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ON n-BOUNDED AND n-CONTINUOUS OPERATOR IN n-NORMED SPACE

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Abstract. In this paper, the concepts of bounded and continuous n-linear operators in n-normed space are discussed. The notions of n-bounded and n-continuous linear operators are then introduced as an extension. This is a generalization of the concepts introduced in [9] and [3]. In addition, the properties of the corresponding spaces of operators are studied to obtain results analogous to the case of normed space. Finally, a sufficient condition for each corresponding space of operators to be a Banach space is given.

Key words: n-normed space, n-Banach space, n-bounded operator, n-continuous operator.

Abstrak. Di dalam makalah ini, konsep operator n-linear di dalam ruang n-norm didiskusikan. Konsep operator linear n-terbatas dan n-kontinu kemudian diperkenalkan sebagai ekstensi. Ini adalah generalisasi dari konsep yang diperkenalkan di [9] dan [3]. Selanjutnya, properti dari ruang operator yang berkaitan dipelajari untuk mendapatkan hasil yang sesuai dengan kasus serupa di ruang norm. Akhirnya, sebuah syarat cukup agar tiap-tiap ruang operator yang berkaitan menjadi ruang Banach diberikan.

Kata kunci: ruang n-norm, ruang n-Banach, operator n-terbatas, operator n-kontinu.

1. Introduction and Preliminaries

Let X be a real vector space with $\dim(X) \geq n$, where n is a positive integer. We allow $\dim(X)$ to be infinite. A real-valued function $\|\cdot, \ldots, \cdot\| : X^n \to \mathbb{R}$ is called an *n-norm* on X^n if the following conditions hold:

(1) $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;
(2) $\|x_1, \ldots, x_n\|$ is invariant under permutations of $x_1, \ldots, x_n$;

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(3) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ for all $\alpha \in \mathbb{R}$ and $x_1, \ldots, x_n \in X$;
(4) $\|x_0 + x_1, x_2, \ldots, x_n\| \leq \|x_0, x_2, \ldots, x_n\| + \|x_1, x_2, \ldots, x_n\|$, for all $x_0, x_1, \ldots, x_n \in X$.

The pair $(X, \|\cdot, \ldots, \cdot\|)$ is then called an *n-normed space*. It also follows from the definition that an n-norm is always non-negative.

A standard example of n-normed space is $X = \mathbb{R}^n$ equipped with the following Euclidean n-norm:

$$\|x_1, \ldots, x_n\|_E := \text{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right),$$

where $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, \ldots, n$.

An important example would be when X is equipped with an inner product $\langle \cdot, \cdot \rangle$. In this case, we can define the *standard n-norm* on X by

$$\|x_1, \ldots, x_n\|_S := \sqrt{\det[\langle x_i, x_j \rangle]}$$

Note that the value of $\|x_1, \ldots, x_n\|_S$ is just the volume of the n-dimensional parallelepiped spanned by $x_1, \ldots, x_n$.

Initially, the theory of 2-normed space was developed by Gähler [6] as an extension of the usual norm. The theory of n-normed space was developed much later by Misiak [11]. A survey of the theory of 2-normed space can be found in [5]. More recent work in various aspects can be found in [7, 8, 9, 10].

Familiar notions such as boundedness and continuity in 2-normed space were then introduced by White in [12]. In [3], Chu et al. then defined the concepts of 2-continuity and 2-isometry as extensions to the usual continuity and isometry in 2-normed space for the purpose of studying the Aleksandrov problem. Related papers in this direction can be found in [1, 2]. In [9], Gozali et al. also introduced the notion of bounded n-linear functionals in n-normed space.

Motivated by all the concepts investigated in the above papers, we will introduce the notions of n-bounded and n-continuous operator in this paper as further extensions of the corresponding notions in [12], [9] and [3]. We will also study their properties. Furthermore, we have the following well-known theorem.

Theorem 1.1. *Let X and Y be normed spaces. Let $B(X, Y)$ be the space of all bounded linear operators from X to Y. Then $(B(X, Y), \|\cdot\|)$ is a normed space, where*

$$\|T\| := \sup_{x \in X, \|x\|=1} \|T(x)\|$$

Furthermore, if Y is a Banach space, then $B(X, Y)$ is a Banach space.

We will also present an extension of the above in the case of n-normed space.

2. Continuous, Bounded n-Linear Operator

Throughout this section, let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space and $(Y, \|\cdot\|)$ be a normed space. Following is an extension of the notion of bounded n-linear functional in n-normed space introduced in [9].

Definition 2.1. An operator $T : X^n \to Y$ is an *n-linear operator* on X if T is linear in each of the variable.
An n-linear operator is *bounded* if there is a constant k such that for all $(x_1, \ldots, x_n) \in X^n$,

$$\|T(x_1, \ldots, x_n)\| \leq k\|x_1, \ldots, x_n\| \tag{2.1}$$

If T is bounded, define $\|T\|$ to be

$$\|T\| := \sup_{\|x_1,\ldots,x_n\| \neq 0} \frac{\|T(x_1, \ldots, x_n)\|}{\|x_1, \ldots, x_n\|} \tag{2.2}$$

or equivalently

$$\|T\| = \sup_{\|x_1,\ldots,x_n\|=1} \|T(x_1, \ldots, x_n)\| \tag{2.3}$$

Note that when $n = 1$, the above reduces to the usual notion of bounded operator in normed space. Below is an example of such operator given in [9].

