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SOLUTION OF LAPLACE'S EQUATION IN A SINGULAR DOMAIN USING MELLIN TRANSFORM

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Abstract. This paper deals with the theory of V. Kondratiev which allows to study the regularity of elliptical problems in corner domains. After having introduced the Mellin transform and the Sobolev spaces to weight, we recall the links with Sobolev spaces. The Mellin trasform represent an important key to study the H^s regularity in corner domains.

Key words and Phrases:Mellin transform, Laplace equation, Regularity, Singularity.

1. Introduction

In mathematics, the Mellin transform is an integral transform that may be regarded as the multiplicative version of the two-sided Laplace transform.

$$M[f(s), s)] = \int_0^\infty x^{s-1} f(x)dx = F(s). \tag{1}$$

This integral transform is closely connected to the theory of Dirichlet series, and is often used in number theory, mathematical statistics, and the theory of asymptotic expansions, it is closely related to the Laplace transform and the Fourier transform, and the theory of the gamma function and allied special functions. Also the Mellin transform is extremely useful for certain applications including solving Laplace equation in polar coordinates, as well as for estimating integrals. We will first consider the generalized Laplace transform. In probability theory, the Mellin transform is an essential tool in studying the distributions of products of random

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variables. The Mellin Transform is widely used in computer science for the analysis of algorithms because of its scale invariance property. In contrast to Fourier and Laplace transformations that were introduced to solve physical problems, Mellin transformation arose in a mathematical context. In fact, the first occurrence of the transformation is found in a memoir by Riemann in which he used it to study the famous Zeta function. Actually, the Mellin transformation can also be placed in another framework, which in some respects conforms more closely to the original ideas of Riemann. The magnitude of the Mellin transform of a scaled function is identical to the magnitude of the original function for purely imaginary inputs. This scale invariance property is analogous to the Fourier Transform's shift invariance property. Mellin transform is method for the exact calculation of one dimensional definite integrals, and illustrates the application. The different types of singularity of a function f are discussed. Every singularity of a holomorphic function is isolated, but isolation of singularities is not alone sufficient to guarantee a function is holomorphic. Many important tools of complex analysis and the residue theorem require that all relevant singularities of the function be isolated. We use the Mellin transform in asymptotic analysis for estimating asymptotically harmonic sums. And also the Mellin transform is an integral transform, which is closely connected [4, 7, 6]. And also is extremely useful for certain applications including solving Laplace equation in polar coordinates, as well as for estimating integrals. We see that if we decompose a regular domain Ω of R^2 into several sectors whose common vertex is inside Ω, the asymptotic types associated with the problem at the limits of each subdomain has a priori nothing to do with the expected regularity at inside (H^{m+2} for an operator of order 2 with data H^m) for the complete problem. Conversely, we can consider do a domain decomposition to solve a problem elliptical in a non-convex sector which will lead to a part principal more singular than what gives a priori the resolution in each subdomain. In accordance with Kondratiev's results, asymptotic types give the main parts up to a certain order of solutions to neighborhoods of $x = 0$. Note that for the solution u two origins:
a) the asymptotic type of the data;
b) the asymptotic type associated with the principal parts of the operators P_0 and $B_{0,\pm}$. Regardless of the data (suppose that the Mellin transforms data is holomorphic in $\Im z < S_1$ with S_1 large) we see that the domain decomposition generates a priori a bad match of asymptotic types between the full domain and each subdomain. In fact we have degrees of freedom in the domain decomposition methods at the level of interface conditions. The idea is therefore to determine the interface conditions which will make the first poles (those closest to $z = 0$). Mellin transforms solutions in each subdomain with those associated with the solution of the complete problem. In addition, we will see the interface conditions are not taken at random, but rather they are chosen so that they work well without wedge then adapt them in the neighborhood of the corner so as to have a good convergence. After having built these operators, we can seek to optimize the coefficients so that we obtain more regular solutions near the corner, in practice this can be done via two approaches:

- The first approach consists in choosing the interface conditions so that the residue on the first pole parasite (associated with the subdomain and not with the complete domain) either null for well-chosen data (corresponding to the development asymptotic of solution of the complete problem).
- The second approach consists in choosing the interface conditions in order to push the most far possible from $z = 0$ the parasitic poles (i.e. raise the exponents of parasitic modes in asymptotic expansions).

