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BELLIGERENT GE-FILTERS IN GE-AGEBRAS

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Abstract. The notion of a belligerent GE-filter in a GE-algebra is introduced, and the relationships between a GE-filter and a belligerent GE-filter will be given. Conditions for a GE-filter to be a belligerent GE-filter are provided. The product and the union of GE-algebras are discussed and its properties are investigated.

Key words and Phrases: Commutative, transitive, left exchangeable, GE-algebra, GE-filter, belligerent GE-filter.

1. Introduction

In mathematics, Hilbert algebras occur in the theory of von Neumann algebras in: Commutation theorem and Tomita–Takesaki theory. The concept of Hilbert algebra was introduced in early 50-ties by L. Henkin and T. Skolem for some investigations of implication in intuicionistic and other nonclassical logics. In 60-ties, these algebras were studied especially by A. Horn and A. Diego from algebraic point of view. Hilbert algebras are an important tool for certain investigations in algebraic logic since they can be considered as fragments of any propositional logic containing a logical connective implication ($\rightarrow$) and the constant 1 which is considered as the logical value "true". Many researchers studied various things about Hilbert algebras (see [2, 3, 4, 5, 6, 7, 9, 10, 11]). As a generalization of Hilbert algebras, R.K. Bandaru et al. [1] introduce the notion of GE-algebras. They studied the various properties and filter theory of Hilbert algebras.

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GE-filters are important substructures in a GE-algebra and play an important role. It is well understood that GE-filters are the kernels of congruences. Filter theory is crucial in the study of any class of logical algebras. From a logical standpoint, different filters correspond to different sets of valid formulas in an appropriate logic. Designing various types of filters in some logical algebra, on the other hand, is also algebraically interesting. With this motivation, we introduce and investigate the concept of a belligerent GE-filter of a GE-algebra in this paper. We study the relation between GE-filter and belligerent GE-filter of a GE-algebra. We provide the conditions under which the set $\overrightarrow{a} := \{x \in X \mid a \leq x\}$ is a GE-filter of a GE-algebra X. Also we introduce the notion of product and union of GE-algebras and investigated their properties. We show that the union of two GE-algebras is again a GE-algebra under certain condition. Finally, we prove that if F_1 and F_2 are GE-filters of GE-algebras X_1 and X_2 respectively then $F_1 \cup F_2$ is a GE-algebra of $X_1 \cup X_2$.

2. Preliminaries

Definition 2.1 ([1]). *A GE-algebra is a non-empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms:*

(GE1) $u * u = 1$,

(GE2) $1 * u = u$,

(GE3) $u * (v * w) = u * (v * (u * w))$

for all $u, v, w \in X$.

In a GE-algebra X, a binary relation "$\leq$" is defined by

$$(\forall x, y \in X)\,(x \leq y \;\Leftrightarrow\; x * y = 1)\,. \tag{1}$$

Proposition 2.2 ([1]). *Every GE-algebra X satisfies the following items.*

$$(\forall u \in X)\,(u * 1 = 1)\,. \tag{2}$$

$$(\forall u, v \in X)\,(u * (u * v) = u * v)\,. \tag{3}$$

$$(\forall u, v \in X)\,(u \leq v * u)\,. \tag{4}$$

Definition 2.3 ([1]). *A GE-algebra X is said to be*

- *transitive if it satisfies:*

$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,. \tag{5}$$

- *commutative if it satisfies:*

$$(\forall x, y \in X)\,((x * y) * y = (y * x) * x)\,. \tag{6}$$

Proposition 2.4. *Every transitive GE-algebra X satisfies the following assertions.*

$$(\forall x, y, z \in X)\,(x * y \leq (y * z) * (x * z))\,. \tag{7}$$

$$(\forall x, y, z \in X)\,(x \leq y \;\Rightarrow\; z * x \leq z * y,\; y * z \leq x * z)\,. \tag{8}$$

Definition 2.5 ([1]). *A subset F of a GE-algebra X is called a GE-filter of X if it satisfies:*

$$1 \in F, \tag{9}$$

$$(\forall x, y \in X)(x * y \in F,\ x \in F \ \Rightarrow\ y \in F). \tag{10}$$

Lemma 2.6 ([1]). *In a GE-algebra X, every filter F of X satisfies:*

$$(\forall x, y \in X)\,(x \leq y,\ x \in F \ \Rightarrow\ y \in F)\,. \tag{11}$$

3. Belligerent GE-filters

Definition 3.1. *A subset F of a GE-algebra X is called a belligerent GE-filter of X if it satisfies* (9) *and*

$$(\forall x, y, z \in X)(x * (y * z) \in F,\ x * y \in F \ \Rightarrow\ x * z \in F). \tag{12}$$

Example 3.2. *Let $X = \{1, a, b, c, d, e, f\}$ be a set with the binary operation "$*$" in Table 1. It is routine to verify that X is a GE-algebra and $F := \{1, a, b, f\}$ is a*

Table 1. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d	e	f
1	1	a	b	c	d	e	f
a	1	1	1	c	e	e	1
b	1	a	1	d	d	d	f
c	1	1	b	1	1	1	1
d	1	a	1	1	1	1	f
e	1	a	b	1	1	1	1
f	1	a	b	e	d	e	1

belligerent GE-filter of X.

