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# CHARACTERISATION OF PRIMITIVE IDEALS OF TOEPLITZ ALGEBRAS OF QUOTIENTS

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**Abstract.** Let $\Gamma$ be a totally ordered abelian group, the topology on primitive ideal space of Toeplitz algebras $\mathrm{Prim}\,\mathcal{T}(\Gamma)$ can be identified through the upwards-looking topology if and only if the chain of order ideals is well-ordered. Let $I$ be an order ideal of such that the chain of order ideals of $\Gamma/I$ is not well-ordered, we show that for any order ideal $J \supsetneqq I$ , the topology on primitive ideal space can be identified through the upwards-looking topology. Also we discuss the closed sets in $\mathrm{Prim}\,\mathcal{T}(\Gamma)$ with the upwards-looking topology and characterize maximal primitive ideals.

*Key words*: Toeplitz algebra, totally ordered group, primitive ideal, quotient, characterisation.

**Abstrak.** Misalkan $\Gamma$ adalah grup abel terurut total, topologi pada ruang ideal primitif dari aljabar Toeplitz $\mathrm{Prim}\,\mathcal{T}(\Gamma)$ dapat diidentifikasi melalui topologi *upwards-looking* jika dan hanya jika rantai dari ideal urutan adalah terurut dengan rapi (*well-ordered*). Misalkan $I$ adalah sebuah ideal urutan sedemikian sehingga rantai dari ideal urutan dari $\Gamma/I$ tidak terurut dengan rapi, diperlihatkan bahwa untuk sembarang ideal $J \supsetneqq I$ , topologi pada ruang ideal primitif dapat diidentifikasi melalui topologi *upwards-looking*. Pada paper ini juga dibahas himpunan-himpunan tutup di $\mathrm{Prim}\,\mathcal{T}(\Gamma)$ di bawah topologi *upwards-looking*, dan karakterisasi dari ideal primitif maksimal.

*Kata kunci*: Aljabar Toeplitz, grup terurut total, ideal primitif, kuosien, karakterisasi

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## 1. **Introduction**

Suppose $\Gamma$ is a totally ordered abelian group. Let $\Sigma(\Gamma)$ be the chain of order ideals of $\Gamma$, and $X(\Gamma)$ denotes the disjoint union

$$\bigsqcup\{\hat{I} : I \in \Sigma(\Gamma)\} = \{(I, \gamma) : I \in \Sigma(\Gamma), \gamma \in \hat{I}\}.$$

Adji and Raeburn shows that every primitive ideal of Toeplitz algebra $\mathcal{T}(\Gamma)$ of $\Gamma$ is of the form

$$\ker Q_I \circ \alpha_\nu^{\Gamma^{-1}}$$

where $I$ is an order ideal of $\Gamma$ and $\nu \in \hat{\Gamma}$. They also showed [**?**, Theorem 3.1] that there is a bijection $L$ of $X(\Gamma)$ onto the primitive ideal space $\operatorname{Prim}\mathcal{T}(\Gamma)$ of Toeplitz algebra $\mathcal{T}(\Gamma)$ given by

$$L(I, \gamma) := \ker Q_I \circ \alpha_\nu^{\Gamma^{-1}} \text{ where } \nu \in \hat{\Gamma} \text{ satisfies } \nu|_I = \gamma.$$

Adji and Raeburn [**?**] introduced a topology in $X(\Gamma)$ which is called the upwards-looking topology. When $\Sigma(\Gamma)$ is isomorphic with a subset of $\mathbb{N} \cup \{\infty\}$, the bijection $L$ is a homeomorphism [**?**, Proposition 4.7], so the usual hull-kernel topology of $\operatorname{Prim}\mathcal{T}(\Gamma)$ can be identified through the upwards-looking topology in $X(\Gamma)$. Later, Raeburn and his collaborators [**?**] showed that $L$ is a homeomorpism if and only if $\Sigma(\Gamma)$ is well-ordered, in the sense that every nonempty subset has a least element.

More recently, Rosjanuardi and Itoh [**?**] characterised maximal primitive ideals of $\mathcal{T}(\Gamma)$. A series of analysis on subsets of $\Sigma(\Gamma)$ implies that any singleton set $\{\gamma\}$ which consists of a character in $\hat{\Gamma}$ is closed. This implies that every maximal primitive ideal of $\mathcal{T}(\Gamma)$ is of the following form

$$L(\Gamma, \gamma) = \ker Q_\Gamma \circ \alpha_\gamma^{\Gamma^{-1}}.$$

Given a totally ordered abelian group $\Gamma$ and an order ideal $I$. In this paper, we apply the method in [**?**] and [**?**] to characterise maximal primitive ideal of Toeplitz algebra $\mathcal{T}(\Gamma/J)$ of quotient $\Gamma/J$ when the chain of order ideal $\Sigma(\Gamma/I)$ is not well-ordered.

