J. Indones. Math. Soc.
Vol. 28, No. 02 (2022), pp. 133–146.

# THE PROPERTIES OF UNIFORM FUZZY MODULES AND SEMIUNIFORM FUZZY MODULES

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**Abstract.** In this paper, the concepts of uniform fuzzy modules and semiuniform fuzzy modules were studied. We discussed the necessary and sufficient conditions between uniform fuzzy modules (and semiuniform fuzzy modules) in fuzzy set theory and uniform modules (and semiuniform modules) in module theory.

*Key words and Phrases*: Uniform modules, Uniform fuzzy modules, Semiuniform modules, Semiuniform fuzzy modules.

## 1. INTRODUCTION

In 1965, Zadeh (see [17]) introduced the notion of fuzzy sets. Afterwards, the theory of fuzzy sets had been widely applied to different fields. The theory of fuzzy subsets were studied in many algebraic structures. In 1971, Rosenfeld (see [14]) first applied this concept to the theory of groupoids and groups. The concept of fuzzy rings, including fuzzy ideals of rings, was initiated by Liu (see [8]) in 1982. The concept of fuzzy modules and fuzzy submodules was introduced by Negoita and Ralescu (see [12]) in 1975. Since then, sevaral authors have studied fuzzy modules (see [3], [7],[10], [12], [13] and [17]). The concept of essential fuzzy modules was introduced by Hadi (see [5]) in 2000. Using this idea, Abbas (see [1]) established the concept of essential fuzzy submodules and uniform fuzzy modules in 2012. Nowadays, many researchers have studied with essential fuzzy submodules and uniform fuzzy modules (see [4], [6]). So all of these give the inspiration to

*2020 Mathematics Subject Classification*: 03B45, 03E72
Received: 30-12-2021, accepted: 21-06-2022.

study about novel properties of uniform fuzzy modules and semiuniform fuzzy modules. The aim of this paper is to study the relationship between uniform modules (semiuniform modules) and uniform fuzzy modules (semiuniform fuzzy modules).

In the second section (Uniform fuzzy modules), we discusses characterizations of uniform fuzzy modules, which establish necessary and sufficient conditions between uniform fuzzy modules and uniform modules. In the third section (Semiuniform fuzzy modules), we introduce the notion of semiuniform fuzzy modules by extending from the concept of uniform fuzzy modules. Moreover, we investigate the relationships between semiuniform fuzzy modules and semiuniform modules.

For the purpose of this paper, $R$ is an associative ring with identity, and $\theta$ is denoted to the zero element in an additive group. We recall some definitions and results of fuzzy sets and fuzzy modules which were necessary for the development of this paper.

**Definition 1.1** ([16])**.** *Let $A$ be a subset of a set $S$. The map $\chi_A : S \to [0,1]$ is called the characteristic map of $A$, if*

$$\chi_A(x) = \begin{cases} 1 & \textit{if } x \in A, \\ 0 & \textit{otherwise,} \end{cases}$$

*for all $x \in S$.*

Note that. For a set $S$.

(1) $\chi_s := \chi_{\{s\}}$ for all $s \in S$.
(2) $\chi_S$ is the largest fuzzy subset of $S$.
(3) $\chi_A \cap \chi_B = \chi_{A\cap B}$ for any subset $A$ and $B$ of a nonempty set $S$,

**Definition 1.2.** *Let $\mu$ be a fuzzy subset of a nonempty set $S$ and $A$ be a subset of $S$. We define a fuzzy subset $\tilde{\mu}_A$ of $S$ by*

$$\tilde{\mu}_A(x) = \begin{cases} \mu(x) & \textit{if } x \in A, \\ 0 & \textit{otherwise,} \end{cases}$$

*for all $x \in S$.*

From Definition 1.2, $\tilde{\mu}_A$ is a fuzzy subset of $\mu$ and if $B \subseteq A$, then $\tilde{\mu}_B$ is a fuzzy subset of $\tilde{\mu}_A$. Let $A, B$ be subsets of a nonempty set $S$ and $\mu$ a fuzzy subset of $S$. Then $\tilde{\mu}_A \cap \tilde{\mu}_B = \tilde{\mu}_{A\cap B}$.

**Definition 1.3** ([16])**.** *Let $\mu$ be a fuzzy subset of a nonempty set $S$. For $t \in [0,1]$, the set*

$$\mu_t = \{x \in S \mid \mu(x) \geq t\}$$

*is called the $t$ level subset of $S$ with respect to $\mu$ ($t$ cut of $\mu$).*

**Proposition 1.4.** *Let $A$ be a subset of a nonempty set $S$ and $t \in (0,1]$. Then $(\chi_A)_t = A$.*

*Proof.* Straightforward. □

**Definition 1.5** ([16]). *Let $\mu$ be a fuzzy subset of a nonempty set $S$. Define the set*

$$\mu^* = \{x \in S \mid \mu(x) > 0\}.$$

*$\mu^*$ is called the support set of $\mu$.*

Let $\mu$ and $\sigma$ be fuzzy subsets of a nonempty set $S$. From Definition 1.3 and Definition 1.5, we have that if $\mu$ is a fuzzy subset of $\sigma$, then $\mu^* \subseteq \sigma^*$ and $\mu_t \subseteq \sigma_t$ for all $t \in (0, 1]$. By [16], we have that for a nonempty set $A$, $(\chi_A)^* = A$.

**Definition 1.6** ([11]). *A fuzzy subset $\mu$ of a right $R$-module $M$ is called a fuzzy module of $M$, if*

(1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ *for all* $x, y \in M$.
(2) $\mu(xr) \geq \mu(x)$ *for all* $x \in M$ *and* $r \in R$.
(3) $\mu(\theta) = 1$ *where* $\theta$ *is the zero element in* $M$.

*A fuzzy subset $\alpha$ of a fuzzy module $\mu$ is called a fuzzy submodule of $\mu$, if $\alpha$ is a fuzzy module of $M$.*

Let $\mu$ be a fuzzy module of a right $R$-module $M$ and $A$ be a submodule of $M$. Then $\tilde{\mu}_A$ is a fuzzy submodule of a fuzzy module $\chi_A$. Moreover, $\chi_\theta$ is the smallest fuzzy module of a right $R$-module $M$ where $\theta$ is the zero element in $M$.

