CHARACTERISATION OF PRIMITIVE IDEALS OF TOEPLITZ ALGEBRAS OF QUOTIENTS

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Abstract. Let $\Gamma$ be a totally ordered abelian group, the topology on primitive ideal space of Toeplitz algebras $\text{Prim} T(\Gamma)$ can be identified through the upwards-looking topology if and only if the chain of order ideals is well-ordered. Let $I$ be an order ideal of such that the chain of order ideals of $\Gamma/I$ is not well-ordered, we show that for any order ideal $J \nsubseteq I$, the topology on primitive ideal space can be identified through the upwards-looking topology. Also we discuss the closed sets in $\text{Prim} T(\Gamma)$ with the upwards-looking topology and characterize maximal primitive ideals.

Key words: Toeplitz algebra, totally ordered group, primitive ideal, quotient, characterisation.

Abstrak. Misalkan $\Gamma$ adalah grup abel terurut total, topologi pada ruang ideal primitif dari aljabar Toeplitz $\text{Prim} T(\Gamma)$ dapat diidentifikasi melalui topologi upwards-looking jika dan hanya jika rantai dari ideal urutan adalah terurut dengan rapi (well-ordered). Misalkan $I$ adalah sebuah ideal urutan sedemikian sehingga rantai dari ideal urutan dari $\Gamma/I$ tidak terurut dengan rapi, diperlihatkan bahwa untuk sembarang ideal $J \nsubseteq I$, topologi pada ruang ideal primitif dapat diidentifikasi melalui topologi upwards-looking. Pada paper ini juga dibahas himpunan-himpunan tutup di $\text{Prim} T(\Gamma)$ di bawah topologi upwards-looking, dan karakterisasi dari ideal primitif maksimal.

Kata kunci: Aljabar Toeplitz, grup terurut total, ideal primitif, kuosien, karakterisasi.
1. Introduction

Suppose $\Gamma$ is a totally ordered abelian group. Let $\Sigma(\Gamma)$ be the chain of order ideals of $\Gamma$, and $X(\Gamma)$ denotes the disjoint union $\bigsqcup\{\hat{I} : I \in \Sigma(\Gamma)\} = \{(I, \gamma) : I \in \Sigma(\Gamma), \gamma \in \hat{I}\}$. Adji and Raeburn shows that every primitive ideal of Toeplitz algebra $T(\Gamma)$ of $\Gamma$ is of the form $\ker Q_I \circ \alpha^{-1}_\nu$ where $I$ is an order ideal of $\Gamma$ and $\nu \in \hat{\Gamma}$. They also showed [?, Theorem 3.1] that there is a bijection $L$ of $X(\Gamma)$ onto the primitive ideal space $\text{Prim} T(\Gamma)$ of Toeplitz algebra $T(\Gamma)$ given by $L(I, \gamma) := \ker Q_I \circ \alpha^{-1}_\nu$ where $\nu \in \hat{\Gamma}$ satisfies $\nu|_I = \gamma$.

Adji and Raeburn [?] introduced a topology in $X(\Gamma)$ which is called the upwards-looking topology. When $\Sigma(\Gamma)$ is isomorphic with a subset of $\mathbb{N} \cup \{\infty\}$, the bijection $L$ is a homeomorphism [?, Proposition 4.7], so the usual hull-kernel topology of $\text{Prim} T(\Gamma)$ can be identified through the upwards-looking topology in $X(\Gamma)$. Later, Raeburn and his collaborators [?] showed that $L$ is a homeomorphism if and only if $\Sigma(\Gamma)$ is well-ordered, in the sense that every nonempty subset has a least element.

More recently, Rosjanuardi and Itoh [?] characterised maximal primitive ideals of $T(\Gamma)$. A series of analysis on subsets of $\Sigma(\Gamma)$ implies that any single-ton set $\{\gamma\}$ which consists of a character in $\hat{\Gamma}$ is closed. This implies that every maximal primitive ideal of $T(\Gamma)$ is of the following form $L(\Gamma, \gamma) = \ker Q_I \circ \alpha^{-1}_\gamma$.