## 2. Upwards-looking Topology

Let $\Gamma$ be a totally ordered abelian group. The Toeplitz algebra $\mathcal{T}(\Gamma)$ of $\Gamma$ is the C*-subalgebra of $B(\ell^2(\Gamma^+))$ generated by the isometries $\{T_x = T_x^\Gamma : x \in \Gamma^+\}$ which are defined in terms of the usual basis by $T_x(e_y) = e_{y+x}$. This algebra is universal for isometric representation of $\Gamma^+$ [**?**, Theorem 2.9].

Let $I$ be an order ideal of $\Gamma$. Then the map $x \longmapsto T_{x+I}^{\Gamma/I}$ is an isometric representation of $\Gamma^+$ in $\mathbb{T}(\Gamma/I)$. Therefore by the universality of $\mathcal{T}(\Gamma)$, there is a homomorphism $Q_I : \mathcal{T}(\Gamma) \longrightarrow \mathcal{T}(\Gamma/I)$ such that $Q_I(T_x) = T_{x+I}^{\Gamma/I}$, and that $Q_I$ is surjective. Suppose $\mathcal{C}(\Gamma, I)$ denotes the ideal in $\mathcal{T}(\Gamma)$ generated by $\{T_u T_u^* - T_v T_v^* :$

$v-u \in I^{+}\}$ and $\mathrm{Ind}_{I^{\perp}}^{\hat{\Gamma}}(\mathcal{T}(\Gamma/I), \alpha^{\Gamma/I})$ is the closed subalgebra of $C(\hat{\Gamma}, \mathcal{T}(\Gamma/I))$ satisfying $f(xh) = {\alpha_h^{\Gamma/I}}^{-1}(f(x))$ for $x \in \hat{\Gamma}, h \in I^{\perp}$. It was proved in [**?**, Theorem 3.1] that there is a short exact sequence of $C^*$-algebras:

$$0 \to \mathcal{C}(\Gamma, I) \to \mathcal{T}(\Gamma) \xrightarrow{\phi_I} \mathrm{Ind}_{I^{\perp}}^{\hat{\Gamma}}(\mathcal{T}(\Gamma/I), \alpha^{\Gamma/I}) \to 0. \tag{1}$$

in which $\phi_I(a)(\gamma) = Q_I \circ (\alpha_\gamma^{\Gamma})^{-1}(a)$ for $a \in \mathcal{T}(\Gamma)$, $\gamma \in \hat{\Gamma}$, and $\alpha_\gamma$ is dual action of $\hat{\Gamma}$ on $\mathcal{T}(\Gamma)$ characterized by $\alpha_\gamma^{\Gamma}(T_x) = \gamma(x)T_x$. The identity representation $T^{\Gamma/I}$ of $\mathcal{T}(\Gamma/I)$ is irreducible [**?**], it follows from [**?**, Proposition 6.16] that $\ker Q_I \circ (\alpha_\gamma^{\Gamma})^{-1}$ is a primitive ideal of $\mathcal{T}(\Gamma)$.

If $X(\Gamma)$ denotes the disjoint union

$$\bigsqcup\{\hat{I} : I \in \Sigma(\Gamma)\} = \{(I, \gamma) : I \in \Sigma(\Gamma), \gamma \in \hat{I}\},$$

it was showed in [**?**, Theorem 3.1] that

$$L(I, \gamma) := \ker Q_I \circ {\alpha_\nu^{\Gamma}}^{-1} \text{ where } \nu \in \hat{\Gamma} \text{ satisfies } \nu|_I = \gamma, \tag{2}$$

is a bijection of $X(\Gamma)$ onto $\mathrm{Prim}\,\mathcal{T}(\Gamma)$.

Using the bijection $L$, Adji and Raeburn describe a new topology on $X$ which corresponds to the hull-kernel topology on $\mathrm{Prim}\,\mathcal{T}(\Gamma)$. This new topology, is later called the *upwards-looking* topology. They topologise $X$ by specifying the closure operation as stated in the following definition.

