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THE ECCENTRIC DIGRAPH OF THE CORONA OF C_n WITH K_m, C_m OR P_m

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Abstract. Let G be a graph with a set of vertices $V(G)$ and a set of edges $E(G)$. The distance from vertex u to vertex v in G, denoted by $d(u, v)$, is the length of the shortest path from vertex u to v. The eccentricity of vertex u in graph G is the maximum distance from vertex u to any other vertices in G, denoted by $e(u)$. Vertex v is an eccentric vertex from u if $d(u, v) = e(u)$. The eccentric digraph $ED(G)$ of a graph G is a graph that has the same set of vertices as G, and there is an arc (directed edge) joining vertex u to v if v is an eccentric vertex from u. In this paper, we answer the open problem proposed by Boland and Miller [1] to find the eccentric digraph of various classes of graphs. In particular, we determine the eccentric digraph of the corona of C_n with K_m, C_m and P_m, with C_n, K_m or P_m are cycle, complete graph and path, respectively.

Key words: Eccentricity, eccentric digraph, corona graph.

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Abstrak. Misal G adalah suatu graf dengan himpunan titik $V(G)$ dan himpunan sisi $E(G)$. Jarak dari titik u ke titik v di G, dinotasikan $d(u,v)$, adalah panjang dari lintasan terpanjang dari titik u ke v. Eksentrisitas titik u dalam graf G adalah jarak maksimum dari titik u ke sebarang titik yang lain di G, dinotasikan $e(u)$. Titik v adalah suatu titik eksentrik dari u jika $d(u,v) = e(u)$. Digraf eksentrik $ED(G)$ dari suatu graf G adalah suatu graf yang mempunyai himpunan titik yang sama dengan himpunan titik G, dan terdapat suatu busur (garis berarah) yang menghubungkan titik u ke v jika v adalah suatu titik eksentrik dari u. Boland dan Miller [1] memperkenalkan digraf eksentrik dari suatu digraf. Mereka juga mengusulkan suatu masalah untuk menemukan digraf ekesentrik dari bermacam kelas dari graf. Dalam makalah ini diselidiki eksentrik digraf dari graf korona C_n dengan K_m, C_m atau P_m, dengan C_n, K_m dan P_m masing-masing adalah graf siklik, graf lengkap dan lintasan.

Kata kunci: Eksentrisitas, digraf eksentrik, graf korona.

1. **Introduction**

Most of the notations and terminologies follow that of Gallian [5], Chartrand and Oellermann [3]. Let G be a graph with a set of vertices $V(G)$ and a set of edges $E(G)$. The *distance* from vertex u to vertex v in G, denoted by $d(u,v)$, is the length of the shortest *path* from vertex u to v. If there is no a *path* joining vertex u and vertex v, then $d(u,v) = \infty$. The *eccentricity* of vertex u in graph G is the maximum *distance* from vertex u to any other vertices in G, denoted by $e(u)$, and so $e(u) = max\{d(u,v) \mid v \in V(G)\}$. *Radius* of a graph G, denoted by $rad(G)$, is the minimum *eccentricity* of every vertex in G. The *diameter* of a graph G, denoted by $diam(G)$, is the maximum *eccentricity* of every vertex in G. If $e(u) = rad(G)$, then vertex u is called central vertex. *Center* of a graph G, denoted by $cen(G)$, is an induced subgraph formed from central vertices of G. Vertex v is an *eccentric* vertex from u if $d(u,v) = e(u)$. The *eccentric digraph* $ED(G)$ of a graph G is a graph having the same set of vertices as G, $V(ED(G)) = V(G)$, and there is an *arc* (directed edge) joining vertex u to v if v is an *eccentric* vertex from u. An *arc* of a digraph D joining vertex u to v and vertex v to u is called a *symmetric arc*. Further, Fred Buckley [2] concluded that almost in every graph G, its *eccentric* digraph is $ED(G) = \overline{G}^*$, where $\overline{G}^*$ is a complement of G which is every edge replaced by a *symmetric arc*. One of the topics in graph theory is to determine the *eccentric* digraph of a given graph. The *eccentric digraph* of a graph was initially introduced by Fred Buckley (Boland and Miller [1]). The class of graph considered here is *corona* of two graphs G_1 and G_2. According to Harary [6], the *corona* $G_1 \odot G_2$ of two graphs G_1 and G_2 was defined as the graph G obtained by taking one copy of G_1(which has n vertices) and n copies of G_2 and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2. Some authors have investigated the problem of finding the *eccentric digraph*. For example, Boland and Miller [1] determined the *eccentric digraph* of a digraph, while Gimbert, et.al [4] found the characterisation of the *eccentric digraphs*. Boland and Miller [1] also proposed an open problem to find the *eccentric digraph* of various classes of graphs. Some results related to this open problem can be found in [7, 8]. In this paper, we also answer the open problem proposed by Boland and Miller [1]. In particular, we determine the *eccentric digraph* of the *corona* of a cycle of n vertices C_n with the classes of graphs K_m, C_m or P_m, where C_m, K_m and P_m are cycle, complete graph and path with m vertices, respectively.