**Lemma 1.7.** *If $\mu$ is a non-zero fuzzy module of a right $R$-module $M$, then there exists $t \in (0, 1]$ such that $\mu_t \neq \{0\}$.*

*Proof.* Clear. □

**Proposition 1.8** ([9]). *Let $\mu$ be a fuzzy module of a right $R$-module $M$. Then $\mu^*$ is a submodules of $M$.*

*Moreover, $\mu$ is a fuzzy module of $M$ if and only if $\mu_t$ is a submodule of $M$ for all $t \in (0, 1]$.*

**Proposition 1.9** ([11]). *Let $\alpha$ and $\beta$ be fuzzy modules of a right $R$-module $M$. Then $\alpha \cap \beta$ is a fuzzy module of $M$.*

If $\alpha$ and $\beta$ are fuzzy modules of a right $R$-module $M$, then $\alpha \cup \beta$ may not be a fuzzy module of $M$. For example, on the $\mathbb{Z}$-module $\mathbb{Z}_6$, $\alpha = \chi_{(2)}$ and $\beta = \chi_{(3)}$ but $\alpha \cup \beta = \chi_{(2)\cup(3)}$ is not fuzzy module because $0 = (\alpha \cup \beta)(3-2) < \min\{\alpha(3), \beta(2)\} = 1$. However, it is true when $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

**Proposition 1.10.** *Let $\mu$ be a fuzzy module of a right $R$-module $M$. Then $\mu^* \neq 0$ if and only if $\mu \neq \chi_\theta$.*

*Proof.* It is obvious. □

**Corollary 1.11.** *Let $\mu$ be a fuzzy module of a right $R$-module $M$ and $A$ be a submodule of $M$. If $\tilde{\mu}_A \neq \chi_\theta$, then $A$ is a non-zero submodule of $M$. The converse holds when $\mu = \chi_M$.*

*Proof.* By Proposition 1.10. □

**Corollary 1.12.** *Let $A$ and $B$ be submodules of a right $R$-module $M$. Then $\chi_A \cap \chi_B \neq \chi_\theta$ if and only if $A \cap B \neq \{\theta\}$.*

*Proof.* Clear. □

Recall that, let $S$ and $T$ be two sets and $: S \to T$ be any function. A fuzzy subset $f$ of $S$ is called $\phi invariant$, if $\phi(x) = \phi(y)$ then $f(x) = f(y)$.

**Proposition 1.13.** *Let $S$, $T$ be nonempty sets and $f : S \to T$ be a one-to-one function. Then every fuzzy subset $\mu$ of $S$ is an $f$-invariant.*

*Proof.* Let $\mu$ be a fuzzy subset of $S$ and $x, y \in X$ such that $f(x) = f(y)$. Since $f$ is one-to-one, $x = y$. Thus, $\mu(x) = \mu(y)$. Therefore, $\mu$ is an $f$-invariant. □

**Proposition 1.14.** *Let $M$ and $N$ be right $R$-modules, $f : M \to N$ be an $R$-homomorphism and $\mu, \nu$ be fuzzy modules on $M$ and $N$, respectively. Then,*

*(1) $f(\chi_{0_M}) = \chi_{0_N}$.*
*(2) If $\rho$ is a fuzzy subset of $M$ which is an $f$-invariant, then $f : \rho \to f(\rho)$ is a fuzzy module homomorphism.*
*(3) If $f$ is a monomorphism, then $f : \mu \to f(\mu)$ is a fuzzy module homomorphism.*
*(4) If $f$ is an isomorphism, then $f^{-1} : f(\rho) \to \rho$ is a fuzzy module homomorphism.*

*Proof.* (1) Let $y \in N \setminus \{0\}$. If $f^{-1}(y) = \varnothing$, then $f(\chi_{0_M})(y) = 0 = \chi_{0_N}(y)$. If $f^{-1}(y) \neq \varnothing$, then $f(\chi_{0_M})(y) = \sup\{\chi_{0_M}(z) \mid z \in f^{-1}(y)\} = 0 = \chi_{0_N}(y)$. So

$$f(\chi_{0_M})(0_N) = \sup\{\chi_{0_M}(z) \mid z \in f^{-1}(0_N)\} = 1 = \chi_{0_N}.$$

(2) For $x \in M$, $\rho(x) = f^{-1}(f(\rho))(x) = f(\rho)(f(x))$.
(3) It follows from (2).
(4) Let $y \in N$. There exists $x \in M$ such that $y = f(x)$ and

$$\rho(f^{-1}(y)) = \rho(x) = f(\rho)(f(x)) = f(\rho)(y).$$

□

**Proposition 1.15.** *If $f : \mu \to \nu$ is a fuzzy module homomorphism from a fuzzy module $\mu$ of a right $R$-module $M$ into a fuzzy module $\nu$ of a right $R$-module $N$, then $f^{-1}(\nu) = \mu$.*

*Proof.* Let $x \in M$. Then $f^{-1}(\nu)(x) = \nu(f(x)) = \mu(x)$. □

**Proposition 1.16.** *Let $\mu, \nu$ be fuzzy modules of right $R$-modules $M$ and $N$, respectively, such that $f : \mu \to \nu$ a fuzzy module isomorphism from $\mu$ into $\nu$. Then, $f^{-1}(\nu) \neq \chi_{0_M}$ if and only if $\nu \neq \chi_{0_N}$. Similarly, $\mu \neq \chi_{0_M}$ if and only if $f(\mu) \neq \chi_{0_N}$.*

*Proof.* Suppose that $\nu = \chi_{0_N}$. Let $x \in M$.

Case 1. $x = 0_M$. Then, $f^{-1}(\nu)(x) = f^{-1}(\nu)(0_M) = \nu(f(0_M)) = \nu(0_N) = 1$.

Case 2. $x \neq 0_M$. Since $f$ is a one to one, $f(x) \neq 0$. So $f^{-1}(\nu)(x) = \nu(f((x))) = 0 = \chi_M(x)$. Hence, $f^{-1}(\nu) = \chi_{0_M}$.
Conversely, suppose that $f^{-1}(\nu) = \chi_{0_M}$. Let $y \in N$.

