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SOME PROPERTIES OF VON NEUMANN REGULAR GRAPHS OF RINGS

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Abstract. Let R be a ring with unity. Taloukolaei and Sahebi [2] introduced the Von Neumann regular graph $G_{Vnr^+}(R)$ of a ring, whose vertex set consists of elements of R and two distinct vertices x and y are adjacent if and only if $x+y$ is a Von Neumann regular element. In this article, we investigate some new properties of $G_{Vnr^+}(R)$ such as the traversability, pancyclic, unicyclic, chordal and perfect. We also investigate the domination parameters of $G_{Vnr^+}(R)$ such as the dominating set, the domination number, the total domination number, the connected domination number and give the condition when the $G_{Vnr^+}(R)$ is an excellent graph. Finally we determine the bondage number.

Key words and Phrases: Von Neumann regular ring; Von Neumann regular graph; Domination parameters.

1. Introduction

Let R be a ring with unity. An element $x \in R$ is called Von Neumann regular if there exists $r \in R$ such that $x = xrx$ (If R is commutative then $x = x^2r$). The set of all Von Neumann regular elements of R is denoted by $Vnr(R)$. Clearly 0 and units of R are Von Neumann regular elements of R. Von Neumann regular ring is a ring where all the elements of R are Von Neumann regular element. Von Neumann regular graphs associated with rings was first introduced by Taloukolaei and Sahebi [2]. The Von Neumann regular graph of a ring R denoted by $G_{Vnr^+}(R)$ is the graph with R as the vertex set and distinct $u, v \in R$ are adjacent if and only if $u + v \in Vnr(R)$. The unit graphs studied in [3] are subgraphs of $G_{Vnr^+}(R)$. In this article, we consider all the rings R to be generated by $Vnr(R)$ as a group,

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and we denote $U(R)$, $\mathfrak{m}$ and I as the set of units, maximal ideal and an ideal of R respectively. We refer [5], for undefined terminology of ring theory.

Let $G = (V(G), E(G))$ be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices in G denoted by $|G|$ is called the order of G, and the number of edges of G denoted by $|E(G)|$ is called the size of G. For $v \in V(G)$, we denote degree of v by $deg(v)$. The minimum degree and maximum degree of G is denoted by $\delta(G)$ and $\Delta(G)$ respectively. A closed trail of length three or more in a graph G is called circuit. A circuit C in a graph G is called an Eulerian circuit if C contains every edge of G. A connected graph G is said to be Eulerian if it contains an Eulerian circuit. A graph G is said to be Hamiltonian if it has a circuit which contains all the vertices of G. A graph G of order $n \geq 3$ is called pancyclic if G has cycles of all lengths from 3 to n. A graph G is said to be unicyclic if G contains exactly one cycle. A simple graph G is said to be a chordal graph if every cycle in G of length 4 and greater has a chord. A graph G is perfect if and only if no induced subgraph of G is an odd cycle of length at least five or the complement of one. A nonempty subset D of V is called a dominating set if every vertex in $V \setminus D$ is adjacent to at least one vertex in D. A subset D of V is called a total dominating set if every vertex in V is adjacent to some vertex in D. A dominating set D is called a connected dominating set if the subgraph induced by D is connected. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. In a similar way we define the total domination number $\gamma_t(G)$ and the connected domination number $\gamma_c(G)$. A Graph G is called excellent if for every vertex $x \in V(G)$, there is a dominating set D which contain x. The bondage number $b(G)$ is the minimum number of edges whose removal increases the domination number. A domatic partition of G is a partition of $V(G)$ into dominating sets in G. The maximum number of classes of a domatic partition of G is called the domatic number of G and is denoted by $d(G)$. A graph G is said to be domatically full if $d(G) = \delta(G) + 1$. We refer [6] and [11] for undefined terminology of graph theory.

The organisation of this paper is as follows: In Section 2, we begin with an obvious remark that $G_{Vnr^+}(R)$ is a complete graph if and only if R is a Von Neumann regular ring, we also investigate the traversibility of $G_{Vnr^+}(R)$ for certain conditions. Finally, we characterize all the rings for which $G_{Vnr^+}(R)$ is pancyclic, unicyclic, chordal and perfect. In Section 3, we attempt to study the domination numbers in the graph $G_{Vnr^+}(R)$ and we prove the domination number of $G_{Vnr^+}(R)$ is 1 if and only if either R is a Von Neumann regular ring or $R \cong \mathbb{Z}_4$. We also determine that domination number is 2 if R is a local ring (not Von Neuman regular ring) and $R \times S$, where R is a Von Neumann regular ring and S is a local ring which is not a Von Neumann regular ring. Finally, we determine the bondage number of $G_{Vnr^+}(R)$.