We establish the relationship between belligerent GE-filter and GE-filter.

Theorem 3.3. *In a GE-algebra, every belligerent GE-filter is a GE-filter.*

Proof. Let F be a belligerent GE-filter of a GE-algebra X. Let $x, y \in X$ be such that $x * y \in F$ and $x \in F$. If we substitute x, y and z with 1, x and y respectively in (12) and use (GE2), then $1 * (x * y) = x * y \in F$ and $1 * x = x \in F$. It follows from (12) that $y = 1 * y \in F$. Hence F is a GE-filter of X. □

The following example shows that the converse of Theorem 3.3 is not true in general.

Example 3.4. *Let $X = \{1, a, b, c, d, e, f\}$ be the GE-algebra in Example 3.2. Then $F := \{1, b\}$ is a GE-filter of X. But it is not a belligerent GE-filter of X since $d * (c * f) = d * 1 = 1 \in F$ and $d * c = 1 \in F$ but $d * f = f \notin F$.*

In a GE-algebra X, consider the following condition:

$$(\forall x, y, z \in X)\,(x * (y * z) \in F \ \Rightarrow \ (x * y) * (x * z) \in F)\,. \tag{13}$$

The following example shows that any GE-filter F of X does not satisfy the condition (13).

Example 3.5. *Consider the GE-algebra X in Example 3.2. Then a GE-filter $F; = \{1, b\}$ of X does not satisfy the condition* (13) *since $d*(c*f) = d*1 = 1 \in F$ but $(d * c) * (d * f) = 1 * f = f \notin F$.*

We explore the conditions for a GE-filter to be a belligerent GE-filter.

Theorem 3.6. *If a GE-filter F of a GE-algebra X satisfies the condition* (13)*, then F is a belligerent GE-filter of X.*

Proof. Let F be a GE-filter of a GE-algebra X which satisfies the condition (13). Let $x, y, z \in X$ be such that $x * (y * z) \in F$ and $x * y \in F$. By the condition (13), we have $(x * y) * (x * z) \in F$ and $x * y \in F$. Since F is a GE-filter of X, it follows from (10) that $x * z \in F$. Therefore F is a belligerent GE-filter of X. □

Consider the following argument for a subset F of a GE-algebra X:

$$(\forall x, y, z \in X)\,(x \in F,\ x * (y * z) \in F \ \Rightarrow \ y * z \in F)\,. \tag{14}$$

The following example shows that any subset F of a GE-algebra X does not satisfy the condition (14).

Example 3.7. *Consider the GE-algebra X in Example 3.2. Then a subset $F := \{1, a, b\}$ of X, which is not a GE-filter of X, does not satisfy the condition* (14) *since $a \in F$ and $a * (b * f) = a * f = 1 \in F$ but $b * f = f \notin F$.*

Theorem 3.8. *If a subset F of a GE-algebra X satisfies* (9) *and* (14)*, then F is a GE-filter of X.*

Proof. Assume that a subset F of a GE-algebra X satisfies (9) and (14). Let $x, y \in X$ be such that $x * y \in F$ and $x \in F$. Using (GE2), we have $x * (1 * y) = x * y \in F$. Hence $y = 1 * y \in F$ by (GE2) and (14). Therefore F is a GE-filter of X. □

The following example shows that any subset F of a GE-algebra X satisfying two conditions (9) and (14) may not be a belligerent GE-filter of X.

Example 3.9. *Let X be the GE-algebra in Example 3.2 and $F := \{1, f\}$. Then F satisfies* (9) *and* (14) *but it is not a belligerent GE-filter of X since $c * (d * b) = c * 1 = 1 \in F$ and $c * d = 1 \in F$ but $c * b = b \notin F$.*

We have the following question.

Question 3.10. *Does any GE-algebra X satisfy the left self-distribution?. That is,*

$$(\forall x, y, z \in X)\,(x * (y * z) = (x * y) * (x * z))\,. \tag{15}$$

The following example gives a negative answer to the Question 3.10.