Case 1. $y = 0_N$. Then, $\nu(y) = \nu(0_N) = \nu(f(0_M)) = f^{-1}(\nu)(0_M) = \chi_{0_M}(0_M) = 1$.

Case 2. $y \neq 0_N$. Since $f$ is an isomorphism, there exists $x \in M \setminus \{0_M\}$ such that $y = f(x)$. Therefore, $\nu(y) = \nu(f(x)) = f^{-1}(\nu)(x) = \chi_{0_M}(x) = 0$. Hence, $\nu = \chi_{0_N}$. □

## 2. Uniform Fuzzy Modules

In this section, we investigate the concepts of uniform fuzzy modules. First, we recall some basic properties of these concepts.

**Definition 2.1** ([15]). *A non-zero submodule $N$ of a right $R$-module $M$ is called an essential (or a large) submodule of $M$, if $N \cap A \neq \{\theta\}$ for any non-zero submodule $A$ of $M$.*

**Definition 2.2** ([15]). *A non-zero right $R$-module $M$ is called a uniform module, if any non-zero submodule of $M$ is an essential submodule of $M$.*

*Equivalently, $M$ is a uniform module, if $A \cap B \neq \{\theta\}$ for all non-zero submodules $A$ and $B$ of $M$.*

**Definition 2.3** ([5]). *Let $\mu$ be a non-zero fuzzy module of a right $R$-module $M$. A non-zero fuzzy submodule $\nu$ of $\mu$ is called an essential fuzzy submodule of $\mu$, if $\nu \cap \gamma \neq \chi_\theta$ for all non-zero fuzzy submodule $\gamma$ of $\mu$.*

**Definition 2.4** ([5]). *A fuzzy module $\mu$ of a right $R$-module $M$ is called a uniform fuzzy module of $M$, if any non-zero fuzzy submodule of $\mu$ is an essential fuzzy submodule of $\mu$.*

*Equivalently, $\mu$ is a uniform fuzzy module of a right $R$-module $M$, if $\alpha \cap \beta \neq \chi_\theta$ for all non-zero fuzzy submodules $\alpha$ and $\beta$ of $\mu$.*

**Example 2.5.** *Consider the right $\mathbb{Z}$-module $(\mathbb{Z}_4, +)$. Let $\mu : \mathbb{Z}_4 \to [0,1]$ be a fuzzy set of $\mathbb{Z}_4$ defined by*

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.5 & \text{if } x = 2, \\ 0.25 & \text{otherwise,} \end{cases}$$

*for all $x \in \mathbb{Z}_4$. By Proposition 1.8, $\mu$ is a non-zero fuzzy module of $\mathbb{Z}_4$. Let $\alpha$ and $\beta$ be non-zero fuzzy submodules of $\mu$. Then, $\alpha^*$ and $\beta^*$ are either $(2) = \{0,2\}$ or $\mathbb{Z}_4$. We can show that $\alpha \cap \beta \neq \chi_0$. Therefore, $\mu$ is a uniform fuzzy module of $\mathbb{Z}_4$.*

**Example 2.6.** *Consider the right $\mathbb{Z}$-module $(\mathbb{Z}_{36}, +)$. By Corollary 1.11, $\chi_{\mathbb{Z}_{36}}$ is a non-zero fuzzy module of $\mathbb{Z}_{36}$. Let $\alpha$ and $\beta$ be fuzzy subsets of $\chi_{\mathbb{Z}_{36}}$ defined by*

$$\alpha(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.5 & \text{if } x = 18, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.5 & \text{if } x = 12, 24, \\ 0 & \text{otherwise.} \end{cases}$$

*It is clear that $\alpha$ and $\beta$ are non-zero fuzzy submodules of $\chi_{\mathbb{Z}_{36}}$. Then, $(\alpha \cap \beta)(0) = 1$ and $(\alpha \cap \beta)(x) = 0$ for all $x \in \mathbb{Z}_{36} \setminus \{0\}$. Therefore, $\chi_{\mathbb{Z}_{36}}$ is not the uniform fuzzy module of $\mathbb{Z}_{36}$.*

It is well known for the next theorem but this study offers different proofs.

**Theorem 2.7.** *Let $\mu$ be a non-zero fuzzy module of a right $R$-module $M$. Then, $\mu^*$ is a uniform submodule of $M$ if and only if $\mu$ is a uniform fuzzy module of $M$.*

*Proof.* Suppose that $\mu^*$ is a uniform submodule of $M$. Let $\alpha$ and $\beta$ be non-zero fuzzy submodules of $\mu$. By Proposition 1.8 and Proposition 1.10, $\alpha^*$ and $\beta^*$ are non-zero submodules of $\mu^*$. However, $\mu^*$ is a uniform submodule of $M$, $\alpha^* \cap \beta^* \neq \{\theta\}$. But $\alpha^* \cap \beta^* = (\alpha \cap \beta)^*$ and by Proposition 1.10, $\alpha \cap \beta \neq \chi_\theta$. Hence, $\mu$ is a uniform fuzzy module of $M$. Conversely, suppose that $\mu$ is a uniform fuzzy module of $M$. Let $A$ and $B$ be non-zero submodules of $\mu^*$. So $\mu(x) > 0$ for all $x \in A$ and $\mu(y) > 0$ for all $y \in B$. Then, $\tilde{\mu}_A(x) > 0$ for all $x \in A$ and $\tilde{\mu}_B(y) > 0$ for all $y \in B$. Thus $\tilde{\mu}_A$ and $\tilde{\mu}_B$ are non-zero fuzzy submodules of $\mu$. Since $\mu$ is a uniform fuzzy module of $M$, $\tilde{\mu}_A \cap \tilde{\mu}_B \neq \chi_\theta$. Thus

$$\{\theta\} = (\chi_\theta)^* \neq (\tilde{\mu}_A \cap \tilde{\mu}_B)^* = (\tilde{\mu}_A)^* \cap (\tilde{\mu}_B)^* \subseteq A \cap B.$$

