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AN IMPLICIT FINITE DIFFERENCE METHOD FOR THIN FILM FLOW

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Abstract. Thin film flow on an inclined plane can be modeled from lubrication theory. Together with boundary conditions, the governing equations can be formulated into a single equation of the thickness of the fluid, and solved numerically by an implicit finite difference method. In analyzing the equation, the linearized equation is considered here, to see the characteristic of the solution as wave propagation and the effect of the physical parameter. We found that the wave propagates by reducing the amplitude, and the wave speed is affected by the steepness of the plane. All of these can be indicated in the linearized equation. Numerical wave-simulation is presented in this paper.

Key words and Phrases: thin film flow, theory of lubrication, single equation, numerical wave-simulation

1. INTRODUCTION

2-D fluid flow on an inclined channel is considered. Since the thickness of the fluid is much smaller than the wave length, the governing equations can be simplified from Navier-Stokes model into lubrication theory. Wiryanto and Febrianti [1] formulated the governing equations into a single equation of the fluid thickness. Time and space are two independent variables involved in the model, so that the fluid thickness can be represented as wave propagation. From this formulation, some works related to the thin film flow are then developed, the model and also numerical method, see [2, 3]. They worked on fully non-linear equation. A finite difference method has been done by Wiryanto and Febrianti [1] for simulating wave

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propagation. Wiryanto [2, 3] then canalized the stability of the numerical method for forward time central space (FTCS), which is conditionally stable.

The numerical simulation showed that the wave propagates by changing the form, and tends to almost shock, but the numerical procedure was not suitable to produce the shock wave. In deformation of the wave, the wave amplitude decreases as the time increases, and the front wave tends steeper to shock. Effect of physical parameters also observed, mainly the inclination of the channel. The surface wave similar to that was also obtained by King, et. al. [4] for steady problem involving air blown above the thin fluid; Needham and Merkin [5], Merkin and Needham [6], Fauzi and Wiryanto [7, 8] who worked on shallow water equations. Analytically, shock wave predicted by Dressler [9], called roll wave, in open inclined channel.

Recently, Wiryanto et. al. [10] developed the model in [1] by involving viscosity and surface tension, so that the model of the single equation such as observed in [1, 2, 3] contains an extra term in fourth derivative of the fluid depth. Using FTCS, that model is able to simulate the wave, but the effect of the surface tension could not be observed, for various values of that physical quantity. They also indicated the numerical procedure was failed for a slightly larger number of the surface tension because of the stability condition. From those, we are then interested in developing the numerical method into implicit one. As a preliminary work, we apply the implicit method to the model without surface tension in this paper, but for linearized model. Our simulation confirms the result in the references.

How relevant the linearized model of that thin film flow, that can be observed, is our concern in this paper. The model is formulated by considering small perturbation of the constant solution. The stability of the solution as the amplitude decrease is much easier to be analyzed. This characteristic can be seen from the coefficient of the equation, so that the physical parameter can be identified. Combined with the numerical solution, the results can be compared and we found that they agree each other.

2. linearized model

The single equation of thin fluid flow is

$$h_t + \frac{\rho}{3\mu}\left[h^3\left(-h_x g\cos\theta + g\sin\theta\right)\right]_x = 0 \qquad (1)$$

The derivation (1) can be seen in Wiryanto and Febrianti [1]. When the model involves surface tension, Wiryanto, et. al, [11] added extra term of fourth derivative of x. The equation is strongly non-linear of fluid thickness h, function of space x and time t, measured from the bottom of the channel. The density and viscosity are denoted by ρ and μ. g is acceleration of gravity, and θ is the angle of the bottom slope. Those parameters are constant, so that (1) has any constant solution $h = h_0$.

From the constant solution h_0, a perturbation η is given on it, so that we write

$$h = h_0 + \epsilon\eta$$

for small parameter ϵ. This form is then substituted to (1), and the equation of $O(\epsilon)$ is

$$\eta_t + \frac{\rho}{3\mu}\left[3h_0^2 g\sin\theta\eta_x - h_0^3 g\cos\theta\eta_{xx}\right] = 0 \tag{2}$$

Next, equation (2) is scaled by $\bar{\eta} = \eta/h_0$, $\bar{x} = x/L$ and $\bar{t} = t/\tau$, with $\tau = \mu L/(\rho h_0^2 g)$ so that we can rewrite (2) without bar $(\bar{.})$

$$\eta_t + a\eta_x + b\eta_{xx} = 0. \tag{3}$$

Here we use notation $a = \sin\theta, b = -\frac{1}{3}\delta\cos\theta$, and $\delta = h_0/L$ as the unity of the fluid thickness, compared to the wavelength.

