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APPLICATIONS OF CERTAIN FUNCTIONS ASSOCIATED WITH LEMNISCATE BERNOULLI

Suzeini Abdul Halim[1] and Rashidah Omar[2]

[1,2]**Institute of Mathematical Sciences,**
Faculty of Science, University of Malaya, Malaysia,
suzeini@um.edu.my
[2]**Faculty of Computer and Mathematical Sciences,**
MARA University of Technology Malaysia,
Ashidah@hotmail.com

Abstract. For $J(\alpha, f(z)) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right]$ $(\alpha \geq 0)$, denote $\mathcal{SL}(\alpha)$ and $\mathcal{SL}^c$ as classes of α-convex and convex functions which respectively satisfy conditions $|\,[J(\alpha, f(z))]^2 - 1| < 1$ and $\left|\left[1 + \frac{zf''(z)}{f'(z)}\right]^2 - 1\right| < 1$. Using established results, namely $1 + \beta zp'(z) \prec \frac{1+Dz}{1+Ez}$, $1 + \frac{\beta zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez}$ and $1 + \frac{\beta zp'(z)}{p^2(z)} \prec \frac{1+Dz}{1+Ez}$ imply $p(z) \prec \sqrt{1+z}$ where $p(z)$ is an analytic function defined on the open unit disk $\mathbf{D}$ with $p(0) = 1$. This article obtains conditions so that analytic functions f belong to the classes $\mathcal{SL}(\alpha)$ and $\mathcal{SL}^c$.

Key words: Convex functions, differential subordination, lemniscate Bernoulli.

Abstrak. Untuk $J(\alpha, f(z)) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right]$ $(\alpha \geq 0)$, nyatakan $\mathcal{SL}(\alpha)$ dan $\mathcal{SL}^c$ sebagai kelas-kelas dari konveks-α and fungsi-fungsi konveks yang secaraberturut-turut memenuhi kondisi $|\,[J(\alpha, f(z))]^2 - 1| < 1$ dan $\left|\left[1 + \frac{zf''(z)}{f'(z)}\right]^2 - 1\right| < 1$. Dengan menggunakan hasil-hasil yang telah diperoleh sebelumnya, yaitu $1 + \beta zp'(z) \prec \frac{1+Dz}{1+Ez}$, $1 + \frac{\beta zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez}$ dan $1 + \frac{\beta zp'(z)}{p^2(z)} \prec \frac{1+Dz}{1+Ez}$ mengakibatkan $p(z) \prec \sqrt{1+z}$ dimana $p(z)$ adalah sebuah fungsi analitis yang didefinisikan pada cakram buka $\mathbf{D}$ dengan $p(0) = 1$. Artikel ini memperoleh kondisi-kondisi sehingga fungsi-fungsi analitis f berada dalam kelas-kelas $\mathcal{SL}(\alpha)$ dan $\mathcal{SL}^c$.

Kata kunci: Fungsi Konveks, subordinasi diferensial, lemniscate Bernoulli.

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1. **Introduction**

Let $\mathcal{A}$ denote the class of all analytic functions f in the open unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ and normalised by $f(0) = 0, f'(0) = 1$. An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)(z \in \mathbf{D})$, if there exists an analytic function w in $\mathbf{D}$ such that $w(0) = 0$ and $|w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbf{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbf{D}) \subset g(\mathbf{D})$. For f in $\mathcal{A}$ and $z \in \mathbf{D}$, the classes of starlike $S^\star$, convex C and α-convex functions are defined respectively by the conditions

$$S^\star = \left\{ z \in \mathbf{D} : \mathfrak{Re} \frac{zf'(z)}{f(z)} > 0 \right\}$$

$$C = \left\{ z \in \mathbf{D} : \mathfrak{Re} \frac{zf''(z)}{f'(z)} > -1 \right\}$$

$$\mathcal{M}_\alpha = \{ z \in \mathbf{D} : \mathfrak{Re}\, [J(\alpha, f(z))] > 0 \}$$

where $J(\alpha, f(z)) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \left[1 + \frac{zf''(z)}{f'(z)}\right], \alpha \geq 0$. Note that $\mathcal{M}_0 = S^\star$ and $\mathcal{M}_1 = C$.