2. Main Results

Given two graphs C_n and M, where M is either a cycle C_m, a complete graph K_m or a path P_m. The set of vertices of them are $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and $V(M) = \{u_1, u_2, \cdots, u_m\}$, respectively. The edges set of C_n is $E(C_n) = \{v_1v_2, v_2v_3, \cdots, v_{n-1}v_n, v_nv_1\}$. The set of edges of M are $E(M) = \{u_1u_2, u_2u_3, \cdots, u_{m-1}u_m, u_mu_1\}$ for $M = C_m$, $E(M) = \{u_iu_j : i, j = 1, 2, \cdots, m\}$ for $M = K_m$ and $E(M) = \{u_1u_2, u_2u_3, \cdots, u_{m-1}u_m\}$ for $M = P_m$. The corona of C_n with M, denoted by $C_n \odot M$ is a graph with $V(C_n \odot M) = V(C_n \cup \bigcup_{v_i \in V(C_n)} V(M_i)$ and $E(C_n \odot M) = E(C_n) \cup \bigcup_{v_i \in C_n} E(M_i) \cup \{(v_i, u_i : v_i \in V(C_n), u_i \in V(M_i)\}$ with $M_i = M$.

The materials of this research are mostly from the papers related to the eccentric digraph. The following result is the eccentric digraphs of the corona $C_n \odot M$ graph. There are two cases the eccentric digraphs of the corona $C_n \odot M$ graph to consider based on the values of n.

Theorem 2.1. *Let $C_n \odot M$ be the corona graph of C_n with M, where M is either one of C_m, K_m or P_m. If n is even then the eccentric digraph $ED(C_n \odot M)$ is the graph G as depicted in Figure 1.*

Proof. It is easy to check that the eccentricity of vertex v_i is $\frac{n}{2} + 1$ and the eccentricity of vertex u_{ij} is $\frac{n}{2} + 2$, $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. The eccentricities of all vertices are used to determine the eccentric vertex of all vertices of the corona graph $C_n \odot M$. The arcs can be obtained by joining every vertex to its eccentric vertex of the corona graph $C_n \odot M$. Table 1 shows the eccentric vertices and arcs of the corona graph $C_n \odot M$.

From Table 1, the arc of the corona graph $C_n \odot M$ vertex is adjacent to its eccentric vertices. The arcs $u_{ij}u_{(\frac{n}{2}+i(mod\ n))j}$ are symmetric and arcs $v_iu_{(\frac{n}{2}+i(mod\ n))j}$ are not symmetric for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$. Therefore the eccentric digraph of the corona graph $C_n \odot M$ with n even can be formed into $\frac{n}{2}K_{m,m} \cup n(\overline{K}_1 \vee \overline{K}_m)$ with $V(\overline{K}_1) = \{v_i\}$, $\overline{K}_m = V(M_{\frac{n}{2}+i(mod\ n)})$ for $i = 1, 2, \cdots, n$.