So $A \cap B \neq \{\theta\}$. Hence, $\mu^*$ is a uniform submodule of $M$. □

**Theorem 2.8.** *Let $\mu$ be a non-zero fuzzy module of a right $R$-module $M$ and $s$ be the maximum element in $(0, 1]$, such that $\mu_s \neq \{\theta\}$. Then, $\mu^*$ is a uniform submodule of $M$ if and only if $\mu_t$ is a uniform submodule of $M$ for all $t \in (0, s]$. Moreover, $\mu$ is a uniform fuzzy module of $M$ if and only if $\mu_t$ is a uniform submodules of $M$ for all $t \in (0, s]$.*

*Proof.* Suppose that $\mu^*$ is a uniform submodule of $M$. Let $t \in (0, s]$. Since $\mu_t$ is a non-zero submodule of $\mu^*$ and $\mu^*$ is a uniform submodule of $M$, $\mu_t$ is a uniform submodule of $M$. Conversely, suppose that $\mu_t$ is a uniform submodule $M$ for all $t \in (0, s]$. Let $A$ and $B$ be non-zero submodules of $\mu^*$. Since $\mu^* = \bigcup\limits_{t \in (0,1]} \mu_t = \bigcup\limits_{t \in (0,s]} \mu_t$, there exist $i, j \in (0, s]$ such that $A \subseteq \mu_i$ and $B \subseteq \mu_j$. Choose $k = \min\{i, j\}$. Then, $A$ and $B$ are submodules of $\mu_k$. Since $\mu_k$ is a uniform submodule of $M$, $A \cap B \neq \{\theta\}$. Hence, $\mu^*$ is a uniform submodule of $M$. □

**Corollary 2.9.** *Let $A$ be a non-zero submodule of a right $R$-module $M$. Then, $A$ is a uniform submodule of $M$ if and only if $\chi_A$ is a uniform fuzzy module of $M$.*

*Proof.* By Theorem 2.7 and $(\chi_A)^* = A$ for any non-zero submodule $A$ of a right $R$-module $M$. □

**Theorem 2.10.** *Let $\mu$ be a non-zero fuzzy module of a right $R$-module $M$, and $A$ be a non-zero submodule of $M$, such that $\tilde{\mu}_A \neq \chi_\theta$. If $A$ is a uniform submodule of $M$, then $\tilde{\mu}_A$ is a uniform fuzzy submodule of $\chi_A$.*

*Proof.* Suppose that $A$ is a uniform submodule of $M$. Since $\tilde{\mu}_A \neq \chi_\theta$, $(\tilde{\mu}_A)^*$ is a non-zero submodule of $A$. But $A$ is a uniform submodule of $M$, so $(\tilde{\mu}_A)^*$ is a uniform submodule of $M$. By Theorem 2.7, $\tilde{\mu}_A$ is a uniform fuzzy module of $M$. Therefore, $\tilde{\mu}_A$ is a uniform fuzzy submodule of $\chi_A$. □

**Corollary 2.11.** *Let $A$ and $B$ be non-zero submodules of a right $R$-module $M$. Then, $\chi_A$ is a uniform fuzzy submodule of $\chi_B$ if and only if $A$ is a uniform submodule of $B$.*

*Proof.* ($\Rightarrow$) By Corollary 2.9.
($\Leftarrow$) By Theorem 2.10. □

**Theorem 2.12.** *Let $R$ be a commutative ring and $\mu$ be a non-zero fuzzy module of a right $R$-module $M$. For $r \in R$, define $r\mu$ by*

$$r\mu(x) = \mu(xr) \quad \text{for all } x \in M.$$

*Then, $r\mu$ is a fuzzy module of $M$ and $\mu$ is a fuzzy submodule of $r\mu$.*

*Proof.* Let $x, y \in M$ and $k \in R$. We have that

(1) $(r\mu)(x-y) = \mu((x-y)r) = \mu(xr-yr) \geq \min\{\mu(xr), \mu(yr)\} = \min\{(r\mu)(x), (r\mu)(y)\}$.
(2) $(r\mu)(xk) = \mu((xk)r) = \mu(x(kr)) = \mu(x(rk)) = \mu((xr)k) \geq \mu(xr) = (r\mu)(x)$.
(3) $(r\mu)(\theta) = \mu(\theta \cdot r) = \mu(\theta) = 1$ where $\theta$ is the zero element in $M$.

Then, $r\mu$ is a fuzzy module of $M$. But $\mu(x) \leq r\mu(x)$ for all $x \in M$. Therefore, $\mu$ is a fuzzy submodule of $r\mu$. □

From Theorem 2.12, if $r$ is the zero element in $R$, then $r\mu = \chi_M$.

**Theorem 2.13.** *Let $R$ be a commutative ring, $\mu$ be a non-zero fuzzy module of a right $R$-module $M$ and $r \in R$. If $r\mu$ is a uniform fuzzy module of $M$, then $\mu$ is a uniform fuzzy submodule of $r\mu$.*

*Proof.* Suppose that $r\mu$ is a uniform fuzzy module of $M$. By Theorem 2.7, $(r\mu)^*$ is a uniform submodule of $M$. Since $\mu^*$ is a non-zero submodule of $(r\mu)^*$, $\mu^*$ is a uniform submodule of $(r\mu)^*$. That is $\mu^*$ is a non-zero submodule of $M$. By Theorem 2.7, $\mu$ is a uniform fuzzy module of $M$. But $\mu$ is a fuzzy submodule of $r\mu$, and $\mu$ is a uniform submodule of $r\mu$. □

The converse of Theorem 2.13 is not true. We can see in the following example.

**Example 2.14.** *Consider the right $\mathbb{Z}$-module $\mathbb{Z}_{12}$, $(4) = \{0, 4, 8\}$ is a uniform submodule of $(2) = \{0, 2, 4, 6, 8, 10\}$. By Corollary 2.9, $\chi_{(4)}$ is a uniform fuzzy module of $\chi_{(2)} = 2\chi_{(4)}$. But $(4) \cap (6) = \{0, 4, 8\} \cap \{0, 6\} = (0)$, thus $2\chi_{(4)}$ is not a uniform fuzzy module of $M$.*

**Theorem 2.15.** *Let $A$ be a uniform submodule of a right $R$-module $M$ and $t \in (0,1]$. Define a fuzzy subset $\mu$ of $M$ by*

$$\mu(x) = \begin{cases} 1 & \text{if } x = \theta, \\ t & \text{if } x \in A \setminus \{\theta\}, \\ 0 & \text{if } x \notin A, \end{cases}$$

*for all $x \in M$. Then, $\mu$ is a uniform fuzzy module of $M$.*

*Proof.* Let $x, y \in M$ and $r \in R$.