Sokól and Stankiewicz [5] introduced the class $SL^\star$ consisting of normalised analytic functions f in $\mathbf{D}$ satisfying the condition $\left| \left[\frac{zf'(z)}{f(z)}\right]^2 - 1 \right| < 1$ for $z \in \mathbf{D}$. Geometrically, a function $f \in SL^\star$ if $\frac{zf'(z)}{f(z)}$ is in the interior of the right half of the lemniscate of Bernoulli $(x^2 + y^2)^2 - 2(x^2 - y^2) = 0$. A function in the class $SL^\star$ is called a Sokól-Stankiewicz starlike function. Alternatively, we can also write $f \in SL^\star \Leftrightarrow \frac{zf'(z)}{f(z)} \prec \sqrt{1+z}$. Some properties of functions in class $SL^\star$ have been studied by [1], [4], [6], [7], [8] and [9]. Next, we denote $S^\star[A, B]$ as the class of Janowski starlike functions defined in Janowski [2] consisting of functions $f \in \mathcal{A}$ satisfying $\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$.

Let $\mathcal{SL}(\alpha)$ and $\mathcal{SL}^c$ denote the classes of α-convex and convex functions which respectively satisfy $|\,[J(\alpha, f(z))]^2 - 1| < 1$ and $\left| \left[1 + \frac{zf''(z)}{f'(z)}\right]^2 - 1 \right| < 1$ $(z \in \mathbf{D})$. It is obvious that $f \in \mathcal{SL}(\alpha) \Leftrightarrow J(\alpha, f(z)) \prec \sqrt{1+z}$ and $f \in \mathcal{SL}^c \Leftrightarrow 1 + \frac{zf''(z)}{f'(z)} \prec \sqrt{1+z}$. Recently in [4] the authors determined condition on β so that $1 + \beta zp'(z), 1 + \frac{\beta zp'(z)}{p(z)}$ and $1 + \frac{\beta zp'(z)}{p^2(z)}$ are subordinated to $\frac{1+Dz}{1+Ez}$ imply $p(z)$ is subordinated to $\sqrt{1+z}$ where $p(z)$ is analytic in $\mathbf{D}$ with $p(0) = 1$. Properties of functions in the class $\mathcal{SL}(\alpha)$ and $\mathcal{SL}^c$ are obtained using results given in [4]. In proving the lemmas, the following proposition is needed.

Proposition 1.1. [3] *Let q be univalent in $\boldsymbol{D}$ and let φ be analytic in a domain containing $q(\boldsymbol{D})$. Let $zq'(z)\varphi[q(z)]$ be starlike. If p is analytic in $D, p(0) = q(0)$ and satisfies $zp'(z)\varphi[p(z)] \prec zq'(z)\varphi[q(z)]$ then $p \prec q$ and q is the best dominant.*

Given that $|E| < 1, |D| \leq 1$ and $D \neq E$, we then have the following lemmas.

Lemma 1.2. *Let* $\beta_0 = \frac{2\sqrt{2}|D-E|}{(1-|E|)}$. *If* $1 + \beta z p'(z) \prec \frac{1+Dz}{1+Ez}$ $(\beta \geq \beta_0)$ *then* $p(z) \prec \sqrt{1+z}$.

Proof. By letting $q(z) = \sqrt{1+z}$ and using Proposition 1.1, it suffices to show

$$s(z) = \frac{1+Dz}{1+Ez} \prec 1 + \beta z q'(z) = 1 + \frac{\beta z}{2\sqrt{1+z}} = h(z).$$

Quite straightforward it can be established that $|s^{-1}[h(z)]| \geq 1$ for $\beta \geq \frac{2\sqrt{2}|(D-E)|}{(1-|E|)}$. Thus the result.

Lemma 1.3. *Let* $\beta_0 = \frac{4|D-E|}{(1-|E|)}$. *If* $1+\beta\frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez}$ $(\beta \geq \beta_0)$ *then* $p(z) \prec \sqrt{1+z}$.

Lemma 1.4. *Let* $\beta_0 = \frac{4\sqrt{2}|D-E|}{(1-|E|)}$. *If* $1 + \beta\frac{zp'(z)}{p^2(z)}$ $(\beta \geq \beta_0) \prec \frac{1+Dz}{1+Ez}$ *then* $p(z) \prec \sqrt{1+z}$.

With appropriate choices of φ in Proposition 1.1, and using similar approach as above Lemma 1.3 and Lemma 1.4 is easily verified. Details of proving these lemmas can be found in [4].

