J. Indones. Math. Soc.
Vol. 28, No. 02 (2022), pp. 215–220.

THE NECESSARY AND SUFFICIENT CONDITION OF $e \rightharpoonup C \leftharpoonup e$ AS A CLEAN R-COALGEBRA

Nikken P. Puspita[1], Indah Emilia Wijayanti[2], Budi Surodjo[2]

[1]**Mathematics Department, Universitas Diponegoro, Semarang, Indonesia, nikkenprima@lecturer.undip.ac.id**

[2]**Department of Mathematics, Universitas Gadjah Mada, Yogyakarta, Indonesia**

Abstract. Let R be a commutative ring with multiplicative identity and C be a coassociative and counital R-coalgebra with the α-condition. A clean comodules defined based on the cleanness on rings and modules. A C-comodule M is a clean comodule if the endomorphism ring of C-comodule M is clean. A clean R-coalgebra C is a clean comodule over itself i.e., if the endomorphism ring of C as a C-comodule is clean. For an idempotent $e \in R$, there are relations between the cleanness of eRe and R. It's motivated us to investigate this condition for coalgebra. For any C, we can construct the R-coalgebra $e \rightharpoonup C \leftharpoonup e$ where e is an idempotent element of dual algebra of C. Here, we show that the clean conditions of C implies the clean property of $e \rightharpoonup C \leftharpoonup e$ and vice versa.

Keywords: coalgebra, clean coalgebra, clean comodule, corner ring

1. INTRODUCTION

Throughout R is a commutative ring with multiplicative identity, (C, Δ, ε) a coalgebra over R, which we assume to be coassociative and counital over multiplication. In the beginning, we need to recall some notions in ring theory, i.e., clean rings, which is introduced by [1]. A ring R is said to be clean where any element of R can be expressed as a sum of e and u, where e is an idempotent and u is a unit in R. The examples of clean rings are Von Neumann regular rings and fields. The clean rings is a subclass of the exchange rings. Some authors studying the property of being clean or the exchange property are [1, 2, 3, 4, 6, 7, 8, 9].

Any R can be considered as a module over itself. Furthermore, the endomorphism ring of R over R (i.e., $End_R(R)$) isomorphic to ring R. Consequently, R is clean if and only if R is also a clean module over R. Some researchers have been

2020 Mathematics Subject Classification: 16T15

Received: 09-10-2021, accepted: 03-01-2022.

developed the cleanness concept of modules from rings by investigated the cleanness on the endomorphism ring of M [10, 11]. Motivated by this, clean modules as a generalization of clean rings have been introduced by [12, 13]. Let M be a module over R. The module M is said to be clean if the endomorphism ring of M over R (denoted by $End_R(M)$) is clean.

In [14], Sweedler introduced the dualization of the algebra over a field. He has introduced the theory of coalgebra over a field. The concept of a coalgebra over a field can be generalized to the concept of a coalgebra over a ring [15]. In [15] reversing the scalar multiplication between M and R, we also have comodules over a coalgebra as a dualization of module. In this paper, the homomorphisms R-module C to R denoted by C^*, i.e., $C^* = Hom_R(C, R)$. Here, C^* is an algebra (ring) over a convolution product

$$f * g = \mu \circ (f \otimes g) \circ \Delta \tag{1}$$

where μ is a multiplication of R and Δ is the comultiplication of C. It is interesting since any right C-comodule M, then M is a left module over the dual algebra C^* by the following multiplication:

$$\rightharpoonup : C^* \otimes_R M \to M, f \otimes m \mapsto (I_M \otimes f) \circ \varrho^M(m) = \sum m_{\underline{0}} f(m_{\underline{1}}). \tag{2}$$

Hence the category of right C-comodule is a subcategory of left C^*-module, since every C-comodule morphism is a C^*-module morphism [15]. Analog for the left side, the category of left C-comodule is the subcategory of right C^*-module.