**Definition 2.1.** [**?**] *The closure $\overline{F}$ of a subset $F$ of $X$ is the set consisting of all pairs $(J, \gamma)$ where $J$ is an order ideal and $\gamma \in \hat{J}$ such that for every open neighbourhood $N$ of $\gamma$ in $\hat{J}$, there exists $I \in \Sigma(\Gamma)$ and $\chi \in N$ for which $I \subset J$ and $(I, \chi|_I) \in F$.*

**Example 2.2.** [**?**, Example 3] *We are going to discuss some description of sets in $X(\Gamma)$ by considering specific cases of $\Gamma$. An observation on $\Gamma := \mathbb{Z} \oplus_{\mathrm{lex}} \mathbb{Z}$ gives interesting results. Let $I$ be the ideal $\{(0, n) : n \in \mathbb{Z}\}$, since $I$ is the only ideal, we have $X(\Gamma) = \hat{0} \sqcup \hat{I} \sqcup \hat{\Gamma}$. Suppose $\lambda_0$ is a character in $\hat{I}$ defined by $(0, n) \mapsto e^{2\pi i n}$, and let $F = \{\lambda_0\}$. Next we consider a character $\gamma$ in $\hat{\Gamma}$ defined by $(m, n) \mapsto e^{2\pi i(m+n)}$. It is clear that $\gamma|_I = \lambda_0$. Then $\gamma \in \bar{F}$, because every open neighbourhood $N$ of $\gamma$ in $\hat{\Gamma}$ contains an element $\lambda$ (which is nothing but $\gamma$ it self) such that its restriction on $I$ gives a character in $F$. It is clear that $\gamma \notin F$, hence $F$ is not closed in the upwards-looking topology for $X(\Gamma)$.*

Adji and Raeburn [**?**] proved that this is the correct topology to identify the hull-kernel topology of $\mathrm{Prim}\,\mathcal{T}(\Gamma)$ when $\Gamma$ is a group such that the set $\Sigma(\Gamma)$ of order ideal is order isomorphic to a subset of $\mathbb{N} \cup \{\infty\}$. In [**?**], Raeburn and his collaborators extended the results in [**?**]. Their main theorem, says that $\mathrm{Prim}\,\mathcal{T}(\Gamma)$ is homeomorphic to $X(\Gamma)$ with the upwards-looking topology if and only if the totally ordered set $\Sigma(\Gamma)$ is well-ordered in the sense that every non-empty subset has a least element. Their technique uses classical Toeplitz operators as well as the universal property of $\mathcal{T}(\Gamma)$ which was the main tool in [?]. Then they described $\operatorname{Prim}\mathcal{T}(\Gamma)$ when parts of $\Sigma(\Gamma)$ are well-ordered.

Rosjanuardi in [?] improved the results in [?] to the case when $\Sigma(\Gamma)$ is not well ordered. In [?, Proposition 6] it is stated that when $\Sigma(\Gamma)$ is isomorphic to a subset of $\{-\infty\}\cup\mathbb{Z}\cup\{\infty\}$, then we can use the upwards-looking topology on $X(\Gamma/I)$ to identify the topology on $\operatorname{Prim}\mathcal{T}(\Gamma/I)$. For general totally abelian group $\Gamma$, as long as there is an order ideal $I$ such that every order ideal $J\supseteq I$ has a successor, the upwards-looking topology is the correct topology for $\operatorname{Prim}\mathcal{T}(\Gamma/I)$ [?, Proposition 8]. In [?, Theorem 9] it was proved that for any quotient $\Gamma/I$ such that the chain $\Sigma(\Gamma/I)$ is isomorphic to a subset $\{-\infty\}\cup\mathbb{Z}\cup\{\infty\}$, for any order ideal $J\supsetneq I$, the upwards-looking topology on $\Sigma(\Gamma/J)$ is the correct topology for $\operatorname{Prim}\mathcal{T}(\Gamma/J)$.

## 3. Characterisation of Primitive Ideals

Example **??** implies that any closed set in the point wise topology is not necessarily closed in the upwards-looking topology. When it is applied to any complement $F^C$ of a set $F$, it arrives to a conclusion that any open set in the point wise topology, is not necessarily open in the upwards-looking topology. This example motivated Rosjanuardi and Itoh [?] to prove more general cases.

Combining results in [?] with ones in [?] give characterisation results for more general cases than in [?].