Given a totally ordered abelian group $\Gamma$ and an order ideal $I$. In this paper, we apply the method in [?] and [?] to characterise maximal primitive ideal of Toeplitz algebra $T(\Gamma/J)$ of quotient $\Gamma/J$ when the chain of order ideal $\Sigma(\Gamma/I)$ is not well-ordered.

2. Upwards-looking Topology

Let $\Gamma$ be a totally ordered abelian group. The Toeplitz algebra $\mathcal{T}(\Gamma)$ of $\Gamma$ is the C*-subalgebra of $B(\mathcal{L}(\Gamma^+))$ generated by the isometries $\{T_x = T_x^* : x \in \Gamma^+\}$ which are defined in terms of the usual basis by $T_x(e_y) = e_{y+x}$. This algebra is universal for isometric representation of $\Gamma^+$ [?, Theorem 2.9].

Let $I$ be an order ideal of $\Gamma$. Then the map $x \mapsto T_{x+I}^{\Gamma/I}$ is an isometric representation of $\Gamma^+$ in $\mathcal{T}(\Gamma/I)$. Therefore by the universality of $\mathcal{T}(\Gamma)$, there is a homomorphism $Q_I : \mathcal{T}(\Gamma) \to \mathcal{T}(\Gamma/I)$ such that $Q_I(T_x) = T_{x+I}^{\Gamma/I}$, and that $Q_I$ is surjective. Suppose $\mathcal{C}(\Gamma, I)$ denotes the ideal in $\mathcal{T}(\Gamma)$ generated by $\{T_uT_u - T_vT_v : u, v \in I\}$.
$v - u \in I^+$ and $\text{Ind}^r_{I^+}(\mathcal{T}(\Gamma/I), \alpha^{\Gamma/I})$ is the closed subalgebra of $C(\hat{\Gamma}, \mathcal{T}(\Gamma/I))$ satisfying $f(xh) = \alpha_h^{\Gamma/I} f(x)$ for $x \in \hat{\Gamma}, h \in I^+$. It was proved in [?, Theorem 3.1] that there is a short exact sequence of $C^*$-algebras:

$$0 \to \mathcal{C}(\Gamma, I) \to \mathcal{T}(\Gamma) \xrightarrow{\phi} \text{Ind}^r_{I^+}(\mathcal{T}(\Gamma/I), \alpha^{\Gamma/I}) \to 0.$$  

(1)

in which $\phi_I(a)(\gamma) = Q_I \circ (\alpha^\Gamma_{I^+})^{-1}(a)$ for $a \in \mathcal{T}(\Gamma)$, $\gamma \in \hat{\Gamma}$, and $\alpha_v$ is dual action of $\hat{\Gamma}$ on $\mathcal{T}(\Gamma)$ characterized by $\alpha_v^\Gamma(T_x) = (x)T_x$. The identity representation $T^\Gamma/I$ of $\mathcal{T}(\Gamma/I)$ is irreducible [?], it follows from [?, Proposition 6.16] that $\ker Q_I \circ (\alpha^\Gamma_{I^+})^{-1}$ is a primitive ideal of $\mathcal{T}(\Gamma)$.

If $X(\Gamma)$ denotes the disjoint union

$$\bigsqcup \{I : I \in \Sigma(\Gamma)\} = \{(I, \gamma) : I \in \Sigma(\Gamma), \gamma \in \hat{\Gamma}\},$$

it was showed in [?, Theorem 3.1] that

$$L(I, \gamma) := \ker Q_I \circ (\alpha^\Gamma_v)^{-1} \text{ where } \nu \in \hat{\Gamma} \text{ satisfies } \nu|_I = \gamma,$$

(2)

is a bijection of $X(\Gamma)$ onto $\text{Prim } \mathcal{T}(\Gamma)$.

Using the bijection $L$, Adji and Raeburn describe a new topology on $X$ which corresponds to the hull-kernel topology on $\text{Prim } \mathcal{T}(\Gamma)$. This new topology, is later called the upwards-looking topology. They topologise $X$ by specifying the closure operation as stated in the following definition.