Example 3.11. *Let $X = \{1, a, b, c, d, e\}$ be a set with the binary operation "$*$" in Table 2.*

Table 2. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	c	e	e
b	1	a	1	d	d	d
c	1	1	b	1	1	1
d	1	a	1	1	1	1
e	1	a	1	1	1	1

Then X is a GE-algebra in which the condition (15) *is not true since $a * (b * c) = a * d = e \neq c = 1 * c = (a * b) * (a * c)$.*

Definition 3.12. *A GE-algebra X is said to be belligerent if X satisfies the left self-distribution, i.e., the condition* (15)*.*

Example 3.13. *Let $X = \{1, a, b, c, d\}$ be a set with the binary operation "$*$" in Table 3. It is routine to verify that X is a belligerent GE-algebra.*

Table 3. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	a	a	1	1
d	1	b	b	1	1

Question 3.14. *Is the following equation established in a (transitive) GE-algebra X?*

$$(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z)). \tag{16}$$

The following example shows that the answer to Question 3.14 is negative.

Example 3.15. *(1) Let $X = \{1, a, b, c, d, e\}$ be a set with the binary operation "$*$" in Table 4. Then it is routine to verify that X is a GE-algebra. But X does not satisfy* (16) *since $b * (c * d) = b * a = e \neq a = c * d = c * (b * d)$.*

(2) Let $X = \{1, a, b, c, d\}$ be a set with the binary operation "$*$" in Table 5. Then it is routine to verify that X is a transitive GE-algebra. But X does not satisfy (16) *since $b * (c * d) = b * a = d \neq a = c * d = c * (b * d)$.*

Definition 3.16. *A GE-algebra X is said to be left exchangeable if it satisfies the condition* (16)*.*

TABLE 4. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	c	1	1
b	1	e	1	1	d	e
c	1	a	b	1	a	a
d	1	1	1	c	1	1
e	1	1	b	1	1	1

TABLE 5. Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	1	1
b	1	d	1	c	d
c	1	a	b	1	a
d	1	1	b	1	1

Example 3.17. *Let $X = \{1, a, b, c\}$ be a set with the binary operation "$*$" in Table 6. Then it is routine to verify that X is a left exchangeable GE-algebra.*

TABLE 6. Cayley table for the binary operation "$*$"

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	1
c	1	1	b	1

Question 3.18. *If a GE-filter F of a GE-algebra X satisfies:*

$$(\forall x, y \in X)\,(x * (x * y) \in F \;\Rightarrow\; x * y \in F)\,, \tag{17}$$

then does F satisfy the condition (13)?

The following example shows that the answer to Question 3.18 is negative.

Example 3.19. *Let $X = \{1, a, b, c, d, e, f\}$ be the GE-algebra in Example 3.2. Then $F := \{1, b\}$ is a GE-filter of X satisfying (17). But it does not satisfy (13) since $d * (c * f) = d * 1 = 1 \in F$ but $(d * c) * (d * f) = 1 * f = f \notin F$.*

Proposition 3.20. *Let F be a GE-filter of a GE-algebra X that satisfies the condition (17). If X is transitive and left exchangeable, then F satisfies the condition (13).*

Proof. Let $x, y, z \in X$ be such that $x * (y * z) \in F$. Using (8) and (5), we have

$$x * (y * z) \leq x * ((x * y) * (x * z)).$$

Since F is a GE-filter of X, it follows from Lemma 2.6 that $x*((x*y)*(x*z)) \in F$. Using the condition (16), we know that

$$x * (x * ((x * y) * z)) = x * ((x * y) * (x * z)) \in F.$$

It follows from (16) and (17) that $(x * y) * (x * z) = x * ((x * y) * z) \in F$. Thus F satisfies the condition (13). □

Using Theorem 3.6 and Proposition 3.20, we have the following theorem.

Theorem 3.21. *Let F be a GE-filter of a transitive and left exchangeable GE-algebra X. If F satisfies the condition* (17)*, then F is a belligerent GE-filter of X.*

Given a point w and a non-empty subset F of a GE-algebra X, we consider a special set:

$$F_w := \{x \in X \mid w * x \in F\}. \tag{18}$$

If F is a GE-filter of a GE-algebra X, then $1, w \in F_w$ for all $w \in X$.

We have the following questions.

Question 3.22. *If F is a GE-filter of a GE-algebra X, then is the set F_w in* (18) *a GE-filter of X?*

The following example gives a negative answer to the Question 3.22.

Example 3.23. *Let $X = \{1, a, b, c, d, e, f\}$ be the GE-algebra in Example 3.2. If we take a GE-filter $F := \{1, b\}$ of X, then $F_d = \{1, b, c, d, e\}$ and it is not a GE-filter of X since $c * a = 1 \in F_d$ and $c \in F_d$ but $a \notin F_d$.*

We suggest conditions that will lead to a positive answer to the Question 3.22.

Theorem 3.24. *If F is a belligerent GE-filter of a GE-algebra X, then the set F_w in* (18) *is a GE-filter of X.*

Proof. Assume that F is a belligerent GE-filter of a GE-algebra X. Let $x, y \in X$ be such that $x * y \in F_w$ and $x \in F_w$. Then $w * (x * y) \in F$ and $w * x \in F$. It follows from (12) that $w * y \in F$, that is, $y \in F_w$. Hence F_w is a GE-filter of X. □

We suggest the conditions under which a GE-filter can be a belligerent GE-filter.