A priori we do not know in advance which method to use, we must each time compare the two methods and see up to what order we can increase the regularity of the solution of the problem to be considered, this is the subject of both following chapters. The domain decomposition method generates two types of corners, there are corners which are inside the full domain and corners which are on the edge of the full domain. The treatment of these two categories of corners is slightly different. In fact in the choice of the conditions of interface one must take into account the nature of the corners [5, 8, 9, 1, 2]. We will show that for corners inside the complete domain or corners on the edge of the complete domain with Neumann condition on the edge, the conditions of connection to the interfaces must not contain a constant term to have a well-posed problem which allows non-zero values at the corner. On the other hand for a decomposition of field which generates corners on the edge of the complete field, with condition of Dirichlet authorizes constant terms in the conditions of connection to the interfaces of the subdomains.

2. Preliminaries

Definition 2.1. [5] *The Mellin transform of* $f(x), 0 < x < \infty$, *is*

$$M[f(s), s)] = \int_0^\infty x^{s-1} f(x)dx = F(s), \tag{2}$$

which is regular for $-a < Re(s) < b$ *for some* a, b *and* d

$$f(x) = M^{-1}[F, x] = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s)ds, \tag{3}$$

where $-a < c < b$.

The Mellin transformation consists in transporting everything we know how to do on the additive group $(\mathbb{R}, +)$ with the Fourier transformation in the multiplicative group $(\mathbb{R}_+^\star, .)$. This can be done in practice with the change of variable $r = e^{-t}, t \in \mathbb{R}$ in the Fourier transformation formula:

$$F(f)(\tau) = \hat{f}(\tau) = \int_{-\infty}^{+\infty} e^{-i\tau t} f(t)dt.$$

We set $f(t) = u(e^{-t})$, then we have:

$$M(u)(\tau) = \int_0^{+\infty} r^{i\tau} u(r) \frac{dr}{r}. \tag{4}$$

The Fourier inversion formula

$$F^{-1}(\hat{f})(t) = \int_{-\infty}^{+\infty} e^{i\tau t} \hat{f}(\tau) \frac{d\tau}{2\pi},$$

leads to

$$M^{-1}(\hat{u})(r) = \int_{-\infty}^{+\infty} r^{-i\tau} \hat{u}(\tau) \frac{d\tau}{2\pi}.$$

Note that we can directly introduce the Mellin transform on $(\mathbb{R}_+^\star, .)$ Using the characters $r \longrightarrow r^{-i\gamma}$ and Haar's measure $\dfrac{dr}{r}$.

2.1. **Isometry and Properties of M.**

Lemma 2.2. *The Mellin transformation is an isometry of* $L^2(\mathbb{R}_+, \frac{dr}{r})$ *in* $L^2(\mathbb{R}, \frac{d\tau}{2\pi})$.

Proof. We know that the Fourier transformation is an isometry of $L^2(\mathbb{R}, dt)$ in $L^2(\ R, \frac{dt}{2\ pi})$, so the Mellin transformation is an isometry of $L^2(\mathbb{R}_+, \frac{dr}{r})$ in $L^2(\mathbb{R}, \frac{d\tau}{2\pi})$. □

2.2. **Properties of M.**

(1) In Mellin transformation, multiplication by a character $r^{i\alpha}$ amounts to frequency translation by $-\alpha$, that is:

$$M(r^{i\alpha} f)(\tau) = M(f)(\tau + \alpha) \ \ for \ \ m\alpha \in \mathbb{R}.$$

This formula is still valid for complex α.