**Definition 2.1.** [?] The closure $\overline{F}$ of a subset $F$ of $X$ is the set consisting of all pairs $(J, \gamma)$ where $J$ is an order ideal and $\gamma \in \hat{J}$ such that for every open neighbourhood $N$ of $\gamma$ in $\hat{J}$, there exists $I \in \Sigma(\Gamma)$ and $\chi \in N$ for which $I \subset J$ and $(I, \chi|_I) \in F$.

**Example 2.2.** [?, Example 3] We are going to discuss some description of sets in $X(\Gamma)$ by considering specific cases of $\Gamma$. An observation on $\Gamma := \mathbb{Z}_{\text{hex}} \mathbb{Z}$ gives interesting results. Let $I$ be the ideal $\{(0, n) : n \in \mathbb{Z}\}$, since $I$ is the only ideal, we have $X(\Gamma) = \emptyset \sqcup \hat{I} \sqcup \hat{\Gamma}$. Suppose $\lambda_0$ is a character in $\hat{I}$ defined by $(0, n) \mapsto e^{2\pi i n}$, and let $F = \{\lambda_0\}$. Next we consider a character $\gamma$ in $\hat{\Gamma}$ defined by $(m, n) \mapsto e^{2\pi i (m+n)}$. It is clear that $\gamma|_I = \lambda_0$. Then $\gamma \in \overline{F}$, because every open neighbourhood $N$ of $\gamma$ in $\hat{\Gamma}$ contains an element $\lambda$ (which is nothing but $\gamma$ it self) such that its restriction on $I$ gives a character in $F$. It is clear that $\gamma \notin F$, hence $F$ is not closed in the upwards-looking topology for $X(\Gamma)$.

Adji and Raeburn [?] proved that this is the correct topology to identify the hull-kernel topology of $\text{Prim } \mathcal{T}(\Gamma)$ when $\Gamma$ is a group such that the set $\Sigma(\Gamma)$ of order ideal is order isomorphic to a subset of $\mathbb{N} \cup \{\infty\}$. In [?], Raeburn and his collaborators extended the results in [?]. Their main theorem, says that $\text{Prim } \mathcal{T}(\Gamma)$ is homeomorphic to $X(\Gamma)$ with the upwards-looking topology if and only if the totally ordered set $\Sigma(\Gamma)$ is well-ordered in the sense that every non-empty subset has a least element. Their technique uses classical Toeplitz operators as well as the
universal property of $\mathcal{T}(\Gamma)$ which was the main tool in [?]. Then they described Prim $\mathcal{T}(\Gamma)$ when parts of $\Sigma(\Gamma)$ are well-ordered.

Rosjanuardi in [?] improved the results in [?] to the case when $\Sigma(\Gamma)$ is not well ordered. In [?, Proposition 6] it is stated that when $\Sigma(\Gamma)$ is isomorphic to a subset of $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, then we can use the upwards-looking topology on $X(\Gamma/I)$ to identify the topology on Prim $\mathcal{T}(\Gamma/I)$. For general totally abelian group $\Gamma$, as long as there is an order ideal $I$ such that every order ideal $J \supseteq I$ has a successor, the upwards-looking topology is the correct topology for Prim $\mathcal{T}(\Gamma/I)$ [?, Proposition 8]. In [?, Theorem 9] it was proved that for any quotient $\Gamma/I$ such that the chain $\Sigma(\Gamma/I)$ is isomorphic to a subset $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, for any order ideal $J \supseteq I$, the upwards-looking topology on $\Sigma(\Gamma/J)$ is the correct topology for Prim $\mathcal{T}(\Gamma/J)$.

3. Characterisation of Primitive Ideals

Example ?? implies that any closed set in the point wise topology is not necessarily closed in the upwards-looking topology. When it is applied to any complement $F^C$ of a set $F$, it arrives to a conclusion that any open set in the point wise topology is not necessarily open in the upwards-looking topology. This example motivated Rosjanuardi and Itoh [?] to prove more general cases.

Combining results in [?] with ones in [?] give characterisation results for more general cases than in [?].