2. Some results of $G_{Vnr^+}(R)$

In this section, we attempt to study some new results of $G_{Vnr^+}(R)$ such as the traversability, pancyclic, unicyclic, chordal and perfect. In order to study some new results of $G_{Vnr^+}(R)$, we state the following Theorem 2.1 from [2].

Theorem 2.1. [2, Proposition 2.2] *Let R be a finite ring and $G_{Vnr^+}(R)$ be a Von Neumann regular graph of R. Then the following hold:*

(1) *If $2x \notin Vnr(R)$, then $deg(x) = |Vnr(R)|$.*
(2) *If $2x \in Vnr(R)$, then $deg(x) = |Vnr(R)| - 1$.*

Therefore $\Delta = |Vnr(R)|$ and $\delta = |Vnr(R)| - 1$.

Remark 2.2. *Let R be a finite ring. Then $G_{Vnr^+}(R)$ is a complete graph if and only if R is Von Neumann regular ring.*

In the following, we characterize the ring whose Von Neumann regular graph $G_{Vnr^+}(R)$ is Eulerian and Hamiltonian.

Remark 2.3. *Let R be a Von Neumann regular ring. Then $G_{Vnr^+}(R)$ is Eulerian if and only if $|R| = odd$.*

Theorem 2.4. *Let $(R, \mathfrak{m})$ be a local ring with $\mathfrak{m} \neq 0$. Then $G_{Vnr^+}(R)$ is not Eulerian.*

Proof. Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{m} \neq 0$. Then by Theorem 2.1 $deg(0) = |Vnr(R)| - 1$. Also there exist $x \in \mathfrak{m}$ such that $deg(x) = |Vnr(R)|$, hence $G_{Vnr^+}(R)$ is not Eulerian. □

Theorem 2.5. *Let R be a finite ring such that $|R| \geq 3$. If $G_{Vnr^+}(R)$ is connected, then $G_{Vnr^+}(R)$ is Hamiltonian.*

Proof. Let R be a finite ring such that $|R| \geq 3$. Then the following two cases complete the proof:
Case 1. If R is a Von Neumann regular ring, then $G_{Vnr^+}(R)$ is a complete graph and so $G_{Vnr^+}(R)$ is Hamiltonian.
Case 2. If R is a non-Von Neumann regular ring, then $G_{Vnr^+}(R)$ is not a complete graph. Now by Theorem 2.1, $deg(x) = |Vnr(R)|$ or $|Vnr(R)| - 1$. Let $x, y \in V(G_{Vnr^+}(R))$ such that $x \nsim y$ in $G_{Vnr^+}(R)$, then we have the following conditions:

(1) $deg(x) + deg(y) = 2|Vnr(R)|$, for all $2x, 2y \notin Vnr(R)$.
(2) $deg(x) + deg(y) = 2|Vnr(R)| - 2$, for all $2x, 2y \notin Vnr(R)$.
(3) $deg(x) + deg(y) = 2|Vnr(R)| - 1$, for $2x \in Vnr(R)$ and $2y \notin Vnr(R)$.

It is clear that $deg(x) + deg(y) \geq |V(G_{Vnr^+}(R))|$ since $2|Vnr(R)| > |R|$. Hence by Ore's theorem of Hamiltonian, $G_{Vnr^+}(R)$ is Hamiltonian. □

Bondy [8] gave us a useful little theorem that relate Hamiltonicity and pancyclicity of a graph, which is stated in the following in a more modified version as per our need to prove the next Theorem:

Theorem 2.6. [8] *Every Hamiltonian non-bipartite graph G of order n with* $\delta(G) \geq \frac{n}{2}$ *is pancyclic.*

Remark 2.7. *We make the following observations about* $G_{Vnr^+}(R)$ *which in turn will help us in proving Theorem 2.8.*

(1) *By using Proposition 4.4 in* [2] *we conclude that* $G_{Vnr^+}(R)$ *is not bipartite for* $|R| \geq 3$.
(2) *Also* $\delta(G_{Vnr^+}(R)) = |Vnr(R)| - 1 \geq \frac{|R|}{2}$.