Theorem 3.25. *For every subset F of a GE-algebra X, if $1 \in F$ and the set F_w in* (18) *is a GE-filter of X for every $w \in X$, then F is a belligerent GE-filter of X.*

Proof. Suppose that $1 \in F$ and the set F_w in (18) is a GE-filter of X for every $w \in X$. Let $x * (y * z) \in F$ and $x * y \in F$. Then $y * z \in F_x$ and $y \in F_x$. Since F_x is a GE-filter of X, we have $z \in F_x$ and so $x * z \in F$. Hence F is a belligerent GE-filter of X. □

Corollary 3.26. *Given a GE-filter F of a GE-algebra X, if the set F_w in* (18) *is a GE-filter of X for every $w \in X$, then F is a belligerent GE-filter of X.*

Theorem 3.27. *If F is a belligerent GE-filter of a GE-algebra X, then the set F_w in* (18) *is the least GE-filter of X containing F and w.*

Proof. Assume that F is a belligerent GE-filter of a GE-algebra X and let $w \in X$. Then F_w is a GE-filter of X (see Theorem **??**TqT23b-200820). it is obvious that F_w contains F and w. Let G be a GE-filter of X containing F and w. If $x \in F_w$, then $w * x \in F \subseteq G$ and so $x \in G$. Hence $F_w \subseteq G$ and F_w is the least GE-filter of X containing F and w. □

The following example shows that the trivial filter $\{1\}$ of a GE-algebra X is not a belligerent GE-filter of X.

Example 3.28. *Let X be the GE-algebra in Example 3.2. Then $F := \{1\}$ is a GE-filter of X but not a belligerent GE-filter of X since $d * (c * f) = d * 1 = 1 \in F$ and $d * c = 1 \in F$ but $d * f = f \notin F$.*

Given an element a of a GE-algebra X, consider the set $\vec{a} := \{x \in X \mid a \leq x\}$. In general, the set $\vec{a}$ is not a GE-filter of X as seen in the following example.

Example 3.29. *Let X be the GE-algebra in Example 3.2. Then $\vec{c} := \{1, a, c, d, e, f\}$ is not a GE-filter of X since $d \in \vec{c}$ and $d * b = 1 \in \vec{c}$ but $b \notin \vec{c}$.*

We provide conditions for the set $\vec{a}$ to be a GE-filter.

Theorem 3.30. *Given an element a in a GE-algebra X, the following are equivalent.*

(i) *The set $\vec{a} := \{x \in X \mid a \leq x\}$ is a GE-filter of X.*
(ii) *X satisfies:*

$$(\forall x, y \in X)\,(a \leq x * y,\ a \leq x \ \Rightarrow\ a \leq y)\,. \tag{19}$$

Proof. Assume that $\vec{a}$ is a GE-filter of X. Let $x, y \in X$ be such that $a \leq x * y$ and $a \leq x$. Then $x * y \in \vec{a}$ and $x \in \vec{a}$. Since $\vec{a}$ is a GE-filter of X, it follows that $y \in \vec{a}$, that is, $a \leq y$. Suppose that X satisfies the condition (19). It is clear that $1 \in \vec{a}$. Let $x, y \in X$ be such that $x * y \in \vec{a}$ and $x \in \vec{a}$. Then $a \leq x * y$ and $a \leq x$ which imply from (19) that $a \leq y$. Hence $y \in \vec{a}$, and therefore $\vec{a}$ is a GE-filter of X. □

Theorem 3.31. *In a GE-algebra X, the following are equivalent.*

(i) *The trivial GE-filter $\{1\}$ is a belligerent GE-filter.*
(ii) *For every $a \in X$, the set $\vec{a} := \{x \in X \mid a \leq x\}$ is a GE-filter of X.*

Proof. Assume that the trivial GE-filter $\{1\}$ is a belligerent GE-filter of X. It is clear that $1 \in \vec{a}$. Let $x, y \in X$ be such that $x * y \in \vec{a}$ and $x \in \vec{a}$. Then $a \leq x * y$ and $a \leq x$, that is, $a * (x * y) = 1 \in \{1\}$ and $a * x = 1 \in \{1\}$. Since $\{1\}$ is a belligerent GE-filter of X, it follows from (12) that $a * y \in \{1\}$. Hence $y \in \vec{a}$. Therefore $\vec{a}$ is a GE-filter of X.