Theorem 2.2. *Let $C_n \odot M$ be the corona graph of C_n with M, where M is either one of C_m, K_m or P_m. If n is odd then the eccentric digraph $ED(C_n \odot M)$ is the graph G as depicted in Figure 2.*

TABLE 1. The eccentric vertices and arc of $C_n \odot M$ vertex, where n even

vertex of $C_n \odot M$	eccentric vertices	arc
v_1	$u_{(\frac{n}{2}+1)j}, j = 1, 2, \cdots, m$	$v_1 u_{(\frac{n}{2}+1)j}$
$\vdots$	$\vdots$	$\vdots$
$v_{\frac{n}{2}}$	$u_{nj}, j = 1, 2, \cdots, m$	$v_{\frac{n}{2}} u_{nj}$
$v_{\frac{n}{2}+1}$	$u_{1j}, j = 1, 2, \cdots, m$	$v_{\frac{n}{2}+1} u_{1j}$
$\vdots$	$\vdots$	$\vdots$
v_n	$u_{\frac{n}{2}j}, j = 1, 2, \cdots, m$	$v_n u_{\frac{n}{2}j}$
$u_{1j}, j = 1, 2, \cdots, m$	$u_{(\frac{n}{2}+1)j}$	$u_{1j} u_{(\frac{n}{2}+1)j}$
$\vdots$	$\vdots$	$\vdots$
$u_{\frac{n}{2}j}, j = 1, 2, \cdots, m$	u_{nj}	$u_{\frac{n}{2}j} u_{nj}$
$u_{(\frac{n}{2}+1)j}, j = 1, 2, \cdots, m$	u_{1j}	$u_{(\frac{n}{2}+1)j} u_{1j}$
$\vdots$	$\vdots$	$\vdots$
$u_{nj}, j = 1, 2, \cdots, m$	$u_{\frac{n}{2}j}$	$u_{nj} u_{\frac{n}{2}j}$

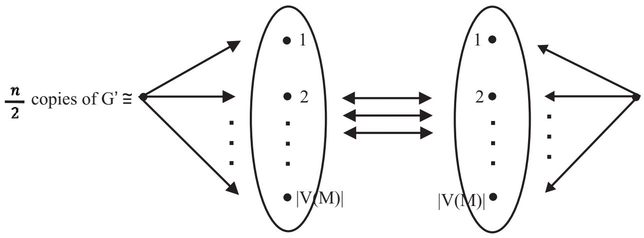

FIGURE 1. The eccentric digraph of $C_n \odot M$ with n even

PROOF. By observation, we obtain the eccentricity of vertex v_i is $\frac{n+1}{2}$ and eccentricity of vertex u_{ij} is $\frac{n+1}{2} + 1$, $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. The eccentricity of all vertices are used to determine the eccentric vertex of all vertices of the corona graph $C_n \odot M$. The arcs can be obtained by joining every vertex to its eccentric vertex of the corona graph $C_n \odot M$. Table 2 shows the eccentric vertices and arcs of the corona graph $C_n \odot M$.

From Table 2, the arc of the corona graph $C_n \odot M$ vertex is adjacent to its eccentric vertices. The symmetric arcs are $u_{ij}u_{(\frac{n+1}{2}+i(mod\ n))j}, u_{ij}u_{(\frac{n+1}{2}+(i-1))j}$ for $i = 1, 2, \cdots, \frac{n+1}{2}$ and $u_{ij}u_{(i-\frac{n+1}{2})j}$, $u_{ij}u_{(i-\frac{n+1}{2}+1)j}$ for $i = \frac{n+1}{2}, \cdots, n$. The arcs $v_i u_{(\frac{n+1}{2}+i(mod\ n))j}, v_i u_{(\frac{n+1}{2}+(i-1))j}$ for $i = 1, 2, \cdots, \frac{n+1}{2}$ and $v_i u_{(i-\frac{n+1}{2}(mod\ n))j}$, $v_i u_{(i-\frac{n+1}{2}+1)j}$ for $i = \frac{n+1}{2}, \cdots, n$ are not symmetric. Therefore the eccentric digraph of the corona graph $C_n \odot M$ with n odd can be formed into $nK_{m,2m} \cup nK_{1,2m}$ with $V(K_1) = \{v_i\}, \overline{K}_m = V(M_i)$ for $i = 1, 2, \cdots, n$, $K_{2m} = M_{i-\frac{n+1}{2}} \cup M_{i-\frac{n+1}{2}+1}$ for $i = \frac{n+1}{2}, \cdots, n$ and $K_{2m} = M_{\frac{n+1}{2}+i(mod\ n)} \cup M_{\frac{n+1}{2}+(i-1)}$ for $i = 1, 2, \cdots, \frac{n+1}{2}$.