Example 2.2. Let $X = \mathbb{R}^n$ equipped with the Euclidean n-norm. Given the standard basis $\{e_1, \ldots, e_n\}$, define $F : X \to \mathbb{R}$ by $F(x_1, \ldots, x_n) = \det[\alpha_{ij}]$, where $x_i = \sum_{j=1}^{n} \alpha_{ij} e_j$, for $i = 1, \ldots, n$. Then F is bounded with $\|F\| = 1$.

Proposition 2.3. *Let $T : X^n \to Y$ be an n-linear operator. T is bounded if and only if for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in X^n$,*

$$\begin{aligned} \|T(x_1, x_2, \ldots, x_n) - T(y_1, y_2, \ldots, y_n)\| \leq & k(\|x_1 - y_1, x_2, \ldots, x_n\| + \|y_1, x_2 - y_2, \ldots, x_n\| \\ & + \ldots + \|y_1, y_2, \ldots, x_{n-1} - y_{n-1}, x_n\| \\ & + \|y_1, y_2, \ldots, y_{n-1}, x_n - y_n\|) \end{aligned} \tag{2.4}$$

Proof. Suppose (2.4) holds. Take $(y_1, \ldots, y_n) = (0, \ldots, 0)$, then the result follows. Conversely if T is bounded, then using n-linearity and triangle inequality,

$$\begin{aligned} & \|T(x_1, \ldots, x_n) - T(y_1, \ldots, y_n)\| \\ & = \|T(x_1 - y_1, x_2, \ldots, x_n) + T(y_1, x_2 - y_2, \ldots, x_n) + \ldots + T(y_1, y_2, \ldots, x_n - y_n)\| \\ & \leq \|T(x_1 - y_1, x_2, \ldots, x_n)\| + \|T(y_1, x_2 - y_2, \ldots, x_n)\| + \ldots + \|T(y_1, y_2, \ldots, x_n - y_n)\| \\ & \leq k(\|x_1 - y_1, x_2, \ldots, x_n\| + \|y_1, x_2 - y_2, \ldots, x_n\| + \ldots + \|y_1, y_2, \ldots, x_n - y_n\|) \end{aligned}$$

as required. □

Observe that if T is a bounded n-linear operator, and $x_1, \ldots, x_n$ are linearly dependent, then $T(x_1, \ldots, x_n) = 0$.

The following gives equivalent formulae for $\|T\|$.

Proposition 2.4. *Let T be a bounded n-linear operator. Then*

$$\|T\| = \inf\{k : \|T(x_1, \dots, x_n)\| \le k\|x_1, \dots, x_n\|, (x_1, \dots, x_n) \in X^n\} \tag{2.5}$$
$$= \sup_{\|x_1, \dots, x_n\| \le 1} \|T(x_1, \dots, x_n)\| \tag{2.6}$$
$$= \inf\{k : (2.4)\ holds\} \tag{2.7}$$

The proof of the above is almost identical to that in [9]. The proof of (2.7) is similar and is immediate from Proposition 2.3

Definition 2.5. An n-linear operator $T : X^n \to Y$ is *continuous* at $(x_1, \dots, x_n) \in X^n$ if for all $\varepsilon > 0$, there is $\delta > 0$, such that $\|T(x_1, \dots, x_n) - T(y_1, \dots, y_n)\| < \varepsilon$ whenever

$$\|x_1 - y_1, x_2, \dots, x_{n-1}, x_n\| < \delta \text{ and } \|y_1, x_2 - y_2, \dots, x_{n-1}, x_n\| < \delta \text{ and } \dots$$
$$\dots \text{ and } \|y_1, y_2, \dots, x_{n-1} - y_{n-1}, x_n\| < \delta \text{ and } \|y_1, y_2, \dots, y_{n-1}, x_n - y_n\| < \delta$$

OR

$$\|x_1 - y_1, y_2, \dots, y_{n-1}, y_n\| < \delta \text{ and } \|x_1, x_2 - y_2, \dots, y_{n-1}, y_n\| < \delta \text{ and } \dots$$
$$\dots \text{ and } \|x_1, x_2, \dots, x_{n-1} - y_{n-1}, y_n\| < \delta \text{ and } \|x_1, x_2, \dots, x_{n-1}, x_n - y_n\| < \delta$$

where $(y_1, \dots, y_n) \in X^n$.
T is continuous (on X^n) if it is continuous at each $(x_1, \dots, x_n) \in X^n$.

The above definition is an extension of that given by White in [12] for 2-normed space. Note that when $n = 1$, the above reduces to the usual notion of continuity in the normed space.

Next, we will relate the notion of boundedness and continuity.