1. We consider two cases:

Case 1.1. $x \in A$ and $y \in A$. Then, $x - y \in A$ and

$$\mu(x-y) = t \geq \min\{\mu(x), \mu(y)\}.$$

Case 1.2. $x \in M \setminus A$ or $y \in M \setminus A$. Since $\mu(x-y) \geq 0 = \min\{\mu(x), \mu(y)\}$, $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$.
Both two cases, we conclude that $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$.

2. We consider two cases:

Case 1.1. $x \in A$. $\mu(xr) = t \geq \mu(x)$.
Case 1.2. $x \notin A$. $\mu(xr) \geq 0 = \mu(x)$.
Then, $\mu(xr) \geq \mu(x)$.

It is clear that $\mu(\theta) = 1$ where $\theta$ is the zero element in $M$.

Thus $\mu$ is a fuzzy module of $M$. Since $\mu^* = A$ is a uniform submodule of $M$ and by Theorem 2.7, $\mu$ is a uniform fuzzy module of $M$. □

**Theorem 2.16.** *Let $\mu$ be a non-zero fuzzy module of a non-zero right $R$-module $M$. If $M$ is a simple module, then $\mu$ is a uniform fuzzy module of $M$.*

*Proof.* Suppose that $M$ is a simple module. Let $\alpha$ and $\beta$ be non-zero fuzzy submodules of $\mu$. Then, $\alpha^*$ and $\beta^*$ are non-zero submodules of $M$ but $M$ is a simple module. Thus $\alpha^* = \beta^* = \mu^* = M$ and $\alpha^* \cap \beta^* = M \neq \{\theta\}$. However, $(\alpha \cap \beta)^* = \alpha^* \cap \beta^*$, $(\alpha \cap \beta)^* \neq \{\theta\}$. Thus $\alpha \cap \beta \neq \chi_\theta$. Therefore, $\mu$ is a uniform fuzzy module of $M$. □

**Theorem 2.17.** *Let $M, N$ be right $R$-modules, $\mu, \nu$ be fuzzy modules of $M$ and $N$, respectively. Let $f$ a fuzzy module isomorphism from $\mu$ into $\nu$. Then, $\mu$ is a uniform fuzzy module of $M$ if and only if $\nu$ is a uniform fuzzy module of $M$.*

*Proof.* Suppose that $\mu$ is a uniform fuzzy module of $M$. Let $\alpha$ and $\beta$ be non-zero fuzzy submodules of $\nu$. Since $f^{-1}(\alpha)$ and $f^{-1}(\beta)$ are non-zero fuzzy submodules of $\mu$ and $\mu$ is a uniform fuzzy module of $M$, $f^{-1}(\alpha) \cap f^{-1}(\beta) \neq \chi_{0_M}$. But $f^{-1}(\alpha \cap \beta) = f^{-1}(\alpha) \cap f^{-1}(\beta)$. Then, $f^{-1}(\alpha \cap \beta) \neq \chi_{0_M}$. There exists $0_M \neq x \in M$ such that $f^{-1}(\alpha \cap \beta)(x) \neq 0$ and thus $0 \neq f^{-1}(\alpha \cap \beta)(x) = (\alpha \cap \beta)(f(x))$. By Proposition 1.16, $\alpha \cap \beta \neq \chi_{0_N}$. Hence, $\nu$ is a uniform fuzzy module of $N$. Conversely, suppose that $\nu$ is a uniform fuzzy module of $N$. Let $\alpha$ and $\beta$ be non-zero fuzzy submodules of $\mu$. Since $f(\alpha)$ and $f(\beta)$ are non-zero fuzzy submodules of $\nu$ and $\nu$ is a uniform fuzzy module of $N$, $f(\alpha) \cap f(\beta) \neq \chi_{0_N}$. But $f(\alpha \cap \beta) = f(\alpha) \cap f(\beta)$, then $f(\alpha \cap \beta) \neq \chi_{0_N}$.

Since $\alpha \cap \beta = f^{-1}(f(\alpha \cap \beta))$ and by Theorem 1.16, $f^{-1}(f(\alpha \cap \beta)) \neq \chi_{0_M}$. Hence, $\alpha \cap \beta \neq \chi_{0_M}$.

$\square$

## 3. Semiuniform Fuzzy Modules

In this section, we introduce the notion of a semiuniform fuzzy module with some of their properties also give the relation between modules and fuzzy modules.

**Theorem 3.1.** *Let $\mu$ be a fuzzy module of a right $R$-module $M$, $t \in (0,1]$ and $x \in M$. Then,*

$$x_t \subseteq \mu \quad \textit{if and only if} \quad x \in \mu_t.$$

*Proof.* If $x \notin \mu_t$, then $\mu(x) < t = x_t(x)$. Thus $\mu \subsetneq x_t$ and hence $x_t \nsubseteq \mu$. Conversely, let $x \in \mu_t$ and $m \in M$.

Case 1. $x \neq m$. Then, $x_t(m) = 0 \leq \mu(m)$.

Case 2. $x = m$. Then, $x_t(m) = t \leq \mu(x) = \mu(m)$.