Moreover, to guarantee that the right C-comodule category (denoted by $\mathbf{M}^C$) to be a full subcategory of the category of left C^*-module (denoted by ${}_{C^*}\mathbf{M}$), we need to study the α-condition. Following [15] the R-coalgebra C satisfies the α-condition provided for any R-module L, the following map is injective, i.e.,

$$\alpha_L : L \otimes_R C \to Hom_R(C^*, L), x \otimes c \mapsto [f \mapsto f(c)x].$$

The α-condition of C implies that the ring of right C-comodule endomorphisms of M is isomorphic to the ring of left C^*-module endomorphisms of M. As the special case when we consider any C is a (right and left) comodule over itself, we have a nice relationship i.e., $End^C(C) \cong {}_{C^*}End(C) \cong C^*$ and ${}^C End(C) \cong End_{C^*}(C) \cong C^*$ [15]. In this paper, we assumed that C satisfies the α condition such that these fact would be used to get our goal.

For any ring R, we can consider R as a trivial coassociative and counital coalgebra over itself by comultiplication $\Delta_T : R \to R \otimes_R R, r \mapsto r \otimes 1$ and counit $\varepsilon_T = I_T$ (the identity map of R). Moreover, any R-module M can be considering as a comodule over R-coalgebra R by trivial coaction $\varrho_T^M : M \to M \otimes_R R, m \mapsto m \otimes 1$. Based on this fact, we have that ring R is clean if and only if the endomorphism ring of R-coalgebra R is a clean ring and the R-module M is clean if and only if the endomorphism ring of R-comodule M is clean. Hence, [16, 17] bring the cleanness concept of the category RING and the category of modules ($\mathbf{M}_R$) to the category of C-comodule ($\mathbf{M}^C$).

The clean comodules also can be considered as a generalization of clean modules since any module is a trivial R-comodule. Clean comodules are defined as below:

Definition 1.1. [16, 17] *Let (C, Δ, ε) be R-coalgebra (coassociative and counital). A right (left) C-comodule M is said to be clean if endomorphism ring of C-comodule M (denoted by $End^C(M)$) is clean.*

Definiton 1.1 means that for any $f \in End^C(M), f = u + e$ where $e \in End^C(M)$ is an idempotent and $u \in End^C(M)$ is a unit. In this paper, since we assume that C satisfies the α-condition, then $End^C(M) \cong {}_{C^*}End(M)$. Thus, M is clean as a comodule over C if and only if M is a clean C^*-module.

Let C with comultiplication $\Delta : C \to C \otimes_R C$. We can consider Δ as a coaction of C. As on the ring theory, we have that C is a comodule over itself [15]. In special case, take $M = C$ (in Definition 1.1). On the following, we introduce a clean coalgebra. On the other hand, clean comodules is a generalization of clean coalgeras by taking $M = C$.

Definition 1.2. [16, 17] *Let R be a ring. An R-coalgebra C is called a clean coalgebra if C is a clean comodule over itself.*

Definition 1.2 means that C is a clean coalgebra if C is clean as a C^*-module. Recall the properties of coalgebras in [15]. Any R-coalgebra C can be consider as a right and left comodule over itself. Therefore, C is a left and right C^*-module. Since we assume that C satisfies the α-condition, we have that C is a clean R-coalgebra if $End_{C^*}(C) \cong {}_{C^*}End(C) \cong C^*$ is a clean ring. In some article, the cleanness concept corner ring has been studied, i.e., R is clean if and only if eRe is clean for e an idemotent element of R.

For any ring R and an idempotent $e \in R$, eRe is called a corner ring. Motivated by the corner on the category $RING$, we introduced the analogous notion of corner coalgebra under the multiplication " $\rightharpoonup$ " and " $\leftharpoonup$ " between C and C^* on Equation 2. Consider an R-coalgebra (C, Δ, ε) and an idempotent $e \in C^*$, we can construct the R-coalgebra $e \rightharpoonup C \leftharpoonup e$ based on the Equation 2 (right and left multiplication) and comultiplication Δ.

