

## ON CLASSES OF ANALYTIC FUNCTIONS CONTAINING GENERALIZATION OF INTEGRAL OPERATOR

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**Abstract.** New classes containing generalization of integral operator are introduced. Characterization and other properties of these classes are investigated. Further, Fekete-Szegö functional for these classes are also given.

*Key words and Phrases:* Integral operator, analytic functions, differential operator, Fekete-Szegö functional, distortion theorem.

**Abstrak.** Pada paper ini diperkenalkan kelas baru yang memuat perumuman dari operator integral. Kemudian karakterisasi dan sifat lain juga dikaji dari kelas. Lebih lanjut, fungsional Fekete-Szegö untuk kelas ini juga diberikan.

*Kata kunci:* Perumuman operator integral, fungsi analitik, operator diferensial, fungsional Fekete-Szegö, teorema distorsi.

### 1. Introduction

Let  $\mathcal{H}$  be the class of functions analytic in  $U$  and  $\mathcal{H}[a, n]$  be the subclass of  $\mathcal{H}$  consisting of functions of the form  $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ . Let  $\mathcal{A}$  be the subclass of  $\mathcal{H}$  consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \quad (1)$$

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Now we introduce a differential operator defined as follows :  $\mathbf{D}_{\lambda,\delta}^{k,\alpha,\beta} : \mathcal{A} \rightarrow \mathcal{A}$  by

$$\mathbf{D}_{\lambda,\delta}^{k,\alpha,\beta} f(z) = z + \sum_{n=2}^{\infty} [(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n) a_n z^n, \quad (2)$$

where  $k \in \mathbb{N}_0, \alpha \geq 0, \lambda \geq 0, \delta \geq 0$ ,  
and

$$C(\delta, n) = \binom{n+\delta-1}{\delta} = \frac{\Gamma(n+\delta)}{\Gamma(n)\Gamma(\delta+1)}.$$

**Remark 1.1.** When  $\alpha = \beta = 1, \lambda = 0, \delta = 0$  or  $\alpha = 0, \lambda = 1, \delta = 0$  we get Sălăgean differential operator [18],  $k = 0$  gives Ruschewyh operator [17],  $\alpha = 0, \delta = 0$  implies Al-Oboudi differential operator of order (k) [1],  $\alpha = \beta = 1, \lambda = 0$  or  $\alpha = 0, \lambda = 1$  operator (2) reduces to Al-shaqsi and Darus differential operator of order (k) [2] and  $\alpha = 0$  poses the differential operator of order (k), which is given by the authors [3]. Note that the differential operator in [3] was initially introduced by Al-Shaqsi and Darus [20]. However, it was overlooked and restated again in [3]. Recently, the authors posed different kind of linear, differential and integral operators (see [4-13,19]).

Given two functions  $f, g \in \mathcal{A}$ ,  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  and  $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$  their convolution or Hadamard product  $f(z) * g(z)$  is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U.$$

And for several functions  $f_1(z), \dots, f_m(z) \in \mathcal{A}$

$$f_1(z) * \dots * f_m(z) = z + \sum_{n=2}^{\infty} (a_{1n} \dots a_{mn}) z^n, \quad z \in U.$$

Analogous to  $\mathbf{D}_{\lambda,\delta}^{k,\alpha,\beta} f(z)$ ,  $z \in U$  we define an integral operator  $\mathbf{J}_{\lambda,\delta}^{k,\alpha} : \mathcal{A} \rightarrow \mathcal{A}$  as

follows.

Let

$$\phi(z) := \frac{z}{1-z} + \frac{\lambda z}{(1-z)^2} - \frac{\lambda z}{1-z}, \quad \lambda \geq 0.$$

$$\begin{aligned} F_k(z) &= \underbrace{\phi(z) * \dots * \phi(z)}_{k\text{-times}} * \underbrace{\frac{z\beta}{(1-z)^2} * \dots * \frac{z\beta}{(1-z)^2}}_{\alpha k\text{-times}} * \left[ \frac{z}{(1-z)^{\delta+1}} \right] \\ &= z + \sum_{n=2}^{\infty} [(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n) z^n, \end{aligned}$$

where  $k, \alpha \in \mathbb{N}_0, \beta \geq 1, \delta \geq 0, \lambda \geq 0$ .