Hence, $x_t \subseteq \mu$. $\square$

**Theorem 3.2.** *Let $\mu$ be a fuzzy module of a right $R$-module $M$ and $\rho$ be a fuzzy submodule of $\mu$. If $\rho$ is a prime fuzzy submodule of $\mu$ such that $\rho^* \neq \mu^*$, then $\rho^*$ is a prime submodule of $\mu^*$.*

*Proof.* Suppose that $\rho$ is a prime fuzzy submodule of $\mu$ such that $\rho^* \neq \mu^*$. Let $r \in R$ and $m \in \mu^*$ such that $mr \in \rho^*$. Then, $\rho(mr) > 0$. Choose $t = \rho(mr)$. So $mr \in \rho_t$. By Proposition 3.1, $(mr)_t \subseteq \rho$. But $(mr)_t = m_t r_t, m_t r_t \subseteq \rho$. Since $\rho$ is a prime fuzzy submodule of $\mu$, $m_t \subseteq \rho$ or $r_t \subseteq (\rho : \mu) = \{s_u \mid \mu s_u \subseteq \rho$ where $s_u$ is a fuzzy ring in $R\}$. Suppose that $m \notin \rho^*$. Since $\rho_t \subseteq \rho^*$, $m \notin \rho_t$. By Proposition 3.1, $m_t \nsubseteq \rho$ and hence $\mu r_t \subseteq \rho$. Let $x \in \mu^* r$. There exists $n \in \mu^*$ such that $x = nr$. Then, $\rho(x) \geq (\mu r_t)(x) = \min\{\mu(n), r_t(r)\} > 0$. Thus $x \in \rho^*$. So $\mu^* r \subseteq \rho^*$, i.e., $r \in (\rho^* : \mu^*)$. Therefore, $\rho^*$ is a prime submodule of $\mu^*$. $\square$

**Question :** Let $\mu$ be a non-zero fuzzy module of a right $R$ module $M$ and $P$ a prime submodule of $M$. Is the $\tilde{\mu}_P$ is a prime fuzzy submodule of $M$ ?

**Example 3.3.** *Consider the right $\mathbb{Z}$-module $(\mathbb{Z}_6, +)$. Then, $(2) = \{0, 2, 4\}$ and $(3) = \{0, 3\}$ are non-zero submodules of $\mathbb{Z}_6$. Defined $\mu : \mathbb{Z}_6 \to [0,1]$ by*

$$\mu(x) = \begin{cases} 1 & \textit{if } x = 0, \\ 0.7 & \textit{if } x = 2, 4, \\ 0.5 & \textit{otherwise}, \end{cases}$$

*for all $x \in \mathbb{Z}_6$. Then, $\mu$ is a fuzzy module of $\mathbb{Z}_6$. However, $\tilde{\mu}_{(3)}$ is not a prime fuzzy submodule of $\mu$.*

**Lemma 3.4.** *Let $M, N$ be right $R$-modules, $k \in (0,1]$ and $m_k$ be a fuzzy singleton of $M$. If $f : M \to N$ is a one-to-one, then $f^{-1}(m_k) = f^{-1}(m)_k$.*

*Moreover, for $t \in (0,1], r_t$ a fuzzy singleton in $R$ if $f : M \to N$ is an isomorphism, then $f^{-1}(m_k r_t) = f^{-1}(m)_k r_t$.*

*Proof.* Let $x \in M$. Then,

$$\begin{aligned} f^{-1}(m_k)(x) &= m_k(f(x)) \\ &= \begin{cases} k & \text{if } f(x) = m, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} k & \text{if } m = f^{-1}(x), \\ 0 & \text{otherwise} \end{cases} \\ &= f^{-1}(m)_k(x). \end{aligned}$$

Thus $f^{-1}(m_k) = f^{-1}(m)_k$ and $f^{-1}(m_k r_t) = f^{-1}((mr)_\lambda)$
$= f^{-1}(mr)_\lambda = (f^{-1}(m)r)_\lambda = f^{-1}(m)_k r_t$ where $\lambda = \min\{k, t\}$. □

**Lemma 3.5.** *Let $M, N$ be right $R$-modules, $k \in (0,1]$ and $m_k$ be a fuzzy singleton of $M$. If $f : M \to N$ is a monomorphism, then $f(m_k) = f(m)_k$. Moreover, $f(m_k r_t) = f(m)_k r_t$ where $r_t$ is a fuzzy ring in $R$.*

*Proof.* Let $y \in N$.

Case 1. $f^{-1}(y) = \varnothing$. Then $y \neq f(m)$ and $f(m_k)(y) = 0 = f(m)_k(y)$.

Case 2. $f^{-1}(y) \neq \varnothing$ and $f(m) \neq y$. Then

$$\begin{aligned} 0 \leq f(m_k)(y) &= \sup\{m_k(z) : z \in f^{-1}(y)\} \\ &= \sup\{m_k(z) : f(z) = y \neq f(m)\} \\ &= \sup\{m_k(z) : z \neq m\} \\ &= 0. \end{aligned}$$

But $f(m)_k(y) = 0$, $f(m_k)(y) = f(m)_k(y)$.

Case 3. $f^{-1}(y) \neq \varnothing$ and $f(m) = y$. Then $f(m_k)(y) = \sup\{m_k(z) : z \in f^{-1}(y) = f^{-1}(f(m))$
$= k = f(m)_k(f(m)) = f(m)_k(y)$. Thus $f(m_k) = f(m)_k$. Moreover, $f(m_k r_t) = f((mr)_\lambda) = f(mr)_\lambda$
$= (f(m)r)_\lambda = f(m)_k r_t$ , where $\lambda = \min\{k, t\}$. □

**Theorem 3.6.** *Let $\mu, \nu$ be fuzzy modules of right $R$-modules $M, N$, respectively and $f : \mu \to \nu$ be a fuzzy module monomorphism from $\mu$ into $\nu$. If $\rho$ is a prime fuzzy submodule of $\mu$, then $f(\rho)$ is a fuzzy prime submodule of $f(\mu)$.*

*Proof.* Suppose that $\rho$ is a prime fuzzy submodule of $\mu$. Then, $\rho \neq \mu$ and there exists $x_0 \in M$ such that $\rho(x_0) < \mu(x_0)$. So $f(\rho)(f(x_0)) = \rho(x_0) < \mu(x_0) = f(\mu)(f(x_0))$. Thus $f(\rho)$ is a fuzzy submodule of $f(\mu)$. Let $r_t$ be a fuzzy singleton in $R$ and $m_k \subseteq f(\mu)$ such that $(m_k r_t) \subseteq f(\rho)$. By Lemma 3.4, $f^{-1}(m)_k r_t =$

$f^{-1}(m_k r_t) \subseteq \rho$. Since $\rho$ is a prime fuzzy submodule of $\mu$, $f^{-1}(m)_k \subseteq \rho$ or $r_t \subseteq (\rho : \mu)$. Suppose that $m_k \nsubseteq f(\rho)$. Then,

$$f^{-1}(m_k)(x) = m_k(f(x)) > f(\rho)(f(x)) = \rho(x)$$

for all $x \in M$, i.e., $f^{-1}(m)_k \nsubseteq \rho$. Thus $r_t \subseteq (\rho : \mu)$. Let $y \in N$.