Definition 1.3. *Let C be a coassociative and counital coalgebra and an idempotent $e \in C^*$. The R-coalgebra $e \rightharpoonup C \leftharpoonup e$ is called a corner R-coalgebra.*

It is interesting to investigate the cleanness properties of corner coalgebra $e \rightharpoonup C \leftharpoonup e$ for $e \in C^*$, and for proving the main results of this paper, here we need to use the Morita Theorem.

In the Morita context, we have already known there is a relationship between the structure of P, P^*, R and $S = End_R(P)$ [18]. The following theorem explains the relationship between P and its dual, in which it is essential to prove our main result.

Theorem 1.4. [18] *Let R be a ring, P is a right R-module, $S = End_R(P)$ and $Q = P^* = Hom_R(P, R)$. If P is a generator in $R - MOD$, then*

(1) $\alpha : Q \otimes_S P \to R$ *is an* (R,R)*-isomorphism;*
(2) $Q \simeq Hom_S({}_SP, {}_SS)$ *as* (R,S)*-bimodules;*
(3) $P \simeq Hom_S(Q_S, S_S)$ *as* (S,R)*-bimodules;*
(4) $R \simeq End({}_SP) \simeq End(Q_S)$ *as rings.*

Theorem 1.5. [18] *Let* R *be a ring,* P *be a right* R*-module,* $S = End_R(P)$ *and* $Q = P^* = Hom_R(P,R)$*. If* P *is finitely generated projective in* $R - MOD$*, then*

(1) $\beta : P \otimes_R Q \to S$ *is an* (S,S)*-isomorphism;*
(2) $Q \simeq Hom_R(P_R, R_R)$ *as* (R,S)*-bimodules;*
(3) $P \simeq Hom_R({}_RQ, {}_RR)$ *as* (S,R)*-bimodules;*
(4) $S \simeq End(P_R) \simeq End({}_RQ)$ *as rings.*

On our main result, we are going to use the Theorem 1.5 and Theorem 1.4 to prove our main theorem. The main theorem of this article is to give the necessary and sufficient condition of clean $e \rightharpoonup C \leftharpoonup e$ where e is an idempotent of C^*.

2. MAIN RESULTS

Given a coassociative and counital R-coalgebra (C, Δ, ε) with the α-condition and e be an idempotent element of C^*. The relationship between the clean ring of R and eRe motivate us to investigate the condition such that R-coalgebra $e \rightharpoonup C \leftharpoonup e$ is clean. In their article, [9] showed that I is an ideal if and only if their exist idempotent set $\{e_1, e_2, \cdots, e_n\}$ such that e_iRe_i is a clean ideal of e_iRe_i, for all i. In trivial cased $I = R$, we have R is clean if and only if e_iRe_i is a clean ring, for all i. We will observe this condition for coalgebra cases and find the sufficient and necessary condition of clean R-coalgebra $e_i \rightharpoonup C \leftharpoonup e_i$, $i = 1$.

Theorem 2.1. *If* C *is a cocommutative clean* R*-coalgebra and* e *is an idempotent in* C^**, then* R*-coalgebra* $e \rightharpoonup C \leftharpoonup e$ *is also clean.*

Proof. Let C be a clean R-coalgebra. The ring ${}_{C^*}End(C) \cong End^C(C) \cong C^*$ is clean. Since C is locally projective, from [15], $e \rightharpoonup C \leftharpoonup e$ is also locally projective and implies

$$End_{(e \rightharpoonup C \leftharpoonup e)^*}(e \rightharpoonup C \leftharpoonup e) \cong (e \rightharpoonup C \leftharpoonup e)^* \cong e * C^* * e.$$