Conversely, suppose that the set $\vec{a}$ is a GE-filter of X for every $a \in X$. Let $x, y, z \in X$ be such that $x * (y * z) \in \{1\}$ and $x * y \in \{1\}$. Then $x * (y * z) = 1$ and $x * y = 1$, i.e., $x \leq y * z$ and $x \leq y$. Hence $y * z \in \vec{x}$ and $y \in \vec{x}$. Since $\vec{x}$ is a GE-filter of X, we have $z \in \vec{x}$, that is, $x \leq z$. Thus $x * z = 1 \in \{1\}$, and therefore $\{1\}$ is a belligerent GE-filter of X. □

4. Product and union of GE-algebras

Let $\mathbb{X}_\alpha := \{(X_\alpha, *_\alpha, 1_\alpha) \mid \alpha \in \Lambda\}$ be a family of GE-algebras where Λ is an index set. Let $\prod \mathbb{X}_\alpha$ be the set of all mappings $\ell : \Lambda \to \bigcup_{\alpha \in \Lambda} X_\alpha$ with $\ell(\alpha) \in X_\alpha$, that is,

$$\prod \mathbb{X}_\alpha := \left\{ \ell : \Lambda \to \bigcup_{\alpha \in \Lambda} X_\alpha \mid \ell(\alpha) \in X_\alpha, \alpha \in \Lambda \right\}. \tag{20}$$

We define a binary operation $\circledast$ on $\prod \mathbb{X}_\alpha$ and the constant $\mathbf{1}$ by

$$\left(\forall \ell, \jmath \in \prod \mathbb{X}_\alpha\right) ((\ell \circledast \jmath)(\alpha) = \ell(\alpha) *_\alpha \jmath(\alpha)) \tag{21}$$

and $\mathbf{1}(\alpha) = 1_\alpha$, respectively, for every $\alpha \in \Lambda$. It is routine to verify that $(\prod \mathbb{X}_\alpha, \circledast, \mathbf{1})$ is a GE-algebra, which is called the product GE-algebra.

The following example illustrates a product GE-algebra.

Example 4.1. *Consider two GE-algebras* $(X_1 = \{1, a, b, c, d\}, *_1, 1)$ *and* $(X_2 = \{1, a, b, c, d, e\}, *_2, 1)$ *with the binary operations* $*_1$ *and* $*_2$ *respectively in the following tables.*

$*_1$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	1	1
b	1	d	1	c	d
c	1	a	b	1	a
d	1	1	b	1	1

$*_2$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	c	e	e
b	1	a	1	d	d	d
c	1	1	b	1	1	1
d	1	a	1	1	1	1
e	1	a	1	1	1	1

Then

$$\begin{aligned} X_1 \times X_2 = \{&(1,1), (1,a), (1,b), (1,c), (1,d), (1,e), (a,1), (a,a), (a,b), (a,c), (a,d), (a,e), \\ &(b,1), (b,a), (b,b), (b,c), (b,d), (b,e), (c,1), (c,a), (c,b), (c,c), (c,d), (c,e), \\ &(d,1), (d,a), (d,b), (d,c), (d,d), (d,e)\} \end{aligned}$$

and $(X_1 \times X_2, \circledast, \mathbf{1})$ *is a GE-algebra in which* $\mathbf{1} = (1,1)$ *and the operation* $\circledast$ *is given by*

$$(\forall (x_1, x_2), (y_1, y_2) \in X_1 \times X_2)((x_1, x_2) \circledast (y1, y_2) = (x_1 *_1 y_1, x_2 *_2 y_2)).$$

Theorem 4.2. *If F_α is a (belligerent) GE-filter of X_α for all $\alpha \in \Lambda$, then $\prod F_\alpha$ is a (belligerent) GE-filter of $\prod \mathbb{X}_\alpha$.*

Proof. It is clear that $\mathbf{1} \in \prod F_\alpha$. Assume that F_α is a GE-filter of X_α for all $\alpha \in \Lambda$. Let $\ell, \jmath \in \prod \mathbb{X}_\alpha$ be such that $\ell \circledast \jmath \in \prod F_\alpha$ and $\ell \in \prod F_\alpha$. Then $\ell(\alpha) *_\alpha \jmath(\alpha) = (\ell \circledast \jmath)(\alpha) \in F_\alpha$ and $\ell(\alpha) \in F_\alpha$ for every $\alpha \in \Lambda$. Since F_α is a GE-filter of X_α, it follows that $\jmath(\alpha) \in F_\alpha$. Hence $\jmath \in \prod F_\alpha$, and therefore $\prod F_\alpha$ is a GE-filter of $\prod \mathbb{X}_\alpha$. Similarly, we can check that if F_α is a belligerent GE-filter of X_α for all $\alpha \in \Lambda$, then $\prod F_\alpha$ is a belligerent GE-filter of $\prod \mathbb{X}_\alpha$. $\square$

Theorem 4.3. *If F is a GE-filter of $\prod \mathbb{X}_\alpha$, then the α-projection F_α of F is a GE-filter of X_α for all $\alpha \in \Lambda$.*