Theorem 2.8. *Let R be a finite ring. Then* $G_{Vnr^+}(R)$ *is pancyclic if and only if* $|R| \geq 3$.

Proof. $\Rightarrow$ Let $G_{Vnr^+}(R)$ be a pancyclic graph. If $|R| = 2$, then $R \cong \mathbb{Z}_2$. Therefore $G_{Vnr^+}(R) = K_2$ which is not a cycle and so it contradict our assumption.
$\Leftarrow$ Assume that $|R| \geq 3$. Then by Theorem 2.5 and Remark 2.7, we see that $G_{Vnr^+}(R)$ is a Hamiltonian and non-bipartite graph with $\delta(G_{Vnr^+}(R)) \geq \frac{|R|}{2}$. Therefore, by Theorem 2.6 the proof follows directly. □

Theorem 2.9. $G_{Vnr^+}(R)$ *is a unicyclic graph if and only if either* $R \cong \mathbb{Z}_3$ *or* $R \cong \dfrac{\mathbb{Z}_2[x]}{x^2}$.

Proof. If either $R \cong \mathbb{Z}_3$ or $R \cong \dfrac{\mathbb{Z}_2[x]}{x^2}$, then $G_{Vnr^+}(R)$ is isomorphic to K_3 and C_4 (cycle graph of length four) respectively. Conversely, let $G_{Vnr^+}(R)$ be a unicyclic graph. From [[5], p.95], we may write $R \cong R_1 \times ... \times R_n$, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$. If $i \geq 2$, then Von Neumann regular elements $(0, ..., 0), (1, 0, ..., 0), ..., (0, 0, ..., 1), (1, 1, ..., 1)$ form a complete sub graph of $G_{Vnr^+}(R)$, which contradict our assumption. Therefore, R is a local ring. If $|R| = 2$, then $G_{Vnr^+}(R)$ is K_2. If $|R| = 3$, then $G_{Vnr^+}(R)$ is K_3. If $|R| = 4$, then R is a Von Neumann regular ring, $\mathbb{Z}_4$, $\dfrac{\mathbb{Z}_2[x]}{x^2}$. The cases that R is a Von Neumann regular ring and $R \cong \mathbb{Z}_4$ are ruled out since $G_{Vnr^+}(R)$ is not unicyclic graph. If $|R| \geq 5$, and R is a Von Neumann regular ring then $G_{Vnr^+}(R)$ is a complete graph, and hence it is not unicyclic. If $|R| \geq 5$, and R is not a Von Neumann regular ring, then there exist $x \in Vnr(R) \setminus \{0, 1\}$ and $y \notin Vnr(R)$ such that the induced subgraphs form by the sets $S_1 = \{0, x, -x\}$ and $S_2 = \{1, -1, y\}$ are two different cycles of length 3 in $G_{Vnr^+}(R)$. Therefore, $G_{Vnr^+}(R)$ is not unicyclic. Hence, the result. □

Theorem 2.10. *Let R be a ring such that* $|R| \geq 4$. *Then* $G_{Vnr^+}(R)$ *is a chordal if and only if either R is a Von Neumann regular ring or* $R \cong \mathbb{Z}_4$.

Proof. $\Rightarrow$ Assume that $G_{Vnr^+}(R)$ is a chordal graph. If $R \cong \dfrac{\mathbb{Z}_2[x]}{x^2}$, then $G_{Vnr^+}(R)$ is C_4. If $|R| > 5$, let us take the set $S = \{0, 1, z, u\}$ where $z \notin Vnr(R)$ and $u \in U(R) \subset Vnr(R)$. Then $< S >= C_4$ is a chordless cycle in $G_{Vnr^+}(R)$, which contradict our assumption. However, if R a Von Neumann regular ring,

then $G_{Vnr^+}(R)$ is a complete graph which implies that $G_{Vnr^+}(R)$ is a chordal graph. Similarly, if $R \cong \mathbb{Z}_4$ then $0 \sim 1 \sim 2 \sim 3 \sim 0$ is a cycle of length 4 with $1 \sim 3$ as the chord. Hence, the result.
$\Leftarrow$ If R is either a Von Neumann regular ring or $R \cong \mathbb{Z}_4$, then it is clear that $G_{Vnr^+}(R)$ is a chordal graph. $\square$