Theorem 2.6. *Let $T : X^n \to Y$ be an n-linear operator. The following statements are equivalent:*

(1) *T is continuous.*
(2) *T is continuous at $(0, \dots, 0) \in X^n$.*
(3) *T is bounded.*

Proof. (1) $\Rightarrow$ (2) is trivial.
(2) $\Rightarrow$ (3): Suppose T is continuous at $(0, \dots, 0) \in X^n$. Then by definition, there is $\delta > 0$ such that $\|T(u_1, \dots, u_n)\| < 1$ whenever $\|u_1, \dots, u_n\| < \delta$. Now let $(x_1, \dots, x_n) \in X^n$. First consider the case when $\|x_1, \dots, x_n\| = 0$. By the continuity at $(0, \dots, 0) \in X^n$, note that there is $\delta_k > 0$ such that $\|T(x_1, \dots, x_n)\| < \frac{1}{k}$ whenever $\|x_1, \dots, x_n\| < \delta_k$. Then since $\|x_1, \dots, x_n\| = 0 < \delta_k$, we have $\|T(x_1, \dots, x_n)\| = 0$, i.e. $T \equiv 0$. Next, if $\|x_1, \dots, x_n\| \ne 0$, then let $u_i = \left(\frac{\delta}{4\|x_1, \dots, x_n\|}\right)^{1/n} x_i$, for $i = 1, \dots, n$. Note that $\|u_1, \dots, u_n\| = \delta/4 < \delta$. Then

$$\|T(u_1, \dots, u_n)\| = \frac{\delta}{4\|x_1, \dots, x_n\|}\|T(x_1, \dots, x_n)\|$$

Therefore

$$\|T(x_1,\dots,x_n)\| = \frac{4}{\delta}\|x_1,\dots,x_n\|\|T(u_1,\dots,u_n)\| \le \frac{4}{\delta}\|x_1,\dots,x_n\|$$

Hence T is bounded.
(3) $\Rightarrow$ (1): Since T is bounded, by Proposition 2.3,

$$\|T(x_1,\dots,x_n) - T(y_1,\dots,y_n)\| \le \|T\|(\|x_1-y_1,x_2\dots,x_n\| + \|y_1,x_2-y_2,\dots,x_n\| + \dots + \|y_1,y_2,\dots,x_n-y_n\|) \quad (2.8)$$

Let $\varepsilon > 0$ be given. Take $\delta = \frac{\varepsilon}{1+n\|T\|}$. If each of the the term in the brackets on the right-hand side of (2.8) is less than δ, then $\|T(x_1,\dots,x_n) - T(y_1,\dots,y_n)\| < \varepsilon$, showing that T is continuous. □

Next, we will study the corresponding space of operators. Let $B(X^n,Y)$ denotes the space of all bounded n-linear operators from X^n into Y.

Theorem 2.7. *$(B(X^n,Y),\|\cdot\|)$ is a normed space with norm given by* (2.2).

Proof. We need to show that $\|\cdot\|$ defined in (2.2) is a norm.
It is clear from the definition of $\|\cdot\|$ that $\|\alpha T\| = |\alpha|\|T\|$. Also,

$$\begin{aligned}\|T_1+T_2\| &\le \sup_{\|x_1,\dots,x_n\|\neq 0} \frac{\|T_1(x_1,\dots,x_n)\| + \|T_2(x_1,\dots,x_n)\|}{\|x_1,\dots,x_n\|} \\ &\le \sup_{\|x_1,\dots,x_n\|\neq 0} \frac{\|T_1(x_1,\dots,x_n)\|}{\|x_1,\dots,x_n\|} + \sup_{\|x_1,\dots,x_n\|\neq 0} \frac{\|T_2(x_1,\dots,x_n)\|}{\|x_1,\dots,x_n\|} \\ &= \|T_1\| + \|T_2\|\end{aligned}$$

Lastly, $\|T\| = 0$ implies $T(x_1,\dots,x_n) = 0$ if $\|x_1,\dots,x_n\| \neq 0$. If $\|x_1,\dots,x_n\| = 0$, then $x_1,\dots,x_n$ are linearly dependent, hence $T(x_1,\dots,x_n) = 0$ by the observation following Proposition 2.3. Hence $T \equiv 0$. Therefore, $\|\cdot\|$ is a norm. □

Finally, we have the following theorem which gives a sufficient condition for $(B(X^n,Y),\|\cdot\|)$ to be a Banach space.

Theorem 2.8. *If $(Y,\|\cdot\|)$ is a Banach space, then $(B(X^n,Y),\|\cdot\|)$ is a Banach space.*

Proof. Let $\{T_k\}$ be a Cauchy sequence in $B(X^n,Y)$. Let $\varepsilon > 0$ be given. Then there exists $N > 0$ such that $\|T_k - T_m\| \le \varepsilon/2$ for all $k,m > N$. By definition, we have

$$\|T_k(x_1,\dots,x_n) - T_m(x_1,\dots,x_n)\| \le \|T_k - T_m\|\|x_1,\dots,x_n\| \quad (2.9)$$

Therefore, for $k,m > N$, we have

$$\|T_k(x_1,\dots,x_n) - T_m(x_1,\dots,x_n)\| \le \frac{\varepsilon}{2}\|x_1,\dots,x_n\| \quad (2.10)$$