Proof. Let $x, y \in X_\alpha$ be such that $x *_\alpha y \in F_\alpha$ and $x \in F_\alpha$. We define ℓ and $\jmath$ as follows:

$$\ell(\gamma) = \begin{cases} x & \text{if } \gamma = \alpha, \\ 1 & \text{if } \gamma \neq \alpha, \end{cases} \quad \text{and} \quad \jmath(\gamma) = \begin{cases} y & \text{if } \gamma = \alpha, \\ 1 & \text{if } \gamma \neq \alpha. \end{cases}$$

Then $\ell(\alpha) *_\alpha \jmath(\alpha) = x *_\alpha y \in F_\alpha$, and so there exists $\varrho \in F$ such that $\varrho(\alpha) = \ell(\alpha) *_\alpha \jmath(\alpha) = (\ell \circledast \jmath)(\alpha)$. Hence $\ell \circledast \jmath \in F$. Also $\ell \in F$ by similar way. Since F is a GE-filter of $\prod \mathbb{X}_\alpha$, it follows that $\jmath \in F$. Hence $y = \jmath(\alpha) \in F_\alpha$. Therefore F_α is a GE-filter of X_α for all $\alpha \in \Lambda$. $\square$

Theorem 4.4. *Let X_1 and X_2 be GE-algebras. If F is a GE-filter of $X_1 \times X_2$, then F is represented by $F = F_1 \times F_2$ where F_α, $\alpha = 1, 2$, is the α-projection of F.*

Proof. It is obvious that $F \subseteq F_1 \times F_2$. Let $\ell \in F_1 \times F_2$. Then ℓ is represented as (a, b) for $a \in F_1$ and $b \in F_2$. It follows that there exist $b' \in F_2$ and $a' \in F_1$ such that $(a, b') \in F$ and $(a', b) \in F$. Using (2), (GE1) and (GE3), we have

$$\begin{aligned}(a, b') \circledast (a' * a, 1) &= (a * (a' * a), b' * 1) = (a * (a' * (a * a)), 1) \\ &= (a * (a' * 1), 1) = (a * 1, 1) = (1, 1) \in F.\end{aligned}$$

Since F is a GE-filter, it follows that $(a' * a, 1) \in F$. Also $(a', b) \circledast (a, b) = (a' * a, 1) \in F$, and so $(a, b) \in F$. This shows that $F_1 \times F_2 \subseteq F$ and the proof is completed. $\square$

The example below describes Theorem 4.4.

Example 4.5. *Consider the product GE-algebra $(X_1 \times X_2, \circledast, \mathbf{1})$ in Example 4.1. Then*

$$F = \{(1, 1), (1, a), (1, b), (b, 1), (b, a), (b, b)\}$$

is a GE-filter of $X_1 \times X_2$ and it is represented as $F = F_1 \times F_2$ where $F_1 = \{1, b\}$ and $F_2 = \{1, a, b\}$ are GE-filters of X_1 and X_2 respectively.

Let $(X_1, *_1, 1)$ and $(X_2, *_2, 1)$ be GE-algebras, and consider their union $X_1 \cup X_2$. Let's call $*$ a binary operation on $X_1 \cup X_2$ as defined as:

$$(\forall x, y \in X_1 \cup X_2) \left(x * y = \begin{cases} x *_1 y \text{ if } x, y \in X_1 \\ x *_2 y \text{ if } x, y \in X_2 \\ y \text{ if } x \text{ and } y \text{ are not belong to same GE-algebra} \end{cases} \right). \tag{22}$$

Question 4.6. *If X_1 and X_2 are GE-algebras, is their union $X_1 \cup X_2$ also a GE-algebra?*

The following example gives a negative answer to the Question 4.6.

Example 4.7. *Consider two GE-algebras $(X_1 := \{1, a, b, c, d, e\}, *_1, 1)$ and $(X_2 := \{1, a, l_1, l_2, l_3, l_4\}, *_2, 1)$ with the binary operations $*_1$ and $*_2$ respectively in the following tables.*

$*_1$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	c	1	1
b	1	e	1	1	d	e
c	1	a	b	1	a	a
d	1	1	1	c	1	1
e	1	1	b	1	1	1

$*_2$	1	a	l_1	l_2	l_3	l_4
1	1	a	l_1	l_2	l_3	l_4
a	1	1	1	l_2	l_4	l_4
l_1	1	a	1	l_3	l_3	l_3
l_2	1	1	l_1	1	1	1
l_3	1	a	1	1	1	1
l_4	1	a	1	1	1	1

Then $X_1 \cup X_2 = \{1, a, b, c, d, e, l_1, l_2, l_3, l_4\}$ and $(X_1 \cup X_2, *, 1)$ is not a GE-algebra under the binary operation $*$ defined by (22) *since $l_2 * (b * a) = l_2 * e = e \neq 1 = l_2 * 1 = l_2 * (b * 1) = l_2 * (b * (l_2 * a))$.*

We look for conditions for the union of two GE-algebras to be a GE-algebra again.