Theorem 2.11. *Let R be any ring. Then $G_{Vnr^+}(R)$ is perfect.*

Proof. The following two cases complete the proof:
Case 1. If R is a Von Neumann regular ring. Then $G_{Vnr^+}(R)$ is a complete graph and so $G_{Vnr^+}(R)$ does not have any induced subgraph of odd cycle of length at least 5.
Case 2. If R is not a Von Neumann regular ring. Let $S \subset V(G_{Vnr^+}(R))$ with $|S| \geq 5$, then there exists at least one $s \in S$ such that $deg_{<S>}(s) > 2$. Therefore, there is no induced cycle subgraph of length at least 5 in $G_{Vnr^+}(R)$. Hence $G_{Vnr^+}(R)$ is perfect. $\square$

3. Domination Parameters of $G_{Vnr^+}(R)$

In this section, we study the domination parameters. The study of domination number has been a topic of interest for graph theorists in the recent past, infact the dominating set problem is a classical NP-complete decision problem and there is no efficient algorithm to find a smallest dominating set of a given graph. We start our discussion on domination parameters of $G_{Vnr^+}(R)$ with the following Lemma.

Lemma 3.1. *If R is a Von Nuemann regular ring, then $\gamma(G_{vnr^+}(R)) = 1$.*

Proof. Let R be a Von Neumann regular ring, then every element is a Von-Neumann regular element. Therefore, for every $x \in R \setminus \{0\}$, we have $x + 0 \in V_{nr}(R)$. So, every $x \neq 0$ is adjacent to 0 and so $D = \{0\}$ is a dominating set of $G_{Vnr^+}(R)$. Hence, $\gamma(G_{Vnr^+}) = 1$ $\square$

Theorem 3.2. $\gamma(G_{Vnr^+}(R)) = 1$ *if and only if either R is Von Neumann regular ring or $R \cong \mathbb{Z}_4$.*

Proof. $\Rightarrow$ Let $\gamma(G_{Vnr^+}(R)) = 1$. Let $D = \{x\}$ be a dominating set of $G_{Vnr^+}(R)$. Hence, $x \sim y$ for all $y \in R$ which implies that $deg(x) = |R| - 1$. By Theorem 2.1 $deg(x) = |Vnr(R)| - 1$ or $|Vnr(R)|$. If $deg(x) = |R| - 1 = |Vnr(R)| - 1$, then R is a Von Neumann regular ring. If $deg(x) = |R| - 1 = |Vnr(R)|$, then $R \cong \mathbb{Z}_4$.
$\Leftarrow$ If R is a Von Neumann regular ring, then by Lemma 3.1 we get the result. If $R \cong \mathbb{Z}_4$, then either $\{1\}$ or $\{3\}$ is the dominating set. Hence, the result hold. $\square$

Remark 3.3. *(1) If $(x_1, x_2, ..., x_n) \in \Pi_{i=1}^{n} R_i$, then $(x_1, x_2, ..., x_n)$ is a Von-Neumann regular element if and only if $x_i \in V_{nr}(R_i)$, for all $i = 1, 2, ...n$.*
(2) If R_i are Von Neumann regular rings, then $\Pi_{i=1}^{n} R_i$ is also a Von Neumann regular ring.

(3) If $R = \Pi_{i=1}^{n} R_i$, where R_i are Von Neumann regular rings, then $\gamma(G_{Vnr^+}(R)) = 1$.

Remark 3.4. *If R is a Von Nuemann regular ring, then we have the following results:*

(1) $\gamma(G_{Vnr^+}) = \gamma_t(G_{Vnr^+}) = \gamma_c(G_{Vnr^+}) = 1$.
(2) By Remark 2.2 we see that $G_{Vnr^+}(R)$ is a complete graph, therefore $D = \{x\}$ is a dominating set of $G_{Vnr^+}(R)$ for all $x \in R$. Hence, we see that $G_{Vnr^+}(R)$ is an excellent graph.

In the following, we determine the domination parameters for a local ring R, which is not Von Neumann regular ring. We begin with the following Lemma.