Using (2.9), since $\{T_k\}$ is Cauchy and Y is a Banach space, we may define $T(x) = \lim_{k\to\infty} T_k(x)$. Then, there exists $M > N$ such that

$$\|T_M(x_1,\dots,x_n) - T(x_1,\dots,x_n)\| \le \frac{\varepsilon}{2}\|x_1,\dots,x_n\| \quad (2.11)$$

Using (2.10) and (2.11), for all $k > M$,

$$\begin{aligned}\|T_k(x_1,\dots,x_n) - T(x_1,\dots,x_n)\| &\le \|T_k(x_1,\dots,x_n) - T_M(x_1,\dots,x_n)\| \\ &\quad + \|T_M(x_1,\dots,x_n) - T(x_1,\dots,x_n)\| \\ &\le \varepsilon\|x_1,\dots,x_n\|\end{aligned}$$

This implies $\|T_k - T\| < \varepsilon$, i.e. $T_k \to T$ as required. It is easy to check that $T \in B(X^n, Y)$, hence the statement is proven. □

3. n-CONTINUOUS, n-BOUNDED LINEAR OPERATOR

We will now generalize the concept of bounded operator by introducing the notion of n-bounded operator.

Throughout this section, let $(X, \|\cdot\|)$ be a normed space and $(X, \|\cdot,\dots,\cdot\|)$ be an n-normed space.

Definition 3.1. An operator $T : (X, \|\cdot\|) \to (X, \|\cdot,\dots,\cdot\|)$ is *n-bounded* if there is a constant k such that for all $x_1,\dots,x_n \in X$,

$$\|T(x_1), x_2,\dots,x_n\| + \|x_1, T(x_2),\dots,x_n\| + \dots + \|x_1, x_2,\dots,T(x_n)\| \le k\|x_1\|\dots\|x_n\|$$

If T is an n-bounded operator, define $\|T\|_n$ by

$$\begin{aligned}\|T\|_n := \inf\{k : &\|T(x_1), x_2,\dots,x_n\| + \|x_1, T(x_2),\dots,x_n\| \\ &+ \dots + \|x_1, x_2,\dots,T(x_n)\| \le k\|x_1\|\dots\|x_n\|,\ x_1, x_2,\dots,x_n \in X\}\end{aligned} \tag{3.1}$$

Again note that when $n = 1$, the above definition reduces to the usual concept of bounded operator. We will give some examples of such operator.

Example 3.2. Let $X = \mathbb{R}^2$ be equipped with the l_1-norm and the Euclidean 2-norm. Define operators T and T' by $T((x_1, x_2)) = (x_1, x_2)$ and $T'((x_1, x_2)) = (0, x_2)$, where $(x_1, x_2) \in \mathbb{R}^2$. Then $\|T\|_2 = 2$ and $\|T'\|_2 = 1$.

Example 3.3. Let $(X, \|\cdot\|)$ be a real inner product space and define $\|x, y\| = (\|x\|^2\|y\|^2 - \langle x, y\rangle^2)^{\frac{1}{2}}$. If $T : X \to X$ is a bounded linear operator, then $\|T(x), y\| + \|x, T(y)\| \le 2\|T\|\|x\|\|y\|$, and so T is a 2-bounded linear operator.

Below are some alternative formulae for $\|T\|_n$.

Proposition 3.4. *Let $T : X \to X$ be an n-bounded linear operator. Then*

$$\begin{aligned}\|T\|_n &= \sup\{\|T(x_1), x_2,\dots,x_n\| + \|x_1, T(x_2),\dots,x_n\| + \dots + \|x_1, x_2,\dots,T(x_n)\| \\ &\qquad : x_1, x_2,\dots,x_n \in X, \|x_1\|\|x_2\|\dots\|x_n\| = 1\} &(3.2)\\ &= \sup\left\{\frac{\|T(x_1), x_2,\dots,x_n\| + \|x_1, T(x_2),\dots,x_n\| + \dots + \|x_1, x_2,\dots,T(x_n)\|}{\|x_1\|\|x_2\|\dots\|x_n\|}\right. \\ &\qquad : x_1, x_2,\dots,x_n \in X, \|x_1\|\|x_2\|\dots\|x_n\| \ne 0\Big\} &(3.3)\end{aligned}$$

Proof. Let $M = \{\text{right-hand side of (3.2)}\}$. Then clearly $M \leq \|T\|_n$ by definition of n-bounded. Let $y_i = \frac{x_i}{\|x_i\|}$ for $i = 1, \ldots, n$. Then we have $\|T(y_1), \ldots, y_n\| + \ldots + \|y_1, \ldots, T(y_n)\| \leq M$. This implies $\|T\|_n \leq M$, and hence $\|T\|_n = M$.