Model of (3) is partial differential equation typical of parabolic. The characteristic of this equation indicates by changing the form of the wave η. But since it involves the first derivative of space, the wave propagates as the time increase. Both characteristics can be seen from the coefficient of each term in (3). The steepness of the bottom plays an important role in observing the wave characteristics. First, we analyze the amplitude of the wave, whether it increases or decreases during the propagation. We can observe by considering a monochromatic wave

$$\eta(x,t) = Ae^{-I(kx-\omega t)},$$

where $I = \sqrt{-1}$, k is the wave number, and ω is the wave frequency, expressed in complex form $\omega = \omega_r + I\omega_i$. So that the wave is written in form

$$\eta(x,t) = Ae^{-\omega_i t}e^{-I(kx-\omega_r t)}, \tag{4}$$

The wave amplitude $Ae^{(-\omega_i t)}$ is time dependent. Decreasing or increasing amplitude can be seen from ω_i relating to the quantities in (3). We substitute (4) into (3), and then it is arranged in real and imaginary parts, we have

$$\omega_i = \frac{1}{3}\delta k^2\cos\theta \tag{5}$$

which is positive since $0 \leq \theta \leq \pi/2$. Therefore, while propagating the wave changes its form by decreasing the amplitude.

3. NUMERICAL SOLUTION

Solution of (3) is proposed to be determined numerically. The chosen method is the implicit finite difference of Crank-Nicolson method. It is formulated by forward time and average center space. We discritize time and space by $t_n = ndt$, for $n = 0, 1, \ldots$ and $x_i = idx$, for $i = 0, 1, \ldots, P$. We use dt and dx as the step size for time and space, respectively. We then denote $\eta_i^n \approx \eta(x_i, t_n)$. From those, the difference equation of (3) is

$$A_1\eta_{i+1}^{n+1} + A_0\eta_i^{n+1} + A_{-1}\eta_{i-1}^{n+1} = B_1\eta_{i+1}^n + B_0\eta_i^n + B_{-1}\eta_{i-1}^n$$

for each i. The value of η with superscript $n+1$ should be determined, where the coefficients are as follows

$$A_1 = \frac{a}{4dx} + \frac{b}{2dx^2}, A_0 = \frac{1}{dt} - \frac{b}{dx^2}, A_{-1} = -\frac{a}{4dx} + \frac{b}{2dx^2},$$
$$B_1 = -\frac{a}{4dx} - \frac{b}{2dx^2}, B_0 = \frac{1}{dt} + \frac{b}{dx^2}, B_{-1} = \frac{a}{4dx} - \frac{b}{2dx^2}.$$

When we observe the wave in domain $x \in [0, L]$, and it is discritized into P number of points, we then have P linear equations for P unknowns. They can be written in

$$A\bar{\eta} = \bar{b}$$

with A is tridiagonal matrix. Gauss-Seidel iteration is then applied to that system. Diagonal dominant is required to get convergent iteration, i.e. $|A_0| \geq |A_1| + |A_{-1}|$. The method we propose here is unconditionally stable. In order to investigate the stability of the scheme, we express

$$\eta_i^n = U^n e^{I\psi i},$$

and substitute it to the finite difference equation so that we have

$$A_1 U^{n+1} e^{I\psi(i+1)} + A_0 U^{n+1} e^{I\psi i} + A_{-1} U^{n+1} e^{I\psi(i-1)}$$
$$= B_1 U^n e^{I\psi(i+1)} + B_0 U^n e^{I\psi i} + B_{-1} U^n e^{I\psi(i-1)}.$$

After some algebraic operations we obtain

$$U^{n+1} = GU^n,$$

where

$$G = \frac{-\frac{b}{4dx^2}\cos\psi + \frac{1}{dt} + \frac{b}{dx^2} - \frac{a}{8dx} I \sin\psi}{\frac{b}{4dx^2}\cos\psi + \frac{1}{dt} - \frac{b}{dx^2} + \frac{a}{8dx} I \sin\psi}$$

as the amplification factor.