And let  $F_k^{(-1)}$  be defined such that

$$\begin{aligned} F_k(z) * F_k^{(-1)} &= \frac{z}{1-z} \\ &= z + \sum_{n=2}^{\infty} z^n. \end{aligned}$$

Note that  $F_k^{(-1)}$  is needed to get the integral operator from differential operator.

Then

$$\begin{aligned} \mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z) &= F_k^{(-1)} * f(z) \\ &= \left[ \underbrace{\phi(z) * \dots * \phi(z)}_{k\text{-times}} * \underbrace{\frac{z\beta}{(1-z)^2} * \dots * \frac{z\beta}{(1-z)^2}}_{\alpha k\text{-times}} * \frac{z}{(1-z)^{\delta+1}} \right]^{(-1)} * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{a_n}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} z^n, \end{aligned} \tag{3}$$

where  $k, \alpha \in \mathbb{N}_0, \beta \geq 1, \lambda \geq 0, \delta \geq 0, z \in U$ .

**Remark 1.2.** When  $\alpha = 0, \lambda = 1, \delta = 0$  we get an integral operator (see [18]),  $k = 0$  gives Noor integral operator [15,16].

Some of relations for this integral operator are discussed in the next lemma.

**Lemma 1.1.** *Let  $f \in \mathcal{A}$ . Then*

- (i)  $\mathbf{J}_{\lambda, 0}^{0, \alpha, \beta} f(z) = f(z),$
- (ii)  $\mathbf{J}_{1, 0}^{1, 0, \beta} f(z) = \int_0^z \frac{f(t)}{t} dt = f(z) * (-\log(1-z)) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}.$

Note that (ii) is the type of Bernardi integral[21].

**Definition 1.1.** *Let  $f(z) \in \mathcal{A}$ . Then  $f(z) \in \mathbf{S}_{\lambda, \delta}^{k, \alpha, \beta}(\mu)$  if and only if*

$$\Re \left\{ \frac{z [\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)]'}{\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)} \right\} > \mu, \quad 0 \leq \mu < 1, \quad z \in U.$$

**Definition 1.2.** *Let  $f(z) \in \mathcal{A}$ . Then  $f(z) \in \mathbf{C}_{\lambda, \delta}^{k, \alpha, \beta}(\mu)$  if and only if*

$$\Re \left\{ \frac{[z (\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z))']'}{(\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z))'} \right\} > \mu, \quad 0 \leq \mu < 1, \quad z \in U.$$

The article is organized as follows: In section 2, we study the characterization and distortion theorems, and other properties of these classes. In section 3, we obtain

sharp upper bound of  $|a_2|$  and of the Fekete-Szegö functional  $|a_3 - \nu a_2^2|$  for the classes  $S_{\lambda, \delta}^{k, \alpha, \beta}(\mu)$  and  $C_{\lambda, \delta}^{k, \alpha, \beta}(\mu)$ . For this purpose we need the following result

**Lemma 1.2.**[14] *Let  $p \in \mathcal{P}$ , that is,  $p$  be analytic in  $U$ , be given by  $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$  and  $\Re\{p(z)\} > 0$  for  $z \in U$ . Then*

$$|p_2 - \frac{p_1^2}{2}| \leq 2 - \frac{|p_1|^2}{2}$$

and  $|p_n| \leq 2$  for all  $n \in \mathbb{N}$ .

## 2. General Properties of $J_{\lambda, \delta}^{k, \alpha, \beta}$

In this section we study the characterization properties and distortion theorems for the function  $f(z) \in \mathcal{A}$  to belong to the classes  $S_{\lambda, \delta}^{k, \alpha, \beta}(\mu)$  and  $C_{\lambda, \delta}^{k, \alpha, \beta}(\mu)$  by obtaining the coefficient bounds.