Now let $N = \{\text{right-hand side of (3.3)}\}$. Then we have

$$\frac{\|T(x_1), \ldots, x_n\|}{\|x_1\| \ldots \|x_n\|} + \ldots + \frac{\|x_1, \ldots, T(x_n)\|}{\|x_1\| \ldots \|x_n\|} = \left\| T\left(\frac{x_1}{\|x_1\|}\right), \ldots, \frac{x_n}{\|x_n\|} \right\| + \ldots + \left\| \frac{x_1}{\|x_1\|}, \ldots, T\left(\frac{x_n}{\|x_n\|}\right) \right\|$$

Hence it follows that $N \leq M$. Moreover, writing M as

$$M = \sup\left\{ \frac{\|T(x_1), x_2, \ldots, x_n\| + \ldots + \|x_1, x_2, \ldots, T(x_n)\|}{\|x_1\|\|x_2\| \ldots \|x_n\|} : \|x_1\| \ldots \|x_n\| = 1 \right\}$$

we see that $M \leq N$ as the set over which the supremum is taken is bigger for N. □

We will now develop properties of n-bounded operator similar to that in previous section. To do so, we introduce the concept of n-continuity.

Definition 3.5. Let $T : (X, \|\cdot\|) \to (X, \|\cdot, \ldots, \cdot\|)$ be an operator. T is *n-continuous* at $x \in X$ if for all $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\|T(x_1) - T(x), x_2 - x, \ldots, x_n - x\| + \|x_1 - x, T(x_2) - T(x), \ldots, x_n - x\| + \ldots + \|x_1 - x, x_2 - x, \ldots, T(x_n) - T(x)\| < \varepsilon \tag{3.4}$$

whenever $\|x_1 - x\|\|x_2 - x\| \ldots \|x_n - x\| < \delta$, where $x_1, \ldots, x_n \in X$.
T is n-continuous (on X) if it is n-continuous at each $x \in X$.

Note that when $n = 1$, the above notion reduces to the usual continuity in normed space.

Theorem 3.6. *Let $T : (X, \|\cdot\|) \to (X, \|\cdot, \ldots, \cdot\|)$ be a linear operator. The following statements are equivalent:*

(1) *T is n-continuous.*
(2) *T is n-continuous at $0 \in X$.*
(3) *T is n-bounded*

Proof. (1) ⇒ (2) is trivial.
(2) ⇒ (3): Suppose T is n-continuous at $0 \in X$. Then by definition, there is a $\delta > 0$ such that $\|T(u_1), \ldots, u_n\| + \ldots + \|u_1, \ldots, T(u_n)\| < 1$ whenever $\|u_1\| \ldots \|u_n\| < \delta$. Let $(x_1, \ldots, x_n) \in X^n$. If $\|x_1\| \ldots \|x_n\| = 0$. Then $\|T(x_1), \ldots, x_n\| + \ldots + \|x_1, \ldots, T(x_n)\| = 0$. If $\|x_1\| \ldots \|x_n\| \neq 0$, let $u_i = \left(\frac{\delta}{4}\right)^{1/n} \frac{x_i}{\|x_i\|}$, for $i = 1, \ldots, n$. Note that $\|u_1\| \ldots \|u_n\| < \delta$. Then we have

$$\|T(u_1), \ldots, u_n\| + \ldots + \|u_1, \ldots, T(u_n)\| = \frac{\delta}{4\|x_1\| \ldots \|x_n\|}(\|T(x_1), \ldots, x_n\| + \ldots + \|x_1, \ldots, T(x_n)\|)$$

Therefore

$$\|T(x_1),\dots,x_n\| + \dots + \|x_1,\dots,T(x_n)\|$$
$$= \frac{4\|x_1\|\dots\|x_n\|}{\delta}(\|T(u_1),\dots,u_n\| + \dots + \|u_1,\dots,T(u_n)\|) < \frac{4}{\delta}\|x_1\|\dots\|x_n\|$$

Hence T is n-bounded.
(3) $\Rightarrow$ (1): Suppose T is n-bounded. Then

$$\|T(x_1-x),\dots,x_n-x\| + \dots + \|x_1-x,\dots,T(x_n-x)\| \le \|T\|_n\|x_1-x\|\dots\|x_n-x\|$$

Let $\varepsilon > 0$ be given. Let $\delta = \frac{\varepsilon}{1+\|T\|_n}$. Then by linearity of T,

$$\|T(x_1)-T(x),\dots,x_n-x\| + \dots + \|x_1-x,\dots,T(x_n)-T(x)\| < \varepsilon$$

whenever $\|x_1-x\|\dots\|x_n-x\| < \delta$. Hence T is n-continuous. □

Now we will study the corresponding space of operators. Let $B_n(X,X)$ denotes the space of all n-bounded linear operators from $(X,\|\cdot\|)$ to $(X,\|\cdot,\dots,\cdot\|)$.

Theorem 3.7. *$(B_n(X,X),\|\cdot\|_n)$ is a normed space with norm given by* (3.1).

Proof. It is easier to check using the formula in (3.3) that $\|\cdot\|_n$ satisfies $\|\alpha T\|_n = |\alpha|\|T\|_n$ and $\|T_1+T_2\|_n \le \|T_1\|_n + \|T_2\|_n$.
Also, $\|T\|_n = 0$ implies $\|T(x_1),\dots,x_n\| = 0$ for all $x_1,\dots,x_n \in X$, hence $T \equiv 0$. Therefore, $(B_n(X,X),\|\cdot\|)$ is a normed space. □

In the following, we need the concept of n-Banach space. A treatment of 2-Banach space can be found in [12]. The notion of n-Banach space and related concepts such as Cauchy sequence and convergence as given below are discussed briefly in [7].