Case 1. $f^{-1}(y) = \varnothing$.

$$(f(\mu)r_t)(y) = f(\mu r_t)(y) = 0 \leq f(\rho)(y).$$

Case 2. $f^{-1}(y) \neq \varnothing$. $x \in M$ exists, such that $y = f(x)$ and $(f(\mu)r_t)(y) = f(\mu r_t)(y) = f(\mu r_t)(f(x)) = (\mu r_t)(x) \leq \rho(x) = f(\rho)(f(x)) = f(\rho)(y)$. So $f(\mu)r_t \subseteq f(\rho)$.

Therefore, $f(\rho)$ is a prime fuzzy submodule of $f(\mu)$. □

**Theorem 3.7.** *Let $\mu, \nu$ be fuzzy modules of right $R$-modules $M, N$, respectively and $f : \mu \to \nu$ be a fuzzy module isomorphism. If $\rho$ is a prime fuzzy submodule of $\nu$, then $f^{-1}(\rho)$ is a fuzzy prime submodule of $\mu$.*

*Proof.* Suppose that $\rho$ is a prime fuzzy submodule of $\nu$. Since $\rho \neq \nu$, there exists $y_0 \in N$ such that $\rho(y_0) < \nu(y_0)$. But $f$ is epimorphism, there exists $x_0 \in M$ such that $y_0 = f(x_0)$ and

$$\begin{aligned} f^{-1}(\rho)(x_0) &= \rho(f(x_0)) = \rho(y_0) \\ &< \nu(y_0) = \nu(f(x_0)) = f^{-1}(\nu)(x_0) = \mu(x_0). \end{aligned}$$

Hence, $f^{-1}(\rho)$ is a proper fuzzy submodule of $\mu$. Let $r_t$ be a fuzzy singleton in $R$ and $m_k \subseteq \mu$ such that $m_k r_t \subseteq f^{-1}(\rho)$. Thus

$$f(m)_k r_t = f(m_k r_t) \subseteq \rho.$$

Since $\rho$ is a prime fuzzy module of $\nu$, $f(m)_k \subseteq \rho$ or $r_t \subseteq (\rho : \nu)$. Suppose that $m_k \nsubseteq f^{-1}(\rho)$. Then, $f(m)_k(f(m)) = k = m_k(m) > f^{-1}(\rho)(m) = \rho(f(m))$. Hence, $f(m)_k \nsubseteq \rho$. So $r_t \subseteq (\rho : \nu)$, i.e., $\nu r_t \subseteq \rho$ and

$$\begin{aligned} (\nu r_t)(f(x)) &= f^{-1}(\nu r_t)(x) \\ &= f^{-1}(\nu r_t)(x) \\ &= f^{-1}(\nu) r_t(x) \\ &= \mu r_t(x) \leq \rho(f(x)) \\ &= f^{-1}(\rho)(x) \end{aligned}$$

for all $x \in M$. Thus $r_t \subseteq (f^{-1}(\rho) : \mu)$. Therefore, $f^{-1}(\rho)$ is a prime fuzzy submodule of $\mu$. □

**Definition 3.8** ([4])**.** *Let $M$ be a right $R$-module. A non-zero submodule $N$ of $M$ is called a semiessential submodule of $M$, if any non-zero prime submodule $P$ of $M$, $P \cap N \neq \{\theta\}$.*

**Definition 3.9** ([4])**.** *A right $R$-module $M$ is called a semiuniform module, if any non-zero submodule of $M$ is a semiessential submodule of $M$.*

**Definition 3.10** ([2]). *Let $\mu$ be a fuzzy module of a right $R$-module $M$. A non-zero fuzzy submodule $\sigma$ of $\mu$ is called a semiessential fuzzy submodule of $\mu$, if any non-zero prime fuzzy submodule $\rho$ of $\mu$, $\rho \cap \sigma \neq \chi_\theta$.*

**Definition 3.11.** *A fuzzy module $\mu$ of a right $R$-module $M$ is called a semiuniform fuzzy module of $M$, if any non-zero fuzzy submodule of $\mu$ is a semiessential fuzzy submodule of $\mu$.*

**Proposition 3.12.** *Every uniform fuzzy module of a right $R$-module $M$ is a semi-uniform fuzzy module of $M$.*

*Proof.* It is clear. □

The converse of Proposition 3.12 is not true.

**Example 3.13.** *Consider $(\mathbb{Z}_{36}, +)$ is the right $\mathbb{Z}$-module and $\mu = \chi_{\mathbb{Z}_{36}}$ is the fuzzy module of $\mathbb{Z}_{36}$. Let $\alpha$ be a non-zero fuzzy submodule of $\mu$ and $\rho$ a non-zero prime fuzzy submodule of $\mu$. Then $\rho^*$ is $(2)$ or $(3)$. So $\rho^* \cap \alpha^* \neq \{0\}$. Hence $\mu$ is a semiuniform fuzzy module of $\mathbb{Z}_{36}$. By example 2.6, $\mu = \chi_{\mathbb{Z}_{36}}$ is not a uniform fuzzy module of $\mathbb{Z}_{36}$.*

**Theorem 3.14.** *Let $\mu$ be a non-zero fuzzy module of a right $R$-module $M$ such that every prime fuzzy module $\rho$ of $\mu$, $\rho^* \neq \mu^*$. If $\mu^*$ is a semiuniform submodule of $M$, then $\mu$ is a semiuniform fuzzy module of $M$.*

*Proof.* Suppose that $\mu^*$ is a semiuniform submodule of $M$. Let $\alpha$ be a non-zero fuzzy submodule of $\mu$ and $\rho$ a non-zero prime fuzzy submodule of $\mu$. Then $\alpha^* \neq \{\theta\}$ and $\rho^* \neq \{\theta\}$. By Theorem 3.2, $\rho^*$ is a prime submodule of $\mu^*$. Since $\mu^*$ is a semiuniform submodule of $M$, $\alpha^* \cap \rho^* \neq \{\theta\}$. However, since $\alpha^* \cap \rho^* = (\alpha \cap \rho)^*$, $(\alpha \cap \rho)^* \neq \{\theta\}$, so $\alpha \cap \rho \neq \chi_\theta$. Therefore, $\mu$ is a semiuniform fuzzy module of $M$. □

For the converse of Theorem 3.14, it is true when $\mu = \chi_M$.