**Theorem 2.1.** *Let  $f(z) \in \mathcal{A}$ . If for  $\alpha \geq 0, \beta \geq 1, \delta \geq 0$  and  $\lambda \geq 0$*

$$\sum_{n=2}^{\infty} \frac{(n-\mu)|a_n|}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} \leq 1 - \mu, \quad 0 \leq \mu < 1, \quad (4)$$

then  $f(z) \in S_{\lambda, \delta}^{k, \alpha, \beta}(\mu)$ . The result (4) is sharp.

**Proof.** Suppose that (4) holds. Since

$$\begin{aligned} 1 - \mu &\geq \sum_{n=2}^{\infty} (n-\mu) \frac{|a_n|}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} \\ &\geq \sum_{n=2}^{\infty} \mu \frac{|a_n|}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} \\ &\quad - \sum_{n=2}^{\infty} \frac{n|a_n|}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} \end{aligned}$$

then this implies that

$$\frac{1 + \sum_{n=2}^{\infty} \frac{n|a_n|}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)}}{1 + \sum_{n=2}^{\infty} \frac{|a_n|}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)}} > \mu,$$

hence

$$\Re \left\{ \frac{z [J_{\lambda, \delta}^{k, \alpha, \beta} f(z)]'}{J_{\lambda, \delta}^{k, \alpha, \beta} f(z)} \right\} > \mu.$$

We also note that the assertion (4) is sharp and the extremal function is given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\mu)}{(n-\mu)} [(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n) z^n.$$

In the same way we can verify the following results

**Theorem 2.2.** *Let  $f(z) \in \mathcal{A}$ . If for  $\alpha \geq 0$ ,  $\beta \geq 1$ ,  $\delta \geq 0$  and  $\lambda \geq 0$*

$$\sum_{n=2}^{\infty} \frac{n(n-\mu)|a_n|}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} \leq 1 - \mu, \quad 0 \leq \mu < 1, \quad (5)$$

then  $f(z) \in \mathbf{C}_{\lambda, \delta}^{k, \alpha, \beta}(\mu)$ . The result (8) is sharp.

**Theorem 2.3.** *Let the hypotheses of Theorem 2.1 be satisfy. Then for  $z \in U$  and  $0 \leq \mu < 1$*

$$|\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)| \geq |z| - \frac{1 - \mu}{2 - \mu}$$

and

$$|\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)| \leq |z| + \frac{1 - \mu}{2 - \mu}.$$

**Proof.** By using Theorem 2.1, one can verify that

$$\begin{aligned} (2 - \mu) \sum_{n=2}^{\infty} \frac{|a_n|}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} \\ \leq \sum_{n=2}^{\infty} \frac{(n-\mu)|a_n|}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} \\ \leq 1 - \mu \end{aligned}$$

then

$$\sum_{n=2}^{\infty} \frac{|a_n|}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} \leq \frac{1 - \mu}{2 - \mu}.$$

Thus we obtain

$$\begin{aligned} |\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)| &\leq |z| + \sum_{n=2}^{\infty} \frac{|a_n|}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} |z|^n \\ &\leq |z| + \sum_{n=2}^{\infty} \frac{|a_n|}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} |z|^2 \\ &\leq |z| + \left[ \frac{1 - \mu}{2 - \mu} \right] |z|^2 \end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned}
|\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)| &= \left| z + \sum_{n=2}^{\infty} \frac{a_n}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} z^n \right| \\
&\geq \left| z - \sum_{n=2}^{\infty} \frac{a_n}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} z^n \right| \\
&\geq |z| - \sum_{n=2}^{\infty} \frac{|a_n|}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} |z|^n \\
&\geq |z| - \sum_{n=2}^{\infty} (n-\mu) [(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n) |a_n| |z|^2 \\
&\geq |z| - \left[ \frac{1-\mu}{2-\mu} \right] |z|^2
\end{aligned}$$

This complete the proof.