**Proposition 3.1.** Suppose that $\Gamma$ is a totally ordered abelian group such that the chain $\Sigma(\Gamma)$ of order ideals in $\Gamma$ is isomorphic to a subset of $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$. For any $I \in \Sigma(\Gamma)$, the maximal primitive ideals of $\mathcal{T}(\Gamma/I)$ are of the form

$$\ker Q_{\Gamma/I} \circ (\alpha_{\Gamma/I})^{-1}.$$  

**Proof.** Let $I \in \Sigma(\Gamma)$. The chain of order ideals in $\Gamma/I$ is

$$I \subset J_1/I \subset J_2/I \subset \ldots,$$

where $J_i \in \Sigma(\Gamma)$ and $I \subset J_i \subset J_{i+1}$ for all $i$. Hence $\Sigma(\Gamma/I)$ is well ordered. Give the set $X(\Gamma/I) := \bigsqcup \{J/I : J \in \Sigma(\Gamma), I \subset J\}$ the upwards-looking topology, hence $L_{\Gamma/I}^\uparrow$ is a homeomorphism of $X(\Gamma/I)$ onto Prim $\mathcal{T}(\Gamma/I)$ by Theorem 3.1 of [?], Proposition 6 of [?], then implies that $X(\Gamma/I)$ is homeomorphic with Prim($\mathcal{T}(\Gamma))$. Theorem 11 of [?] then gives the result.

**Proposition 3.2.** Suppose that $\Gamma$ is a totally ordered abelian group, and let $I$ be an order ideal in $\Gamma$ such that every order ideal $J \supseteq I$ has a successor. Then the maximal primitive ideals of $\mathcal{T}(\Gamma/I)$ are of the form

$$\ker Q_{\Gamma/I} \circ (\alpha_{\Gamma/I})^{-1}.$$  

**Proof.** Let $I \in \Sigma(\Gamma)$ such that every order ideal $J \supseteq I$ has a successor. Since each nontrivial element of $\Sigma(\Gamma/I)$ is of the form $J/I$ for $J \in \Sigma(\Gamma)$ and $J \supseteq I$, every element of $\Sigma(\Gamma/I)$ has a successor. This implies that $\Sigma(\Gamma/I)$ is well ordered. Give the set $X(\Gamma/I) := \bigsqcup \{J/I : J \in \Sigma(\Gamma), I \subset J\}$ the upwards-looking topology, hence
$L^{\Gamma/I}$ is a homeomorphism by Theorem 3.1 of [?]. The result then follows from Theorem 9 of [?].

**Theorem 3.3.** Suppose that $\Gamma$ is a totally ordered abelian group, and $I \in \Sigma(\Gamma)$ such that $\Sigma(\Gamma/I) \cong \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$. Let $J \in \Sigma(\Gamma)$ such that $J \not\supset I$. Then the maximal primitive ideals of $\mathcal{T}(\Gamma/J)$ are of the form $\ker Q_{\Gamma/J} \circ (\alpha^I_{\Gamma/J})^{-1}$.

**Proof.** Since every nontrivial ideal of $\Gamma/I$ is of the form $J/I$ where $J \in \Sigma(\Gamma)$ and $J \not\supset I$, and for ideals $J_1, J_2$ such that $J_1/I \subseteq J_2/I$ implies $J_1 \subseteq J_2$, then may write

$$\Sigma(\Gamma/I) := \{ I = J_{-\infty} \subseteq \ldots \subseteq J_k/I \subseteq J_{k+1}/I \subseteq \ldots \subseteq \Gamma = J_\infty \}.$$ 

Now consider the subset $I := \{ I = J_{-\infty} \subseteq \ldots \subseteq J_k \subseteq \ldots \subseteq J_\infty = I \}$ of $\Sigma(\Gamma)$. If $J \neq I$ is an element of $I$, i.e $J \in \Sigma(\Gamma)$ such that $J \not\supset I$, the set

$$\Sigma(\Gamma/J) = \{ J \subseteq K_1/J \subseteq K_2/J \ldots \}$$

is well ordered. Hence $L^{\Gamma/J}$ is a homeomorphism of $X(\Gamma/J)$ onto $\text{Prim} \mathcal{T}(\Gamma/J)$ by Theorem 3.1 of [?]. The result is then follow from Theorem 9 of [?].

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**References**