Lemma 3.5. *Let $(R, \mathfrak{m} \neq 0)$ be a local ring and let $x \in R$. If $x \notin \mathfrak{m}$, then $x \in V_{nr}(R)$ and if $x \in \mathfrak{m}$, then $1 + x \in V_{nr}(R)$.*

Theorem 3.6. *Let $(R, \mathfrak{m})$ be a local ring with $\mathfrak{m} \neq 0$ such that $R \ncong \mathbb{Z}_4$. Then $\gamma(G_{vnr^+}(R)) = 2$.*

Proof. Let R be a local ring with $\mathfrak{m} \neq 0$ as the maximal ideal. If $R \cong \mathbb{Z}_4$, then by Theorem 3.2 $\gamma(G_{vnr^+}(R)) = 1$. Therefore, we assume that $R \ncong \mathbb{Z}_4$. If $x \notin \mathfrak{m}$, then by Lemma 3.5 $x \in V_{nr}(R)$ therefore, $x \sim 0$ in $(G_{vnr^+}(R))$. If $x \in \mathfrak{m}$, then again by Lemma 3.5 $1 + x \in V_{nr}(R)$ therefore, $x \sim 1$ in $(G_{Vnr^+}(R))$. Hence, $D = \{0, 1\}$ is a dominating set in $G_{vnr^+}(R)$. This yields $\gamma(G_{vnr^+}(R)) = 2$. □

Theorem 3.7. *Let $(R, \mathfrak{m})$ be a local ring with $\mathfrak{m} \neq 0$. Then the following conditions hold:*

(1) $\gamma(G_{Vnr^+}(R)) = \gamma_t(G_{Vnr^+}(R)) = \gamma_c(G_{Vnr^+}(R)) = 2$.
(2) $G_{Vnr^+}(R)$ is an excellent graph.

Proof. (1) The proof follows directly from Theorem 3.6.
(2) If $x \in Vnr(R)$, then $D = \{0, x\}$ is a dominating set in $G_{Vnr^+}(R)$. If $x \notin Vnr(R)$, then $D = \{1, x\}$ is a dominating set of $G_{Vnr^+}(R)$. Therefore, for every $x \in R$ we obtain a dominating set of $G_{Vnr^+}(R)$ which contain x. Hence, we conclude that $G_{Vnr^+}(R)$ is an exellent graph.

□

In the following, we find the domination parameters for the ring $R' = R \times S$, where R is any Von Neumann regular ring and S is a local ring which is not a Von Neumann regular ring.

Theorem 3.8. *Let R be a Von Nuemann regular ring and S be any local ring such that S is not a Von Neumann regular ring. Then $\gamma(G_{Vnr^+}(R \times S)) = 2$.*

Proof. Let $(x, y) \in R \times S$. If $(x, y) \in Vnr(R \times S)$, then $(0, 0) + (x, y) = (x, y) \in Vnr(R \times S)$ i.e. $(0, 0) \sim (x, y)$. If $(x, y) \notin Vnr(R \times S)$, then it implies that $y \notin Vnr(S)$. Therefore, by Remark 3.5 $(0, 1) + (x, y) = (x, 1 + y) \in Vnr(R \times S)$ and so $(x, y) \sim (0, 1)$. Hence, $D = \{(0, 0), (0, 1)\}$ is a dominating set in $G_{Vnr^+}(R \times S)$. Thus, $\gamma(G_{Vnr^+}(R \times S)) = 2$. □

Theorem 3.9. *Let R be a Von Nuemann regular ring and S be any local ring which is not a Von Neumann regular ring. Then the following conditions hold:*

(1) $\gamma(G_{Vnr^+}(R \times S)) = \gamma_t(G_{Vnr^+}(R \times S)) = \gamma_c(G_{Vnr^+}(R \times S)) = 2$.
(2) $G_{Vnr^+}(R \times S)$ *is an excellent graph.*

Proof. (1) From Theorem 3.8 we obtained that $D = \{(0,0),(0,1)\}$ is a dominating set of $G_{Vnr^+}(R \times S)$. Also $(0,0) \sim (0,1)$, thus $\gamma_t(G_{Vnr^+}(R \times S)) = \gamma_c(G_{Vnr^+}(R)) = 2$.

(2) If $(x,y) \in Vnr(R \times S)$, then $D = \{(0,0),(x,y)\}$ is a dominating set of $G_{Vnr^+}(R \times S)$, and if $(x,y) \notin Vnr(R \times S)$, then $D = \{(0,1),(x,y)\}$ is a dominating set of $G_{Vnr^+}(R \times S)$. Therefore, for every $(x,y) \in R \times S$ we have a dominating set $\{(0,0),(x,y)\}$ in $G_{Vnr^+}(R \times S)$, which contain (x,y). Hence, $G_{Vnr^+}(R \times S)$ is an excellent graph.