**Theorem 2.4.** *Let the hypotheses of Theorem 2.21 be satisfy. Then for  $z \in U$  and  $0 \leq \mu < 1$*

$$|\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)| \geq |z| - \frac{(1-\mu)}{2(2-\mu)} |z|^2$$

and

$$|\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)| \leq |z| + \frac{(1-\mu)}{2(2-\mu)} |z|^2.$$

**Theorem 2.5.** *Let the hypotheses of Theorem 2.1 be satisfy. Then*

$$\frac{(n-\mu)}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} \geq 1, \quad \forall n \geq 2 \text{ and } 0 \leq \mu < 1$$

implies

$$|f(z)| \geq |z| - (1-\mu) |z|^2$$

and

$$|f(z)| \leq |z| + (1-\mu) |z|^2.$$

**Proof.** In virtue of Theorem 2.1, we have

$$\sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \frac{(n-\mu) |a_n|}{[(\beta n)^\alpha + (n-1)(\beta n)^\alpha \lambda]^k C(\delta, n)} \leq (1-\mu)$$

then

$$\sum_{n=2}^{\infty} |a_n| \leq (1-\mu).$$

Thus we obtain

$$\begin{aligned} |f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n z^n \right| \\ &\leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^n \\ &\leq |z| + (1 - \mu) |z|^2 \end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned} |f(z)| &\geq \left| z - \sum_{n=2}^{\infty} a_n z^n \right| \\ &\geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n \\ &\geq |z| - (1 - \mu) |z|^2. \end{aligned}$$

This complete the proof.

**Theorem 2.6.** *Let the hypotheses of Theorem 2.2 be satisfy. Then*

$$\frac{(n - \mu)}{[(\beta n)^\alpha + (n - 1)(\beta n)^\alpha \lambda]^k C(\delta, n)} \geq 1, \forall n \geq 2 \text{ and } 0 \leq \mu < 1$$

poses

$$|f(z)| \geq |z| - \frac{(1 - \mu)}{2} |z|^2$$

and

$$|f(z)| \leq |z| + \frac{(1 - \mu)}{2} |z|^2.$$

### 3. Fekete-Szegö for the Classes $\mathbf{S}_{\lambda, \delta}^{k, \alpha, \beta}(\mu)$ and $\mathbf{C}_{\lambda, \delta}^{k, \alpha, \beta}(\mu)$

In this section we determine the sharp upper bound for  $|a_2|$  for the classes  $\mathbf{S}_{\lambda, \delta}^{k, \alpha, \beta}(\mu)$  and  $\mathbf{C}_{\lambda, \delta}^{k, \alpha, \beta}(\mu)$ . Moreover, we calculate the Fekete-Szegö  $|a_3 - \nu a_2^2|$  functional for them.

**Theorem 3.1.** *Let the hypotheses of Theorem 2.1 be satisfy. Then*

$$|a_2| \leq \frac{2(1 - \mu)[(\beta 2)^\alpha (1 + \lambda)]^k C(\delta, 2)}{1 + \mu}$$

and for all  $\nu \in \mathbb{C}$  the following bound is sharp

$$\begin{aligned} |a_3 - \nu a_2^2| &\leq 2(1 - \mu)[(3\beta)^\alpha (1 + 2\lambda)]^k C(\delta, 3) \\ &\max \left\{ 1, \left| 1 + \frac{2(1 - \mu)}{(1 + \mu)[(3\beta)^\alpha (1 + 2\lambda)]^k C(\delta, 3)} \right. \right. \\ &\quad \left. \left. - 2\nu \frac{1 - \mu}{(1 + \mu)^2} \frac{\{[(2\beta)^\alpha (1 + \lambda)]^k C(\delta, 2)\}^2}{[(3\beta)^\alpha (1 + 2\lambda)]^k C(\delta, 3)} \right| \right\}. \end{aligned}$$