2. Main Results

Theorem 2.1. *Let* $\beta_0 = \frac{2\sqrt{2}(|D-E|)}{1-|E|}$, $|E| < 1$, $|D| \leq 1$, $D \neq E$, $\beta \geq \beta_0$ *and* $f \in A$. *If* f *satisfies*

$$1+\beta\left\{(1-\alpha)\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1\right) + \alpha\frac{zf''(z)}{f'(z)}\left(\frac{z[f''(z)]'}{f''(z)} - \frac{zf''(z)}{f'(z)} + 1\right)\right\} \prec \frac{1+Dz}{1+Ez}$$

then $f \in \mathcal{SL}(\alpha)$

Proof. With $p(z) = (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right] + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right]$, we have

$$p'(z) = (1-\alpha)\left\{\frac{f(z)[zf''(z) + f'(z)] - z[f'(z)]^2}{[f(z)]^2}\right\} + \alpha\left\{\frac{f'(z)[z(f''(z))' + f''(z)] - z[f''(z)]^2}{[f'(z)]^2}\right\}$$

$$= (1-\alpha)\frac{f'(z)}{f(z)}\left\{\frac{zf''(z)}{f'(z)} + 1 - \frac{zf'(z)}{f(z)}\right\} + \alpha\frac{f''(z)}{f'(z)}\left\{\frac{z[f''(z)]'}{f''(z)} + 1 - \frac{zf''(z)}{f'(z)}\right\}$$

and from Lemma 1.2, the result implies $f \in \mathcal{SL}(\alpha)$.

For the special case $\alpha = 1$, Theorem 2.1 gives the result for the class $\mathcal{SL}^c$.

Corollary 2.2. *Let* $\beta_0 = \frac{2\sqrt{2}(|D-E|)}{1-|E|}$, $|E| < 1$, $|D| \leq 1$, $D \neq E$, $\beta \geq \beta_0$ *and* $f \in A$*. If* f *satisfies*

$$1 + \beta \frac{zf''(z)}{f'(z)} \left\{ \frac{z[f''(z)]'}{f''(z)} - \frac{zf''(z)}{f'(z)} + 1 \right\} \prec \frac{1 + Dz}{1 + Ez}$$

then $f \in \mathcal{SL}^c$.

Proof. With $p(z) = 1 + \frac{zf''(z)}{f'(z)}$, it follows that

$$zp'(z) = \frac{z^2[f''(z)]'}{f'(z)} + \frac{zf''(z)}{f'(z)} - \frac{z^2[f''(z)]^2}{[f'(z)]^2}$$

$$= \frac{zf''(z)}{f'(z)} \left\{ \frac{z[f''(z)]'}{f''(z)} + 1 - \frac{zf''(z)}{f'(z)} \right\},$$

and applying Lemma 1.2 gives $1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Dz}{1+Ez}$, hence $f \in \mathcal{SL}^c$.

Theorem 2.3. *Let* $\beta_0 = \frac{4|D-E|}{(1-|E|)}$ *and* $\beta \geq \beta_0$*. Suppose* $f \in A$ *and*

$$1+\beta \left\{ \frac{(1-\alpha)zf'(z)\left[1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right]}{(1-\alpha)zf'(z) + \alpha f(z)\left[1 + \frac{zf''(z)}{f'(z)}\right]} + \frac{\alpha zf''(z)\left[1 - \frac{zf''(z)}{f'(z)} + \frac{z[f''(z)]'}{f''(z)}\right]}{(1-\alpha)f'(z)\frac{zf'(z)}{f(z)} + \alpha[f'(z) + zf''(z)]} \right\} \prec \frac{1 + Dz}{1 + Ez}$$

then $f \in \mathcal{SL}(\alpha)$.

Proof. Let $p(z) = (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right] + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right]$.

$$\frac{p'(z)}{p(z)} = \frac{(1-\alpha)\frac{f'(z)}{f(z)}\left\{\frac{zf''(z)}{f'(z)} + 1 - \frac{zf'(z)}{f(z)}\right\} + \alpha\frac{f''(z)}{f'(z)}\left\{\frac{z[f''(z)]'(z)}{f''(z)} + 1 - \frac{zf''(z)}{f'(z)}\right\}}{(1-\alpha)\left[\frac{zf'(z)}{f(z)}\right] + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right]}$$

$$= \frac{(1-\alpha)f'(z)\left\{zf''(z) + f'(z) - \frac{zf'(z)f'(z)}{f(z)}\right\}}{f'(z)\left\{(1-\alpha)zf'(z) + \alpha f(z)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}} + \frac{\alpha f''(z)f(z)\left\{\frac{z[f''(z)]'}{f''(z)} + 1 - \frac{zf''(z)}{f'(z)}\right\}}{f(z)\left\{(1-\alpha)f'(z)\frac{zf'(z)}{f(z)} + \alpha[f'(z) + zf''(z)]\right\}}$$

