$ and $C(A,S) \neq \emptyset$.*

Theorem 3.6. *In addition to the hypotheses of Proposition* (*Proposition* 3.5)*, if both the pairs* (A,S) *and* (B,T) *are occasionally weakly compatible, then the self maps A, B, S and T have a unique common fixed point in X.*

Proof. By Proposition 3.4 (*Proposition* 3.5), we get $C(A,S) \neq \emptyset$ and $C(B,T) \neq \emptyset$. The rest of the proof follows on the same lines as of Theorem 3.3. □

Corollary 3.7. *Let (X, F, t) be a complete Menger space with $t(x,y) = min\{x,y\}$ for all $x, y \in [0,1]$ and A, B, S and T be selfmaps on X and satisfying the condition* (1)*,* (i) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$*,* (ii) *one of A, B, S and T is continuous,* (iii) *the pairs* (A,S) *and* (B,T) *are satisfying the condition either compatible of type* (A) *or compatible maps or weakly compatible maps. Then A, B, S and T have a unique common fixed point in X.*

Corollary 3.8. *Let A and S be self maps of Menger spaces (X, F, Δ) such that*

$$(F_{Ax,By}(kt))^2 \geq min\begin{Bmatrix}(F_{Sx,Ty}(t))^2, F_{Sx,Ax}(t)F_{Ty,Ax}(t), F_{Sx,Ax}(t)F_{Ty,By}(t), \\ F_{Sx,By}(t)F_{Ty,Ax}(t), F_{Sx,By}(t)F_{Ty,By}(t)\end{Bmatrix},$$

for all $x, y \in X$ and $t > 0$. (i) *the pair* (A,S) *satisfies property* ($E.A.$) *and* (ii) $S(X)$ *is a closed subspace of X.Then $C(A,S) \neq \emptyset$. Further, if the pair (A,S) is owc on X, then the maps A and S have a unique common fixed point in X.*

In the following, we will take an example to support our result of Theorem 3.3.

Example 3.9. *Let $X = [0, 20)$ be the set with the metric d defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$. Define*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|} & if\ t > 0 \\ 0 & if\ t = 0 \end{cases}$$

for all $x, y \in X$. Clearly, (X, F, t) be a complete Menger space. Let A, B, S and T be self maps on X and define by

$$Ax = \begin{cases} 0, & x = 0; \\ 4, & x \neq 0. \end{cases} \qquad Bx = \begin{cases} 0, & x = 0; \\ 6, & x \neq 0. \end{cases}$$

$$Sx = \begin{cases} 0, & x = 0; \\ 8, & x \in (0, 10); \\ x - 6, & x \in [10, 20). \end{cases} \qquad Tx = \begin{cases} 0, & x = 0; \\ 4, & x \in (0, 10); \\ x - 4, & x \in [10, 20). \end{cases}$$

Clearly $B(X) = \{0, 6\} \subset \{0\} \cup [4, 14) = S(X)$. $T(X)$ is a closed subspace of X. The self maps A, B, S and T satisfies the inequality (1) with $k \in (0, 1)$. Now consider the sequence $\{x_n\}$ defined as $\{x_n\} = 4 + \frac{1}{n}$, $n \geq 1$. Now Ax_n, $Sx_n \to 4$ as $n \to \infty$. So that the pair (B, T) satisfies property (E.A). The pairs (A, S) and (B, T) are owc at the point $x = 0$. Hence,the selfmaps A, B, S and T satisfy all the conditions of Theorem 3.3 and 0 is the unique common fixed point of A, B, S and T.

4. CONCLUSION

In this paper, we use verious condition apply to prove the existence of the proposition and theorem convert in fixed point in Menger space. Example is constructed to support our results. We using compatible, occasionally weakly compatible maps and common property (E.A) in quadratic inequality in PM-space and the results extend in the resent literature.

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