**Proof.** Since  $f \in \mathbf{S}_{\lambda, \delta}^{k, \alpha, \beta}(\mu)$  then the condition

$$\Re \left\{ \frac{z[\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)]'}{\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)} \right\} > \mu, \quad 0 \leq \mu < 1, \quad z \in U$$

is equivalent to

$$z[\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)]' = (1 - \mu)p(z)\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z), \quad z \in U,$$

for some  $p \in \mathcal{P}$ . Equating coefficients we obtain  $a_2 = \frac{(1-\mu)Ap_1}{1+\mu}$ ,  $a_3 = (1-\mu)B(p_2 + \frac{(1-\mu)}{2+\mu}p_1^2)$  where  $A := [(2\beta)^\alpha(1+\lambda)]^k C(\delta, 2)$ ,  $B := [(3\beta)^\alpha(1+2\lambda)]^k C(\delta, 3)$  and further, for  $C := \frac{(1-\mu)^2}{1+\mu} + \frac{(1-\mu)B}{2} - \nu(\frac{(1-\mu)A}{1+\mu})^2$  and by using Lemma 1.2 we have  $|a_3 - \nu a_2^2| \leq H(x) = 2(1-\mu)B + (C - \frac{(1-\mu)B}{2})x^2$ ,  $x := |p_1|$ . Consequently, we receive

$$|a_3 - \nu a_2^2| = \begin{cases} H(0) = 2(1-\mu)B, & C \leq \frac{(1-\mu)B}{2} \\ H(2) = 4C, & C > \frac{(1-\mu)B}{2}. \end{cases}$$

Equality is attained for functions given by

$$\frac{z[\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)]'}{\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)} = \frac{1 + z^2(1 - 2\mu)}{1 - z^2}$$

and

$$\frac{z[\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)]'}{\mathbf{J}_{\lambda, \delta}^{k, \alpha, \beta} f(z)} = \frac{1 + z(1 - 2\mu)}{1 - z}$$

respectively.

For  $\mu = 0$  we receive the following corollary.

**Corollary 3.1.** *Let the assumptions of Theorem 3.1 hold. Then for  $\mu = 0$*

$$|a_2| \leq 2[(2\beta)^\alpha(1+\lambda)]^k C(\delta, 2)$$

and

$$|a_3 - \nu a_2^2| \leq 2[(3\beta)^\alpha(1+2\lambda)]^k C(\delta, 3) \max \left\{ 1, \left| 1 + \frac{2}{[(3\beta)^\alpha(1+2\lambda)]^k C(\delta, 3)} - 2\nu \frac{\{[(2\beta)^\alpha(1+\lambda)]^k C(\delta, 2)\}^2}{[(3\beta)^\alpha(1+2\lambda)]^k C(\delta, 3)} \right| \right\}.$$

In the similar manner we can prove the following result.

**Theorem 3.2.** *Let the hypotheses of Theorem 2.2 be satisfy. then*

$$|a_2| \leq \frac{2(1-\mu)[(2\beta)^\alpha(1+\lambda)]^k C(\delta, 2)}{3+\mu}$$



and for all  $\nu \in \mathbb{C}$  the following bound is sharp

$$|a_3 - \nu a_2^2| \leq \frac{2(1-\mu)}{3(2+\mu)} [(3\beta)^\alpha (1+2\lambda)]^k C(\delta, 3) \\ \max \left\{ 1, \left| \frac{1}{2} + \frac{2(1-\mu)}{3+\mu} \right. \right. \\ \left. \left. - \frac{3\nu(1-\mu)}{(3+\mu)^2} \{ [(2\beta)^\alpha (1+\lambda)]^k C(\delta, 2) \}^2 \right| \right\}.$$

For  $\mu = 0$  we receive the following corollary.

**Corollary 3.2.** *Let the assumptions of Theorem 3.2 hold. Then for  $\mu = 0$*

$$|a_2| \leq \frac{2}{3} [(2\beta)^\alpha (1+\lambda)]^k C(\delta, 2)$$

and

$$|a_3 - \nu a_2^2| \leq \frac{1}{3} [(3\beta)^\alpha (1+2\lambda)]^k C(\delta, 3) \max \left\{ 1, \left| \frac{1}{2} + \frac{2}{3} - \frac{\nu}{3} \{ [(2\beta)^\alpha (1+\lambda)]^k C(\delta, 2) \}^2 \right| \right\}.$$

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