TABLE 2. The eccentric vertices and arc of $C_n \odot M$ vertex, where n odd

vertex of $C_n \odot M$	eccentric vertices	arc
v_1	$u_{\frac{n+1}{2}j}, u_{(\frac{n+1}{2}+1)j}; j = 1, \cdots, m$	$v_1u_{\frac{n+1}{2}j}, v_1u_{(\frac{n+1}{2}+1)j}$
v_2	$u_{(\frac{n+1}{2}+1)j}, u_{(\frac{n+1}{2}+2)j}; j = 1, \cdots, m$	$v_2u_{(\frac{n+1}{2}+1)j}, v_2u_{(\frac{n+1}{2}+2)j}$
$\vdots$	$\vdots$	$\vdots$
$v_{\frac{n+1}{2}-1}$	$u_{(n-1)j}, u_{nj}; j = 1, \cdots, m$	$v_{\frac{n+1}{2}-1}u_{(n-1)j}, v_{\frac{n+1}{2}-1}u_{nj}$
$v_{\frac{n+1}{2}}$	$u_{nj}, u_{1j}; j = 1, \cdots, m$	$v_{\frac{n+1}{2}}u_{nj}, v_{\frac{n+1}{2}}u_{1j}$
$v_{\frac{n+1}{2}+1}$	$u_{1j}, u_{2j}; j = 1, \cdots, m$	$v_{\frac{n+1}{2}+1}u_{1j}, v_{\frac{n+1}{2}+1}u_{2j}$
$\vdots$	$\vdots$	$\vdots$
v_n	$u_{\frac{n-1}{2}j}, u_{\frac{n+1}{2}j}; j = 1, \cdots, m$	$v_nu_{\frac{n-1}{2}j}, v_nu_{\frac{n+1}{2}j}$
$u_{1j}, j = 1, \cdots, m$	$u_{\frac{n+1}{2}j}, u_{(\frac{n+1}{2}+1)j}$	$u_{1j}u_{\frac{n+1}{2}j}, u_{1j}u_{(\frac{n+1}{2}+1)j}$
$u_{2j}, j = 1, \cdots, m$	$u_{(\frac{n+1}{2}+1)j}, u_{(\frac{n+1}{2}+2)j}$	$u_{2j}u_{(\frac{n+1}{2}+1)j}, u_{2j}u_{(\frac{n+1}{2}+2)j}$
$\vdots$	$\vdots$	$\vdots$
$u_{(\frac{n+1}{2}-1)j}, j = 1, \cdots, m$	$u_{(n-1)j}, u_{nj}$	$u_{(\frac{n+1}{2}-1)j}u_{(n-1)j}, u_{(\frac{n+1}{2}-1)j}u_{nj}$
$u_{\frac{n+1}{2}j}, j = 1, \cdots, m$	u_{nj}, u_{1j}	$u_{\frac{n+1}{2}j}u_{nj}, u_{\frac{n+1}{2}j}u_{1j}$
$u_{(\frac{n+1}{2}+1)j}, j = 1, \cdots, m$	u_{1j}, u_{2j}	$u_{(\frac{n+1}{2}+1)j}u_{1j}, u_{(\frac{n+1}{2}+1)j}u_{2j}$
$\vdots$	$\vdots$	$\vdots$
$u_{nj}, j = 1, \cdots, m$	$u_{\frac{n-1}{2}j}, u_{\frac{n+1}{2}j}$	$u_{nj}u_{\frac{n-1}{2}j}, u_{nj}u_{\frac{n+1}{2}j}$

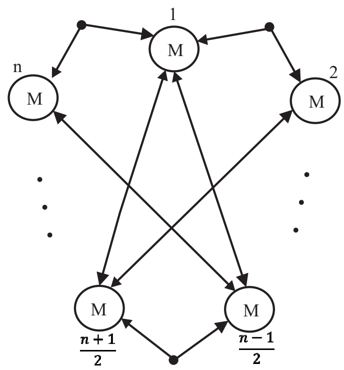

FIGURE 2. The eccentric digraph of $C_n \odot M$ with n odd

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