Definition 3.8. A sequence $\{x_n\}$ in an n-normed space $(X,\|\cdot,\dots,\cdot\|)$ is said to *converge* to an $x \in X$ if

$$\lim_{k\to\infty}\|x_1,\dots,x_{n-1},x_k-x\| = 0$$

for all $x_1,\dots,x_{n-1} \in X$.

Definition 3.9. A sequence $\{x_n\}$ in an n-normed space $(X,\|\cdot,\dots,\cdot\|)$ is a *Cauchy sequence* if

$$\lim_{k,l\to\infty}\|x_1,\dots,x_{n-1},x_k-x_l\| = 0$$

for all $x_1,\dots,x_{n-1} \in X$.

Definition 3.10. If every Cauchy sequence in an n-normed space $(X,\|\cdot,\dots,\cdot\|)$ converges to an $x \in X$, then X is said to be *complete*. A complete n-normed space is called an *n-Banach space*.

Theorem 3.11. *$(B_n(X,X),\|\cdot\|_n)$ is a Banach space if $(X,\|\cdot,\dots,\cdot\|)$ is an n-Banach space.*

Proof. Let $\{T_k\}$ be a Cauchy sequence in $B_n(X,X)$. Let $\varepsilon > 0$ be given. Then there exists $N > 0$ such that $\|T_k - T_m\|_n \le \varepsilon/2$ for all $k, m > N$. By definition, we have

$$\|(T_k - T_m)(x_1), \ldots, x_n\| + \ldots + \|x_1, \ldots, (T_k - T_m)(x_n)\| \le \|T_k - T_m\|_n \|x_1\| \ldots \|x_n\| \tag{3.5}$$

Therefore, for $k, m > N$, we have

$$\|T_k(x_1) - T_m(x_1), \ldots, x_n\| + \ldots + \|x_1, \ldots, T_k(x_n) - T_m(x_n)\| \le \frac{\varepsilon}{2}\|x_1\| \ldots \|x_n\| \tag{3.6}$$

Since $\{T_k\}$ is Cauchy, by definition of $\|\cdot\|_n$, we have $\|T_k(x_1) - T_m(x_1), x_2, \ldots, x_n\|$, $\ldots, \|x_1, x_2, \ldots, T_k(x_n) - T_m(x_n)\| \to 0$ as $k, m \to \infty$ for all $x_1, \ldots, x_n \in X$. This implies the sequence $\{T_k(x)\}$ is Cauchy in $(X, \|\cdot, \ldots, \cdot\|)$ for all $x \in X$.
Since $(X, \|\cdot, \ldots, \cdot\|)$ is an n-Banach space, we may define $T(x) = \lim_{k\to\infty} T_k(x)$ in the sense of n-norm. By definition of convergence, there exists $M = M(x_1, \ldots, x_n) > N$ such that

$$\|T_M(x_1) - T(x_1), \ldots, x_n\| + \ldots + \|x_1, \ldots, T_M(x_n) - T(x_n)\| \le \frac{\varepsilon}{2}\|x_1\| \ldots \|x_n\| \tag{3.7}$$

for all $x_1, \ldots, x_n \in X$.
Using (3.6) and (3.7) and triangle inequality for n-norm, for all $k > M$ and $x_1, \ldots, x_n \in X$,

$$\|T_k(x_1) - T(x_1), \ldots, x_n\| + \ldots + \|x_1, \ldots, T_k(x_n) - T(x_n)\| \le \varepsilon\|x_1\| \ldots \|x_n\|$$

This implies $\|T_k - T\|_n < \varepsilon$, i.e. $T_k \to T$ as required. It is easy to check that $T \in B_n(X,X)$, hence the statement is proven. □

4. Other Notions of n-Continuous, n-Bounded Operator

There is another notion of continuity in n-normed spaces other than those introduced in the previous sections. In [3], Chu et al. introduced the notion of 2-continuous mapping to study the Aleksandrov problem in 2-normed space. Motivated by this paper, we will generalize this concept.

Let $(X, \|\cdot, \ldots, \cdot\|)$ and $(Y, \|\cdot, \ldots, \cdot\|)$ be n-normed spaces.

Definition 4.1. An operator $T : (X, \|\cdot, \ldots, \cdot\|) \to (Y, \|\cdot, \ldots, \cdot\|)$ is *n-bounded of type-I* if there is a constant k such that for all $x_1, \ldots, x_n \in X$,

$$\|T(x_1), \ldots, T(x_n)\| \le k\|x_1, \ldots, x_n\|$$

If T is an n-bounded of type-I operator, define $[T]_n$ by

$$[T]_n := \sup_{\|x_1,\ldots,x_n\| \ne 0} \frac{\|T(x_1), \ldots, T(x_n)\|}{\|x_1, \ldots, x_n\|} \tag{4.1}$$

Below are some examples of such operator.