$$= \frac{(1-\alpha)f'(z)\left\{\frac{zf''(z)}{f'(z)} + \frac{f'(z)}{f'(z)} - \frac{z[f'(z)]^2}{f'(z)f(z)}\right\}}{(1-\alpha)zf'(z) + \alpha f(z)\left[1 + \frac{zf''(z)}{f'(z)}\right]} + \frac{\frac{\alpha f''(z)f(z)}{f(z)}\left\{\frac{z[f''(z)]'}{f''(z)} + 1 - \frac{zf''(z)}{f'(z)}\right\}}{(1-\alpha)f'(z)\frac{zf'(z)}{f(z)} + \alpha[f'(z) + zf''(z)]}$$

$$= \frac{(1-\alpha)f'(z)\left\{\frac{zf''(z)}{f'(z)} + 1 - \frac{zf'(z)]}{f(z)}\right\}}{(1-\alpha)zf'(z) + \alpha f(z)\left[1 + \frac{zf''(z)}{f'(z)}\right]} + \frac{\alpha f''(z)\left\{\frac{z[f''(z)]'}{f''(z)} + 1 - \frac{zf''(z)}{f'(z)}\right\}}{(1-\alpha)f'(z)\frac{zf'(z)}{f(z)} + \alpha[f'(z) + zf''(z)]}$$

Applying Lemma 1.3 gives $f \in \mathcal{SL}(\alpha)$.

For $\alpha = 1$ we have the result for the class $\mathcal{SL}^c$.

Corollary 2.4. *Let* $\beta_0 = \frac{4|D-E|}{(1-|E|)}$, $\beta \geq \beta_0$ *and* $f \in A$. *If*

$$1 + \beta\frac{zf''(z)}{[f'(z) + zf''(z)]}\left\{\frac{z[f''(z)]'}{f''(z)} - \frac{zf''(z)}{f'(z)} + 1\right\} \prec \frac{1+Dz}{1+Ez}$$

then $f \in \mathcal{SL}^c$.

Proof. For $p(z) = 1 + \frac{zf''(z)}{f'(z)}$, hence

$$\frac{zp'(z)}{p(z)} = \frac{zf''(z)}{f'(z)}\left[\frac{f'(z) - zf''(z)}{f'(z)}\right]\left[\frac{f'(z)}{f'(z) + zf''(z)}\right] + \left[\frac{z^2\left[f''(z)\right]'}{f'(z)}\right]\left[\frac{f'(z)}{f'(z) + zf''(z)}\right]$$

$$= \frac{zf''(z)}{[f'(z) + zf''(z)]} - \frac{[zf''(z)]^2}{f'(z)[f'(z) + zf''(z)]} + \frac{z^2[f''(z)]'}{[f'(z) + zf''(z)]}$$

$$= \frac{zf''(z)}{[f'(z) + zf''(z)]}\left\{\frac{z[f''(z)]'}{f''(z)} + 1 - \frac{zf''(z)}{f'(z)}\right\}$$

and based on Lemma 1.3, $f \in \mathcal{SL}^c$.

Theorem 2.5. *Let* $\beta_0 = \frac{4\sqrt{2}|D-E|}{(1-|E|)}$, $\beta \geq \beta_0$ *and* $f \in A$.

$$1+\beta z \left\{ \frac{(1-\alpha)[f'(z)]^3 f(z)\left[\frac{zf''(z)}{f'(z)}+1-\frac{zf'(z)}{f(z)}\right]+\alpha[f(z)]^2 f'(z)f''(z)\left[\frac{z[f''(z)]'}{f''(z)}+1-\frac{zf''(z)}{f'(z)}\right]}{[(1-\alpha)f'(z)zf'(z)+\alpha f(z)[f'(z)+zf''(z)]]^2} \right\}$$

$$\prec \frac{1+Dz}{1+Ez} \Rightarrow f \in \mathcal{SL}(\alpha).$$

Proof. Using Lemma 1.4 with $p(z) = (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right] + \alpha\left[1+\frac{zf''(z)}{f'(z)}\right]$ gives the desired result.

Corollary 2.6. *Let* $\beta_0 = \frac{4\sqrt{2}|D-E|}{(1-|E|)}$, $\beta \geq \beta_0$ *and* $f \in A$.

$$1+\beta\frac{zf'(z)f''(z)}{[f'(z)+zf''(z)]^2}\left\{\frac{z[f''(z)]'}{f''(z)}+1-\frac{zf''(z)}{f'(z)}\right\} \prec \frac{1+Dz}{1+Ez} \Rightarrow f \in \mathcal{SL}^c.$$

Proof. Set $p(z) = 1+\frac{zf''(z)}{f'(z)}$ and in a similar manner the result is easily obtained.

3. Concluding Remark

For $\alpha = 0$, Theorem 2.1, Theorem 2.3 and Theorem 2.5 reduce to the results for $f \in SL^\star$.

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