□

Next we obtain a relationship between the domination number of a ring R and its quotient ring R/I, where I is any ideal of R.

Theorem 3.10. *Let R be any ring and I be a proper ideal of R, then* $\gamma(G_{Vnr^+}(R/I)) \leq \gamma(G_{Vnr^+}(R))$.

Proof. Let $\gamma(G_{Vnr^+}(R)) = n$ and $D = \{u_1, u_2, ..., u_n\}$ be the dominating set for $G_{Vnr^+}(R)$. Choose $u_{i_1}+I, u_{i_2}+I, ..., u_{i_m}+I$ from the set $\{u_1+I, u_2+I, ..., u_n+I\}$ such that they are distinct from each other, then obviously $\{u_{i_1}, u_{i_2}, ..., u_{i_m}\} \subseteq \{u_1, u_2, ..., u_n\}$ and $m \leq n$. We claim that $D' = \{u_{i_1}+I, u_{i_2}+I, ..., u_{i_m}+I\}$ is a dominating set for $G_{Vnr^+}(R/I)$. Let $v+I \in (R/I) \setminus D'$, then $v \in R \setminus D$. Therefore, there exists $u_l \in D$ such that $v \sim u_l$ in $G_{Vnr^+}(R)$. Now By [2, Proposition 2.6] $v+I \sim u_l+I$ in $G_{Vnr^+}(R/I)$ and so $u_l+I = u_{i_j}+I$. Hence, $v+I \sim u_{i_j}+I$ in $G_{Vnr^+}(R/I)$. This gives that $\gamma(G_{Vnr^+}(R/I)) \leq |D'| = m \leq n = \gamma(G_{Vnr^+}(R))$. □

In order to prove the Theorem 3.12, we state the following Theorem.

Theorem 3.11. [7, Theorem 3.1 and proposition 4.1] *Let G and H be two graphs, then $\gamma(G \times H) \geq \gamma(G) + \gamma(H) - 1$. Moreover if the equality holds and $\gamma(G) = 1$, then $G = K_1$ and H is an edgeless graph.*

Theorem 3.12. *Let R be a ring and R' be a Von Neumann regular ring. Then* $\gamma(G_{Vnr^+}(R \times R')) \geq \gamma(G_{Vnr^+}(R))$.

Proof. By Theorem 3.11 we see that

$$\gamma(G_{Vnr^+}(R) \times G_{Vnr^+}(R')) \geq \gamma(G_{Vnr^+}(R)) + \gamma(G_{Vnr^+}(R')) - 1$$

Now, by Lemma 3.1 $\gamma(G_{Vnr^+}(R')) = 1$ and so the result hold. □

Next we attempt to find the domination number $R(+)M$, where R is a ring and M is an R-module. But before, we recall the definiton of $R(+)M$ and its properties.

Definition 3.13. *Let R be a commutative ring and M be an R module. The set $R(+)M = \{(r, m)|r \in R$ and $m \in M\}$ under the addition and multiplication defined as $(r, m) + (r', n) = (r + r', m + n)$ and $(r, m)(r', n) = (rr', rn + r'm)$ for $(r, m), (r', n) \in R(+)M$ is a commutative ring. This ring is called the idealization of M in R.*

Remark 3.14.
(1) Let $W_r = \{(r, m)|m \in M\}$, then for distinct $r, r' \in R$, $W_r \cap W_{r'} = \emptyset$ and we have $R(+)M = \bigcup_{r \in R} W_r$.
(2) $Vnr(R(+)M) = \{(r, rm)|\ r \in Vnr(R)$ and $m \in M\}$

Lemma 3.15. [2, Lemma 4.9] *For arbitrary elements $r, r' \in R$ and a module M over R, the following statements are equivalent:*

(1) The vertex r is adjacent to r' in $G_{Vnr^+}(R)$.
(2) Every element of W_r is adjacent to every element of $W_{r'}$ in $G_{Vnr^+}(R(+)M)$.