Theorem 4.8. *Let $(X_1, *_1, 1)$ and $(X_2, *_2, 1)$ be GE-algebras with $X_1 \cap X_2 = \{1\}$. If a binary operation $*$ on $X_1 \cup X_2$ is defined by* (22)*, then $(X_1 \cup X_2, *, 1)$ is a GE-algebra. Moreover, if X_1 and X_2 are commutative (resp., transitive), then so is $X_1 \cup X_2$.*

Proof. It is clear that (GE1) and (GE2) are established. Let $x, y, z \in X_1 \cup X_2$. If $x, y \in X_1$ and $z \in X_2$, then $x * (y * z) = x * z = z$ and $x * (y * (x * z)) = x * (y * z) = x * z = z$. If $x, z \in X_1$ and $y \in X_2$, then $x * (y * z) = x *_1 z$ and $x * (y * (x * z)) = x * (y * (x *_1 z))x * (x *_1 z) = x *_1 (x *_1 z) = x *_1 z$. If $y, z \in X_1$ and $x \in X_2$, then $x * (y * z) = x * (y *_1 z) = y *_1 z$ and $x * (y * (x * z)) = x * (y * z) = x * (y *_1 z) = y *_1 z$. Similarly, we know that (GE3) is established for the cases:

- $x, y \in X_2$ and $z \in X_1$,
- $x, z \in X_2$ and $y \in X_1$,
- $y, z \in X_2$ and $x \in X_1$.

Hence $(X_1 \cup X_2, *, 1)$ is a GE-algebra. Assume that X_1 and X_2 are commutative. If $x, y \in X_i$, then $(x * y) * y = (x *_i y) *_i y = (y *_i x) *_i x = (y * x) * x$ for $i = 1, 2$. If x and y are not belong to same GE-algebra, then $(x * y) * y = 1 = (y * x) * x$.

Hence $X_1 \cup X_2$ is commutative. Assume that X_1 and X_2 are transitive. If $x \in X_1$ and $y, z \in X_2$, then $x * y = y \leq z *_2 y = x * (z * y) = (z * x) * (z * y)$ by (4). If $y \in X_1$ and $x, z \in X_2$, then $x * y = y \leq y = (z * x) * (z * y)$. If $z \in X_1$ and $x, y \in X_2$, then $x * y = x *_2 y$ and $(z * x) * (z * y) = x *_2 y$. Similarly, we can check that the condition (5) for the cases:

- $x \in X_2$ and $y, z \in X_1$,
- $y \in X_2$ and $x, z \in X_1$,
- $z \in X_2$ and $x, y \in X_1$.

Therefore $X_1 \cup X_2$ is transitive. □

Corollary 4.9. *Let $(X_1, *_1, 1)$ and $(X_2, *_2, 1)$ be commutative GE-algebras with $X_1 \cap X_2 = \{1\}$. If a binary operation $*$ on $X_1 \cup X_2$ is defined by (22), then $(X_1 \cup X_2, *, 1)$ is a Hilbert-algebra.*

The following example describes Theorem 4.8.

Example 4.10. *Consider two GE-algebras X_1 and X_1, where $(X_1 := \{1, a, b, c, d\}, *_1, 1)$ and $(X_2 := \{1, l_1, l_2, l_3, l_4\}, *_2, 1)$ with the binary operations $*_1$ and $*_2$ respectively in the following tables.*

$*_1$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	c
b	1	a	1	d	d
c	1	a	1	1	1
d	1	a	1	1	1

$*_2$	1	l_1	l_2	l_3	l_4
1	1	l_1	l_2	l_3	l_4
l_1	1	1	l_2	l_3	l_3
l_2	1	l_1	1	l_4	l_4
l_3	1	l_1	1	1	1
l_4	1	l_1	l_2	1	1

Then $X_1 \cup X_2 = \{1, a, b, c, d, l_1, l_2, l_3, l_4\}$ and $(X_1 \cup X_2, *, 1)$ is a GE-algebra under the binary operation $*$ defined by (22). The binary operation $*$ on $X_1 \cup X_2$ is described by the next Cayley table.