The stability condition for the scheme is $|G| < 1$. From the above formulation we obtain

$$|G| = \frac{\sqrt{(-\frac{b}{4dx^2}\cos\psi)^2 + \frac{1}{dt^2} + \frac{2b}{dtdx^2}(\frac{-\cos\psi}{4} + 1) + (\frac{a}{8dx}\sin\psi)^2}}{\sqrt{(-\frac{b}{4dx^2}\cos\psi)^2 + \frac{1}{dt^2} - \frac{2b}{dtdx^2}(-\frac{\cos\psi}{4} + 1) + (\frac{a}{8dx}\sin\psi)^2}}.$$

Since $b = -\frac{1}{3}\delta\cos\theta < 0$ and $3/4 \leq -\cos\psi/4 + 1 \leq 5/4$, the numerator is smaller than the denominator. So that we obtain $|G| < 1$ for any ψ. Thus, the implicit scheme is unconditionally stable.

Now, we simulate some wave propagations. Most of our calculation uses $dx = 0.1$ and $dt = 0.1$ with observation domain $x \in [0, 100]$. As the initial wave, we use

$$\eta(x, 0) = \sin(0, 1\pi(x - 10))$$

for $x \in [10, 20]$ and zero outside the interval. The bottom inclination is set $\theta = 5$ degree, and $\delta = 1$. As the result, we present in Figure 1.(a). We plot wave profiles at some values of t. We can see that the wave propagates slowly, followed by amplitude decreasing as time increasing, and the shape changes from sinusoidal into flatter. Similar result can be seen in [2] calculated using FTCS method for the case of the linearized model. Each time step, the Gauss-Seidel method gets converge with no more than 15 iterations needed, with error tolerance 10^{-6} or more smaller. We also observe that using bigger step time dt, the simulation of wave deformation is carried out successfully.

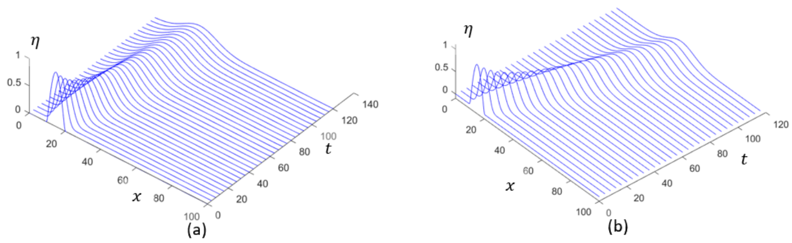

FIGURE 1. The wave simulation calculated from initial value $\eta(x,0) = \sin(0.1\pi(x-10))$

We compare the calculation for different value of the bottom inclination $\theta = 5$ degree and $\theta = 15$ degree as shown in Figure 1.(a) and Figure 1.(b). The wave propagation to the right is faster for bigger θ. This value corresponds to the coefficient of η_x, which is increasing for larger θ. Plot of $\eta(x, 120)$ is then shown in Figure 2.(a) for different inclinations, $\theta = 5, 15$ and 20 degrees, as indicated. Meanwhile, reducing the amplitude for those values of θ is slightly the same one. We show in Figure 2.(b), the plot of maximum value of η versus time t, for $\theta = 5$ and $\theta = 20$, bigger inclination slightly reduce the amplitude.

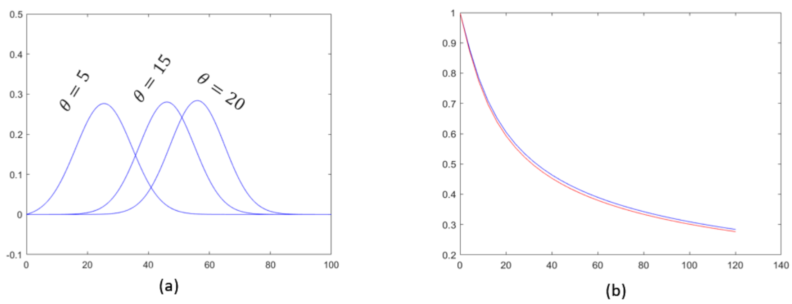

FIGURE 2. (a) Plot of $\eta(x, 120)$ for different inclination, $\theta = 5, 15$ and 20 degrees. (b) Plot η maximum versus t, $\theta = 5$ degree (lower curve) and $\theta = 20$ degree (upper curve)

In Figure 3.(a), we show the simulation of wave propagation similar to Figure 1.(b) of $\theta = 15$ degree, and the same initial wave, but for $\delta = 0.1$. We found that the wave amplitude decreases slower than using $\delta = 1$, as shown in Figure 1.(b).