This means, the cleanness of R-coalgebra $e \rightharpoonup C \leftharpoonup e$ can be determined by the structure of ring $e * C^* * e$.
The R-coalgebra C is cocommutative if and only if the dual algebra C^* is commutative (by [15]). Since C^* is a clean ring and commutative, C^* is a strongly clean ring. By [9], $e * C^* * e$ is also strongly clean. Hence, $(e \rightharpoonup C \leftharpoonup e)^* \cong e * C^* * e$ is a clean ring. Consequently, $e \rightharpoonup C \leftharpoonup e$ is a clean R-coalgebra. For the converse, the following condition must be added in order to get the sufficient and necessary condition of the clean coalgebra $e \rightharpoonup C \leftharpoonup e$.

Theorem 2.2. *Let C be a cocommutative R-coalgebra and e is a full idempotent in C^* (i.e., $e * e = e$ and $C^* * e * C^* = C^*$ where $*$ is the convolution product of C^*). If $e \rightharpoonup C \leftharpoondown e$ is a clean R-coalgebra, then C is clean.*

Proof. Suppose that $e \rightharpoonup C \leftharpoondown e$ is a clean R-coalgebra. Then by [15],

$$End_{(e \rightharpoonup C \leftharpoondown e)^*}(e \rightharpoonup C \leftharpoondown e) \simeq (e \rightharpoonup C \leftharpoondown e)^* \cong e * C^* * e$$

is a clean ring. We will use the Morita Context and Morita Equivalences to prove that C^* is a clean ring (see Example 18.30 page 490 on [18]).
Given the ring C^* with full idempotent e then $e * C^*$ is a progenerator (finitely generated projective generator) C^*-module. Put $P = e * C^* = \{e * f | f \in C^*\}$, $Q = Hom_{C^*}(P, C^*) \cong C^* * e$ and $End_{C^*}(P) \cong e * C^* * e$. In [18], the Morita context associated with P is $(e * C^*, C^*, C^* * e, e * C^* * e, \alpha, \beta)$, with the isomorphism

$$\alpha : C^* * e \otimes_{e * C^* * e} e * C^* \to C^*, f * e \otimes e * f' \mapsto f * e * f'$$
$$\beta : e * C^* \otimes_{C^*} C^* * e \to e * C^* * e, e * f \otimes f' * e \mapsto e * f f' * e.$$

By Morita Theorem, there is mutually inverse category equivalences:

$$F = - \otimes_{C^*} (C^* * e) :_{C^*} \mathbf{M} \to_{e * C^* * e} \mathbf{M} \text{ and}$$
$$G = - \otimes_{e * C^* * e} (e * C^*) :_{e * C^* * e} \mathbf{M} \to_{C^*} \mathbf{M}.$$

Therefore, the ring C^* is Morita equivalence with ring $e * C^* * e$, denoted by $C^* \approx e * C^* * e$. Since C is a cocommutative coalgebra, C^* is commutative. Thus, $e * C^* * e$ is also commutative. By consequences of Morita context, since $C^* \approx e * C^* * e$ and $C^*, e * C^* * e$ is commutative, the ring $C^* \approx e * C^* * e$. By consequences of Morita context and Theorem 1.5, we have an isomorphism ring as below:

$$\begin{aligned} e * C^* * e &= End_{e * C^*}(C^*) \\ &= End_{e * (C^* * e * C^*)}(C^* * e * C^*), \text{ (since } e \text{ is a full idempotent)} \\ &= End_{C^* * e * C^*}(C^* * e * C^*), \text{ (by commutativity of } *) \\ &= End_{C^*}(C^*) \\ &= C^*. \end{aligned}$$

Therefore, the ring $e * C^* * e \cong C^*$. As a consequence the fact that $e * C^* * e$ is a clean ring, C^* is also clean and C is a clean R-coalgebra. The conclusion of the Theorem 2.1 and Theorem 2.2 yields the sufficient and necessary condition of R-coalgebra $e \rightharpoonup C \leftharpoondown e$ to be clean. Assume that C satisfies the commutativity property in coalgebra theory and $e \in C^*$ is a full idempotent. If C is clean, then $e \rightharpoonup C \leftharpoondown e$ is also clean and true for the converse.