$*$	1	a	b	c	d	l_1	l_2	l_3	l_4
1	1	a	b	c	d	l_1	l_2	l_3	l_4
a	1	1	b	c	c	l_1	l_2	l_3	l_4
b	1	a	1	d	d	l_1	l_2	l_3	l_4
c	1	a	1	1	1	l_1	l_2	l_3	l_4
d	1	a	1	1	1	l_1	l_2	l_3	l_4
l_1	1	a	b	c	d	1	l_2	l_3	l_4
l_2	1	a	b	c	d	l_1	1	l_4	l_4
l_3	1	a	b	c	d	l_1	1	1	1
l_4	1	a	b	c	d	l_1	l_2	1	1

Theorem 4.11. *If F_1 and F_2 are (belligerent) GE-filters of X_1 and X_2 respectively, then the union $F_1 \cup F_2$ is a (belligerent) GE-filter of $X_1 \cup X_2$.*

Proof. It is clear that $1 \in F_1 \cup F_2$. Let $x, y \in X_1 \cup X_2$ be such that $x * y \in F_1 \cup F_2$ and $x \in F_1 \cup F_2$. If $x * y \in F_i$ and $x \in F_i$, then $y \in F_i \subseteq F_1 \cup F_2$ for $i = 1, 2$. Assume that $x * y \in F_1$ and $x \in F_2$. If $y \in X_2$, then $x * y \in X_2$ by (4) and Lemma 2.6. Hence $x * y = 1$, i.e., $x \leq y$ since $x * y \in F_1 \subseteq X_1$ and $X_1 \cap X_2 = \{1\}$. It follows from

(2.6) that $y \in F_2$. If $y \in X_1$, then $x * y = y \in F_1$. Hence $y \in F_1 \cup F_2$. Therefore $F_1 \cup F_2$ is a GE-filter of $X_1 \cup X_2$. Now, suppose that F_1 and F_2 are belligerent. Let $x, y, z \in X_1 \cup X_2$ be such that $z * (y * x) \in F_1 \cup F_2$ and $z * y \in F_1 \cup F_2$. If x and y are not belong to the same GE-algebra, then $z * x = z * (y * x) \in F_1 \cup F_2$. Suppose that x and y are contained in X_1. If $z \in X_1$, then $z * (y * x) = z *_1 (y *_1 x) \in F_1$ and $z * y = z *_1 y \in F_1$. Since F_1 is a belligerent GE-filter of X_1, it follows that $z * x = z *_1 x \in F_1$. If $z \in X_2$, then $z * (y * x) = y *_1 x \in F_1$ and $z * y = y \in F_1$, which imply that $x \in F_1$ Since $x \leq z * x$ by (4), it follows that $z * x \in F_1$. Similarly, if $x, y \in X_2$, then $z * x \in F_2$. Thus $z * x \in F_1 \cup F_2$, and $F_1 \cup F_2$ is a belligerent GE-filter of $X_1 \cup X_2$. □

The following example illustrates Theorem 4.11.

Example 4.12. *In Example 4.10, we can observe that* $F_1 = \{1, a\}$ *and* $F_2 = \{1, l_1, l_2\}$ *are (belligerent) GE-filters of* X_1 *and* X_2 *respectively, and their union* $F_1 \cup F_2 = \{1, a, l_1, l_2\}$ *is also a (belligerent) GE-filter of* $X_1 \cup X_2$.

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REFERENCES

[1] Bandaru, R., Saeid, A. B., and Jun, Y. B., "On GE-algebras", *Bull. Sect. Log.*, **50:1** (2021), 81–96. https://doi.org/10.18778/0138-0680.2020.20

[2] Borzooei, R. A. and Shohani, J., "On generalized Hilbert algebras", *Ital. J. Pure Appl. Math.*, **29** (2012), 71–86.

[3] Chajda, I. and Halas, R., "Congruences and idealas in Hilbert algebras", *Kyungpook Math. J.*, **39** (1999), 429–432.

[4] Chajda, I., Halas, R. and Jun, Y. B., "Annihilators and deductive systems in commutative Hilbert algebras", *Comment. Math. Univ. Carolinae*, **43:3** (2002), 407–417.

[5] Diego, A., *Sur les algebres de Hilbert*, Collection de Logique Mathematique, Edition Hermann, Serie A, XXI, 1966.

[6] Jun, Y. B., "Commutative Hilbert algebras", *Soochow J. Math.*, **22:4** (1996), 477–484.

[7] Jun, Y. B. and Kim, K. H., H-filters of Hilbert algebras, *Sci. Math. Jpn.*, e-2005, 231- 236.

[8] Najafi, A. and Borumand Saeid, A., "Fuzzy points in BE-algebras", *J. Mahani Math. Research Center*, **8:1-2** (2019). 69-80. DOI: 10.22103/jmmrc.2019.12457.1065

[9] Nasab, A. S. and Borumand Saeid, A., "Semi maximal filter in Hilbert algebras", *J. Intell. Fuzzy Syst.*, **30** (2016), 7–15. DOI:10.3233/IFS-151706

[10] Nasab, A. S. and Borumand Saeid, A., "Stonean Hilbert algebra", *J. Intell. Fuzzy Syst.*, **30** (2016), 485–492. DOI:10.3233/IFS-151773

[11] Nasab, A. S. and Borumand Saeid, A., "Study of Hilbert algebras in point of filters", *An St. Univ. Ovidius Constanta*, **24:2** (2016), 221–251. DOI: 10.1515/auom-2016-0039