**Theorem 3.15.** *Let $A$ be a non-zero submodule of a right $R$-module $M$. Then, $\chi_A$ is a semiuniform fuzzy submodule of $\chi_M$ if and only if $A$ is a semiuniform submodule of $M$.*

*Proof.* Suppose that $\chi_A$ is a semiuniform fuzzy submodule of $\chi_M$. Let $B$ be a non-zero submodule of $A$, and $P$ a non-zero prime submodule of $A$. By Proposition 1.10, $\chi_B$ and $\chi_P$ are non-zero fuzzy submodules of $\chi_A$. Since $P$ is a prime submodule of $A$, $\chi_P$ is a prime fuzzy submodule of $\chi_A$. Since $\chi_A$ is a semiuniform fuzzy submodule of $\chi_M$, $\chi_B \cap \chi_P \neq \chi_\theta$. By Corollary 1.12 , $B \cap P \neq \{\theta\}$. Hence, $A$ is a semiuniform submodule of $M$. Conversely, by Theorem 3.14. □

**Theorem 3.16.** *Let $\mu, \nu$ be fuzzy modules of right $R$-modules $M, N$, respectively. Also, let $f : \mu \to \nu$ a fuzzy module isomomorphism from $\mu$ into $\nu$. Then, $\mu$ is a semiuniform fuzzy module of $M$ if and only if $\nu$ is a semiuniform fuzzy module of $N$.*

*Proof.* Suppose that $\mu$ is a semiuniform fuzzy module of $M$. Let $\alpha$ be a non-zero fuzzy module of $\nu$, and $\rho$ a non-zero prime fuzzy submodule of $\nu$. Then, $f^{-1}(\alpha)$ is a non-zero fuzzy submodule of $\mu$ and $f^{-1}(\rho)$ is a non-zero prime fuzzy submodule of $\mu$. Since $\mu$ is a semiuniform fuzzy module of $M$, $f^{-1}(\alpha) \cap f^{-1}(\rho) \neq \chi_{0_M}$. But $f^{-1}(\alpha \cap \rho) = f^{-1}(\alpha) \cap f^{-1}(\rho)$, $f^{-1}(\alpha \cap \rho) \neq \chi_{0_M}$ and then $\alpha \cap \rho \neq \chi_{0_N}$. Hence, $\nu$ is a semiuniform fuzzy module of $N$. Conversely, suppose that $\nu$ is a semiuniform fuzzy module of $N$. Let $\alpha$ be a non-zero fuzzy submodule of $\mu$, and $\rho$ a non-zero prime fuzzy submodule of $\mu$. Then, $f(\alpha)$ is a non-zero fuzzy submodule of $\nu$, and $f(\rho)$ is a non-zero prime fuzzy submodule of $\nu$. Since $\nu$ is a semiuniform fuzzy module of $N$, $f(\alpha) \cap f(\rho) \neq \chi_{0_N}$. But $f(\alpha \cap \rho) = f(\alpha) \cap f(\rho)$, $f(\alpha \cap \rho) \neq \chi_{0_N}$. So $\alpha \cap \rho \neq \chi_{0_M}$. Hence, $\mu$ is a semiuniform module of $M$. □

Next, we can find condition of semiuniform fuzzy module and uniform fuzzy module coincide in the case of $\chi_M$ where $M$ is a right $R$-module.

**Theorem 3.17.** *Let $M$ be a right $R$-module and $P$ be a non-zero prime submodule of $M$ such that any non-zero submodule of $M$ containing $P$. Then $\chi_M$ is a semiuniform fuzzy module of $M$ if and only if $\chi_M$ is a uniform fuzzy module.*

*Proof.* Suppose that $\chi_M$ is a semiuniform fuzzy module of $M$. Let $\alpha$ and $\beta$ be non-zero fuzzy submodules of $M$. Then $\alpha^*$ and $\beta^*$ are non-zero submodules of $M$. By assumption, $P \subseteq \alpha^*$. But $M$ is a semiuniform module, $\{\theta\} \neq P \cap \beta^* \subseteq \alpha^* \cap \beta^*$. So $\alpha^* \cap \beta^* \neq \{\theta\}$. But $\alpha^* \cap \beta^* = (\alpha \cap \beta)^*$, $(\alpha \cap \beta)^* \neq \{\theta\}$. Hence $\alpha \cap \beta \neq \chi_\theta$. Conversely, by Theorem 3.12. □

**Example 3.18.** *Consider $(\mathbb{Z}_{p^2}, +)$ is a right $\mathbb{Z}$-module where $p$ is a prime number. There are only submodules $\{0\}, (p) = \{0, p, 2p, 3p, \ldots, (p-1)p\}$ and $\mathbb{Z}_{p^2}$ of $\mathbb{Z}_{p^2}$. Since $(p)$ is a non-zero prime submodule of $\mathbb{Z}_{p^2}$, $\mathbb{Z}_{p^2}$ is a semiuniform fuzzy module and by Theorem 3.17, $\chi_{\mathbb{Z}_{p^2}}$ is a uniform fuzzy module of $\mathbb{Z}_{p^2}$.*

## 4. CONCLUDING REMARKS

In this paper, the research was able to distinguish the properties between uniform modules and uniform fuzzy modules. This was done by establishing the support set of fuzzy modules, $t$ level subset of a right $R$-module with respect to a fuzzy module and understanding the characteristic map of a right $R$-module. The research was able to define Theorems 2.7, Theorem 2.8, Corollary 2.9, and furthermore as described above. The research was also successful in expanding these concepts to further understand the relationships between semiuniform module and semiuniform fuzzy modules. This was also done through studying the characteristic map of a right $R$-module which was established by analyzing Theorem 3.15.

**Acknowledgement.** The authors would like to thank the referee for his/her helpful comments.

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