Example 4.2. Let $T : (X, \|\cdot, \dots, \cdot\|) \to (Y, \|\cdot, \dots, \cdot\|)$ be a dilation, i.e. $T(x) = cx$ for all $x \in X$, where $c \in \mathbb{R}$. Then T is n-bounded of type-I.

Example 4.3. Let $X = C^1[0, 1]$, equipped with 2-norm defined by $\|f, g\| = \sup_{t\in[0,1]} W(f, g)(t)$ for $f, g \in X$, where $W(f, g)$ is the Wronskian of f and g. Let $T : X \to X$ be defined by $T(x) = y$, where $y(t) = tx(t)$. Then $\|T(f), T(g)\| \leq \|f, g\|$ for all $f, g \in X$, and so T is 2-bounded.

Definition 4.4. Let $T : X \to Y$ be an operator. T is *n-continuous of type-I* at $x \in X$ if for all $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\|T(x_1) - T(x), T(x_2) - T(x), \dots, T(x_n) - T(x)\| < \varepsilon \tag{4.2}$$

whenever $\|x_1 - x, x_2 - x, \dots, x_n - x\| < \delta$, where $x_1, \dots, x_n \in X$.
T is n-continuous of type-I (on X) if it is n-continuous of type-I at each $x \in X$.

When $n = 1$, the above reduces to the usual notion of continuity in normed space. When $n = 2$, it reduces to the notion of 2-continuity introduced by Chu et al. in [3].

Theorem 4.5. *Let $T : X \to Y$ be a linear operator. Then the following statements are equivalent:*

(1) *T is n-continuous of type-I.*
(2) *T is n-continuous of type-I at $0 \in X$.*
(3) *T is n-bounded of type-I*

Proof. (1) $\Rightarrow$ (2) is trivial.
(2) $\Rightarrow$ (3): Suppose T is n-continuous of type-I at $0 \in X$. Then by definition there is a $\delta > 0$ such that $\|T(u_1), \dots, T(u_n)\| < 1$ whenever $\|u_1, \dots, u_n\| < \delta$. Now let $x_1, \dots, x_n \in X$. If $\|x_1, \dots, x_n\| = 0$, then $x_1, \dots, x_n$ are linearly dependent. By linearity of T, $T(x_1), \dots, T(x_n)$ are also linearly dependent, hence $\|T(x_1), \dots, T(x_n)\| = 0$. Next, if $\|x_1, \dots, x_n\| \neq 0$, let $u_i = \left(\frac{\delta}{4\|x_1,\dots,x_n\|}\right)^{1/n} x_i$, for $i = 1, \dots, n$. Note that $\|u_1, \dots, u_n\| < \delta$. Then we have

$$\|T(u_1), \dots, T(u_n)\| = \frac{\delta}{4\|x_1, \dots, x_n\|}\|T(x_1), \dots, T(x_n)\|$$

Therefore,

$$\|T(x_1), \dots, T(x_n)\| = \frac{4\|x_1, \dots, x_n\|}{\delta}\|T(u_1), \dots, T(u_n)\| < \frac{4}{\delta}\|x_1, \dots, x_n\|$$

Hence, T is n-bounded of type-I.
(3) $\Rightarrow$ (1): The proof is similar to the corresponding part in Theorem 3.6. $\square$

We have shown that our notions of n-bounded of type-I and n-continuous of type-I are equivalent. However, as our natural definition of $[\cdot]$ arising from the above is not a norm, we could not have the analogue of Theorem 2.8 here.

Motivated by the definitions introduced in the previous section, we will modify the definitions slightly. Subsequently, let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space.

Definition 4.6. Let $T : (X, \|\cdot, \ldots, \cdot\|) \to (X, \|\cdot, \ldots, \cdot\|)$ be an operator. T is *n-bounded of type-II* if there is a constant k such that for all $x_1, \ldots, x_n \in X$,

$$\|T(x_1), x_2, \ldots, x_n\| + \|x_1, T(x_2), \ldots, x_n\| + \ldots + \|x_1, x_2, \ldots, T(x_n)\| \\ \leq k\|x_1, x_2, \ldots, x_n\| \tag{4.3}$$

If T is an n-bounded of type-II operator, define $\| T \|_n$ by

$$\| T \|_n = \sup\left\{\frac{\|T(x_1), x_2, \ldots, x_n\| + \ldots + \|x_1, x_2, \ldots, T(x_n)\|}{\|x_1, x_2, \ldots, x_n\|}, \|x_1, x_2, \ldots, x_n\| \neq 0\right\} \tag{4.4}$$

Below is an example of such operator.

Example 4.7. Let $T : X \to X$ be a dilation, i.e. $T(x) = cx$ for all $x \in X$, where $c \in \mathbb{R}$. Then T is n-bounded of type-II.

Definition 4.8. Let $T : X \to X$ be an operator. T is *n-continuous of type-II* at $x \in X$ if for all $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\|T(x_1) - T(x), x_2 - x, \ldots, x_n - x\| + \|x_1 - x, T(x_2) - T(x), \ldots, x_n - x\| + \\ \ldots + \|x_1 - x, x_2 - x, \ldots, T(x_n) - T(x)\| < \varepsilon \tag{4.5}$$

whenever $\|x_1 - x, x_2 - x, \ldots, x_n - x\| < \delta$, where $x_1, \ldots, x_n \in X$.
T is n-continuous of type-II (on X) if it is n-continuous of type-II at each $x \in X$.