This can be seen as that parameter corresponds to the coefficient of η_{xx}, where this term represents similar like in heat conduction equation. Increasing or decreasing of the value of η depends on the coefficient of that term. The decreasing amplitude is shown in Figure 3.(b). Smaller parameter δ produces wave simulation with a slightly decreasing in amplitude. We can see the comparison of that in Figure 3.(b) for $\delta = 1$ and $\delta = 0.1$.

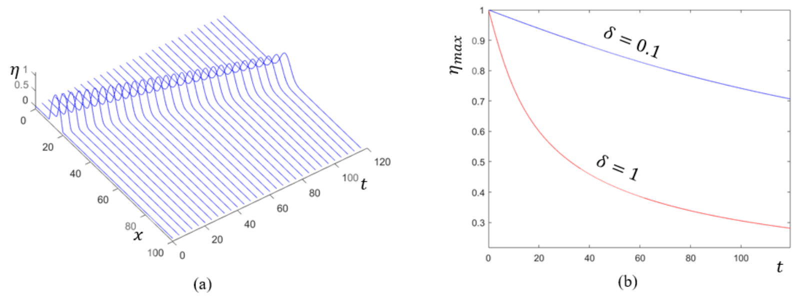

FIGURE 3. (a) Simulation of wave propagation for initial value $\eta(x, 0) = \sin(0.1\pi(x - 10))$ calculated using $\theta = 15$ degree and $\delta = 0.1$. (b) Plot of η_{max} versus t for the same θ, but $\delta = 1$ and $\delta = 0.1$ as indicated.

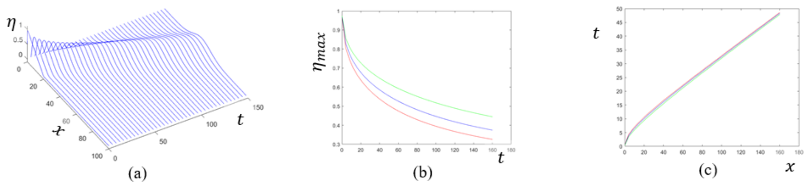

FIGURE 4. (a) Simulation of wave propagation for initial value $\eta(x, 0) = (20 - x)/20$ calculated using $\theta = 15$ degree and $\delta = 0.1$. (b) Plot of η_{max} versus t for the same θ, different parameters $\delta = 1$ and $\delta = 0.1$ as indicated. (c) Plot η_{max} in the $x - t$ plane.

For another initial value, we present the animation of the numerical solution of (3), using initial wave

$$\eta(x, 0) = (20 - x)/20,$$

for $x \in (0, 20)$ and zero outside the interval. Plot of the animation is shown in Figure 4.(a), as the result of the calculation using $\theta = 15$ degree and $\delta = 1$. The wave changes its shape followed by amplitude decreasing. The reducing amplitude is shown in Figure 4.(b) as the time increase. We can compare that plot for different parameter $\delta = 1, 0.75$ and 0.5, as indicated in Figure 4.(b). In $x - t$ plane, the position of the maximum value of η at each time t is slightly at the same position for different values of δ. We show in Figure 4.(c), plot of η_{max} in the $x-t$ plane. The maximum value of η at each step time t_n is collected and ploted in the $x-t$ plane.

4. CONCLUDING REMARKS

A linear model of thin film flow has been formulated and solved numerically by forward time central average space. Gauss-Seidel iteration was then used to calculate the solution. So that the model can simulate the surface wave, and the wave characteristic can be observed, i.e., the propagation of the wave by changing the form followed by amplitude decreasing. Two parameters play an important role in its propagation, i.e., the bottom inclination and the ratio between the fluid depth and the wave-length. Comparing to FTCS, the method can be used to calculate the solution for bigger step time. This is expected that the method can also be applied to the model involving surface tension.

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