3. CONCLUDING REMARKS

This paper yields the condition for guarantee the R coalgebra $e \rightharpoonup C \leftharpoondown e$ related to R-coalgebra C. Hence, from the assumption that R-coalgebra C is cocomutative and $e \in C^*$ is a full idempotent, C is clean if $e \rightharpoonup C \leftharpoonup e$ is a clean R-coalgebra and $e \rightharpoonup C \leftharpoonup e$ is clean if C is a clean R-coalgebra.

REFERENCES

[1] Nicholson, W.K., "Lifting Idempotents and Exchange Rings" *Trans. Amer. Math. Soc.*, **229**(1977) 269-278.

[2] Warfield Jr. R.B., "Exchange rings and decompositions of modules" *Math. Ann.* **199** (1972) 31-36.

[3] Crawley, P., and Jónnson, B., "Refinements for Infinite Direct Decompositions Algebraic System" *Pacific J. Math.*, **14(3)** (1964) 797-855.

[4] Camillo, V.P. and Yu, H.P., "Exchange Rings, Units and Idempotents" *Comm. Algebra*, **22(12)** (1994) 4737-4749.

[5] Han, J. and Nicholson, W.K., "Extension of Clean Rings" *Comm. Algebra* **29(6)** (2001) 2589-2595.

[6] Anderson, D.D. and Camillo, V.P., "Commutative Rings Whose Element are a sum of a Unit and Idempotent", *Comm. Algebra* **30(7)** (2002) 3327-3336.

[7] Tousi, M. and Yassemi, S., "Tensor Product of Clean Rings" *Glasgow Math. J* **47** (2005) 501–503.

[8] McGovern, W. Wm., "Characterization of commutative clean rings" *Int. J. Math. Game Theory Algebra* **15(40)** (2006) 403-413.

[9] Chen, H. and Chen, M., "On Clean Ideals" *IJMMS* **62** (2002) 3949-3956.

[10] Nicholson, W.K and Varadarajan,K., "Countable Linear Transformations are Clean" *Proceedings of American Mathematical Socienty* **126** (1998) 61-64.

[11] Nicholson, W.K., Varadarajan, K. and Zhou, Y., "Clean Endomorphism Rings" *Archiv der Mathematik* **83** (2004) 340-343.

[12] Camillo, V.P., Khurana, D., Lam, T.Y., Nicholson, W.K. and Zhou, Y., "Continous Modules are Clean" *J. Algebra* **304** (2006) 94-111.

[13] Camillo, V.P., Khurana, D., Lam, T.Y., Nicholson, W.K., and Zhou, Y., "A Short Proof that Continous Modules are Clean" *Contemporary Ring Theory 2011, Proceedings of the Sixth China-Japan-Korea International Conference on Ring Theory* (2012) 165-169.

[14] Sweedler, M.E., *Hopf Algebra, Mathematics Lecture Note Series*, W.A. Benjamins, Inc., 1969

[15] Brzeziński, T. and Wisbauer, R., *Corings and Comodules*, Cambridge University Press, 2003.

[16] Puspita., N. P., Wijayanti., I.E., and Surodjo, B., "Graded Modules as a Clean Comodule", *Journal of Mathematics Research* **12(6)** (2020) 66-73.

[17] Puspita., N. P., Wijayanti., I.E., and Surodjo, B., "Clean Coalgebra and Clean Comodules from Finitely Generated Projective Modules" *Algebra and Discrete Mathematics Journal* **31(2)** (2021).

[18] Lam, T.Y., *Graduated Texts in Mathematics: Lectures on Modules and Rings*, Springer-Verlag Inc., 1994.