Theorem 4.9. *Let $T : X \to X$ be a linear operator. Then the following statements are equivalent:*

(1) *T is n-continuous of type-II.*
(2) *T is n-continuous of type-II at $0 \in X$.*
(3) *T is n-bounded of type-II.*

Proof. (1) $\Rightarrow$ (2) is trivial.
(2) $\Rightarrow$ (3): The proof is similar to the corresponding part of Theorem 3.6. Suppose T is n-continuous of type-II at $0 \in X$. Then by definition, there is a $\delta > 0$ such that $\|T(u_1), \ldots, u_n\| + \ldots + \|u_1, \ldots, T(u_n)\| < 1$ whenever $\|u_1, \ldots, u_n\| < \delta$. Let $x_1, \ldots, x_n \in X$. Now, let $u_i = \left(\frac{\delta}{4\|x_1, \ldots, x_n\|}\right)^{1/n} x_i$, for $i = 1, \ldots, n$. Note that $\|u_1, \ldots, u_n\| < \delta$. Then we have

$$\|T(u_1), \ldots, u_n\| + \ldots + \|u_1, \ldots, T(u_n)\| \\ = \frac{\delta}{4\|x_1, \ldots, x_n\|}(\|T(x_1), \ldots, x_n\| + \ldots + \|x_1, \ldots, T(x_n)\|)$$

Therefore

$$\|T(x_1), \ldots, x_n\| + \ldots + \|x_1, \ldots, T(x_n)\| \\ = \frac{4\|x_1, \ldots, x_n\|}{\delta}(\|T(u_1), \ldots, u_n\| + \ldots + \|u_1, \ldots, T(u_n)\|) < \frac{4}{\delta}\|x_1, \ldots, x_n\|$$

Hence T is n-bounded of type-II.
(3) $\Rightarrow$ (1): The proof is similar to the corresponding part of Theorem 3.6. □

We can now study the corresponding space of operators. Let $B^n_n(X,X)$ denotes the space of all n-bounded of type-II linear operators from X to X.

Theorem 4.10. *$(B^n_n(X,X), \|\!| \cdot \|\!|_n)$ is a normed space with norm given by (4.4).*

Proof. The proof is similar to Theorem 3.7. □

Theorem 4.11. *$(B^n_n(X,X), \|\!| \cdot \|\!|_n)$ is a Banach space if $(X, \|\cdot,\ldots,\cdot\|)$ is an n-Banach space.*

Proof. The proof is similar to Theorem 3.11 □

Remark 4.12. It appears that the type-I definition is somewhat more natural and allows us to get a larger class of operator satisfying such conditions. On the other hand, the type-II definition seems restrictive but allows us to get an analogue of Theorem 2.8. Better definition which achieves these two remains to be seen.

Remark 4.13. Yet another way to resolve this in some special cases is by realizing the n-normed space as a normed space via the method described in [7]. In particular for finite-dimensional case, the convergence and completeness in n-norm is equivalent to that in the derived norm, and our results in section 3 can be used.

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REFERENCES

1. X.Y. Chen, M.M. Song, *Characterizations on isometries in linear n-normed spaces*, Nonlinear Anal. TMA **72** (2010), 1895–1901.
2. H.Y. Chu, C. Park, W. Park, *The Aleksandrov problem in linear 2-normed spaces*, J. Math. Anal. Appl. **289** (2004), 666–672.
3. H.Y. Chu, S.H. Ku, D.S. Kang, *Characterizations on 2-isometries*, J. Math. Anal. Appl. **340** (2008), 621–628.
4. H.Y. Chu, K.H. Lee, C.K. Park, *On the Aleksandrov problem in linear n-normed spaces*, Nonlinear Anal. TMA **59** (2004), 1001–1011.
5. R. Freese and Y.J. Cho, *Geometry of linear 2-normed spaces*, Nova Science Publ., New York, 2001.
6. S. Gähler, *Lineare 2-normierte räume*, Math. Nachr. **28** (1964), 1–43.
7. H. Gunawan and Mashadi, *On n-normed spaces*, Int. J. Math. Math. Sci. **27** (2001), 631–639.
8. H. Gunawan, *The space of p-summable sequences and its natural n-norms*, Bull. Austral. Math. Soc. **64** (2001), 137–147.
9. S.M. Gozali, H. Gunawan and O. Neswan, *On n-norms and n-bounded linear functionals in a Hilbert space*, Ann. Funct. Anal. **1** (2010), 72–79.
10. S.S. Kim and Y.J. Cho, *Strict convexity in linear n-normed spaces*, Demonstratio Math. **29** (1996), no. 4, 739–744.
11. A. Misiak, *n-inner product spaces*, Math. Nachr. **140** (1989), 299–319.
12. A. White, *2-Banach spaces*, Math. Nachr. **42** (1969), 43–60.