Theorem 3.16. *Let R be a ring and M be an R-module, then the following conditions hold:*

(1) If R is a Von Neumann regular ring, then $\gamma(G_{Vnr^+}(R(+)M)) = 1$.
(2) If (R, m) is a local ring and $m \neq 0$, then $\gamma(G_{Vnr^+}(R(+)M)) = 2$.

Proof. (1) If R is a Von Neumann regular ring, then $r \sim 0$, for all $r \in R$. Then by Lemma 3.15, we see that all elements in W_0 are adjacent to all elements in W_r. Therefore, $D = \{(0, 0)\}$ is the dominating set in $G_{Vnr^+}(R(+)M)$ and so $\gamma(G_{Vnr^+}(R(+)M)) = 1$.
(2) If R is a local ring and $r \in R$. Let $r \in Vnr(R)$, then $r \sim 0$, this implies that all elements in W_0 are adjacent to all elements in W_r. If $r \notin Vnr(R)$, then $r \sim 1$, this implies that all elements in W_1 are adjacent to all elements in W_r. Therefore, $D = \{(0, 0), (1, 0)\}$ is a dominating set in $G_{Vnr^+}(R(+)M)$ and so $\gamma(G_{Vnr^+}(R(+)M)) = 2$.

□

Next we attempt to determine the domatic number of connected Von Neumann regular graph $G_{Vnr^+}(R)$ of ring.

Remark 3.17. *If R is a Von Neumann regular ring then, $\{x\}$ is a dominating set for all $x \in R$, therefore $d(G_{Vnr^+}(R)) = |R|$.*

Theorem 3.18. *Let $(R, \mathfrak{m})$ be a local ring with $\mathfrak{m} \neq 0$ such that $R \ncong \mathbb{Z}_4$. Then $d(G_{Vnr^+}(R)) = \lfloor \frac{|R|}{2} \rfloor$.*

Proof. Let $(R, \mathfrak{m})$ be a local ring with $\mathfrak{m} \neq 0$. Then $S = \{x_1, x_2\}$ is a dominating set in $G_{Vnr^+}(R)$, where $x_2 \notin x_1 + \mathfrak{m}$. Hence, $d(G_{Vnr^+}(R)) = \lfloor \frac{|R|}{2} \rfloor$. □

Remark 3.19. *The above Theorem 3.18 fails if $R \cong \mathbb{Z}_4$. Here, dominating sets of $G_{Vnr^+}(R)$ are $\{1\}, \{3\}, \{0, 2\}$ and so $d(G_{Vnr^+}(R)) = 3$. Moreover, the graph $G_{Vnr^+}(R)$ is domatically full.*

We end up this article by discussing the bondage number of Von Neumann regular graph $G_{Vnr^+}(R)$.

Remark 3.20. *If R is a Von Neumann regular ring and so $G_{Vnr^+}(R)$ is a complete graph. Therefore,* $b(G_{Vnr^+}(R)) = \lceil \frac{|R|}{2} \rceil$.

Theorem 3.21. *Let $(R, \mathfrak{m})$ be a local ring with $\mathfrak{m} \neq 0$ and $|R| \neq 4$. Then* $b(G_{Vnr^+}(R)) = \delta(G_{Vnr^+}(R))$.

Proof. Let $(R, \mathfrak{m})$ be a local ring such that $\mathfrak{m} \neq 0$. Let $|R| = 4$. If $R \cong \mathbb{Z}_4$, then $Vnr(\mathbb{Z}_4) = \{0, 1, 3\}$. Now, if we remove the edge $(1, 3)$ from the graph $G_{Vnr^+}(R)$, then it increases the domination number by 1. So $b(G_{Vnr^+}(R)) = 1 \neq \delta(G_{Vnr^+}(R))$. If $R \cong \dfrac{\mathbb{Z}_2[x]}{x^2}$, then $G_{Vnr^+}(\dfrac{\mathbb{Z}_2[x]}{x^2})$ is C_4 and so, $b(G_{Vnr^+}(R)) \neq \delta(G_{Vnr^+}(R))$. Therefore, we consider $|R| \neq 4$. Then $\delta(G_{Vnr^+}(R)) = |Vnr(R)| - 1$; Obviously $deg(0) = \delta(G_{Vnr^+}(R))$. Now, to increase the domination number, we have to remove all the edges from 0. This yields $b(G_{Vnr^+}(R)) = \delta(G_{Vnr^+}(R))$. □

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