**Proposition 3.1.** *Suppose that $\Gamma$ is a totally ordered abelian group such that the chain $\Sigma(\Gamma)$ of order ideals in $\Gamma$ is isomorphic to a subset of $\{-\infty\}\cup\mathbb{Z}\cup\{\infty\}$. For any $I\in\Sigma(\Gamma)$, the maximal primitive ideals of $\mathcal{T}(\Gamma/I)$ are of the form*

$$\ker Q_{\Gamma/I}\circ(\alpha_\gamma^{\Gamma/I})^{-1}.$$

Proof. Let $I\in\Sigma(\Gamma)$. The chain of order ideals in $\Gamma/I$ is

$$I\subset J_1/I\subset J_2/I\subset ...,$$

where $J_i\in\Sigma(\Gamma)$ and $I\subset J_i\subset J_{i+1}$ for all $i$. Hence $\Sigma(\Gamma/I)$ is well ordered. Give the set $X(\Gamma/I):=\bigsqcup\{\widehat{J/I}: J\in\Sigma(\Gamma),\ I\subset J\}$ the upwards-looking topology, hence $L^{\Gamma/I}$ is a homeomorphism of $X(\Gamma/I)$ onto $\operatorname{Prim}\mathcal{T}(\Gamma/I)$ by Theorem 3.1 of [?]. Proposition 6 of [?], then implies that $X(\Gamma/I)$ is homeomorphic with $\operatorname{Prim}(\mathcal{T}(\Gamma))$. Theorem 11 of [?] then gives the result.

**Proposition 3.2.** *Suppose that $\Gamma$ is a totally ordered abelian group, and let $I$ be an order ideal in $\Gamma$ such that every oreder ideal $J\supseteq I$ has a successor. Then the maximal primitive ideals of $\mathcal{T}(\Gamma/I)$ are of the form*

$$\ker Q_{\Gamma/I}\circ(\alpha_\gamma^{\Gamma/I})^{-1}.$$

Proof. Let $I\in\Sigma(\Gamma)$ such that every order ideal $J\supseteq I$ has a successor. Since each nontrivial element of $\Sigma(\Gamma/I)$ is of the form $J/I$ for $J\in\Sigma(\Gamma)$ and $J\supsetneq I$, every element of $\Sigma(\Gamma/I)$ has a successor. This implies that $\Sigma(\Gamma/I)$ is well ordered. Give the set $X(\Gamma/I):=\bigsqcup\{\widehat{J/I}: J\in\Sigma(\Gamma),\ I\subset J\}$ the upwards-looking topology, hence

$L^{\Gamma/I}$ is a homeomorphism by Theorem 3.1 of [?]. The result then follows from Theorem 9 of [?].

**Theorem 3.3.** *Suppose that $\Gamma$ is a totally ordered abelian group, and $I \in \Sigma(\Gamma)$ such that $\Sigma(\Gamma/I) \cong \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$. Let $J \in \Sigma(\Gamma)$ such that $J \supsetneqq I$. Then the maximal primitive ideals of $\mathcal{T}(\Gamma/J)$ are of the form*

$$\ker Q_{\Gamma/J} \circ (\alpha_\gamma^{\Gamma/J})^{-1}.$$

Proof. Since every nontrivial ideal of $\Gamma/I$ is of the form $J/I$ where $J \in \Sigma(\Gamma)$ and $J \supsetneq I$ and for ideals $J_1, J_2$ such that $J_1/I \subseteq J_2/I$ implies $J_1 \subseteq J_2$, then may write

$$\Sigma(\Gamma/I) := \{I = J_{-\infty} \subseteq ... \subseteq J_k/I \subseteq J_{k+1}/I \subseteq ... \subseteq \Gamma = J_\infty\}.$$

Now consider the subset

$$\mathcal{I} := \{I = J_{-\infty} \subseteq ... \subseteq J_k \subseteq J_{k+1} \subseteq ... \subseteq J_\infty = \Gamma\}$$

of $\Sigma(\Gamma)$. If $J \neq I$ is an element of $\mathcal{I}$, i.e $J \in \Sigma(\Gamma)$ such that $J \supsetneqq I$, the set

$$\Sigma(\Gamma/J) = \{J \subseteq K_1/J \subseteq K_2/J...\}$$

is well ordered. Hence $L^{\Gamma/J}$ is a homeomorphism of $X(\Gamma/J)$ onto $\operatorname{Prim} \mathcal{T}(\Gamma/J)$ by Theorem 3.1 of [?]. The result is then follow from Theorem 9 of [?].

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