(2) Derivation formulas for the Mellin transformation:

$$M(ir\,\partial_r f)(\tau) = \tau M(f)(\tau),$$

$$M(i\ln(r)f)(\tau) = \partial_\tau M(f)(\tau),$$

(3) Expansion formulas:

$$M(f(r^c))(\tau) = c^{-1} M(f)(c^{-1}\tau), c \in \mathbb{R}_+^\star,$$

$$M(f(c\,r)(\tau) = c^{-i\tau} M(f)(\tau), c \in \mathbb{R}_+^\star.$$

(4) Convolution product:

We define the convolution product in a multiplicative group by:

$$f * g(r) = \int_0^{+\infty} f(\frac{r}{r'}) g(r') \frac{dr'}{r'}.$$

So we have:

$$M(f * g) = M(f)M(g)$$

(5) $M(fg) = M(f) *_{2\pi} M(g)$

Proof. (1) We work with functions of class C^∞ and with compact support in $(0,+\infty)$ and we extend the identities by density.

By using the integral definition of the Mellin transformation (4) we obtain the 1 ere equality that is to say:

$$M(r^{i\alpha} f)(\tau) = \int_0^{+\infty} r^{i(\alpha+\tau)} f(r)\frac{dr}{r} = M(f)(\tau+\alpha), \tag{5}$$

(2) • According to (4) we have:

$$M(ir\partial_r f)(\tau) = \int_0^{+\infty} ir^{i\tau} r\partial_r f(r)\frac{dr}{r} = i\int_0^{+\infty} r^{i\tau}\partial_r f(r)dr.$$

By integration by part we obtain: $M(ir\partial_r f)(\tau) = \tau M(f)(\tau)$, for any function f of class C^∞ and with compact support in $(0,+\infty)$

• According to (4) we have

$$M(i\ln(r)f)(\tau) = \int_0^{+\infty} r^{i\tau} i\ln(r) f(r)\frac{dr}{r},$$

just notice that

$$\partial_\tau r^{i\tau} = i\ln(r) r^{i\tau},$$

then we get:

$$\begin{aligned} M(i\ln(r)f)(\tau) \qquad &= \int_0^{\infty} \partial_\tau r^{i\tau} f(r)\frac{dr}{r} \\ &= \partial_\tau\left(\int_0^{+\infty} r^{i\tau} f(r)\frac{dr}{r}\right). \end{aligned}$$

So, $M(i\ln(r)f)(\tau) = \partial_\tau M(f)(\tau)$.

(3) • To show the dilation formula it suffices to set the change of variable $r' = r^c$. Which gives $\frac{dr'}{r'} = c\frac{dr}{r}$

So according to (4) we have:

$$M(f(r^c))(\tau) = \int_0^{+\infty} r^{i\tau} f(r^c)\frac{dr}{r} = \int_0^{+\infty} r'^{\frac{i\tau}{c}} f(r')\frac{1}{c}\frac{dr'}{r'} = c^{-1}M(f)(\frac{\tau}{c}).$$

• The second dilation formula is proved by setting as a change of variable $r' = c\ r$ we easily obtain the result.

(4) According to (4) we can write:

$$M(f*g)(\tau) = \int_0^{+\infty} r^{i\tau}(f*g)(r)\frac{dr}{r}, \tag{6}$$

using the definition of the convolution product

$$(f*g)(r) = \int_0^{+\infty} f(\frac{r}{r'})g(r')\frac{dr'}{r'},$$

therefore equality (6) is equivalent to:

$$M(f*g)(\tau) = \iint_{(0,+\infty)^2} r^{i\ tau} f(\frac{r}{r'})g(r')\frac{dr'}{r'}\frac{dr}{r}. \quad (7)$$

We set the change of variable $y = \frac{r}{r'}$ then we have: $\frac{dy}{y} = \frac{dr}{r}$.

So, with this change of variable and the equality (7) we get the result ie:

$$M(f*g)(\tau) = \Big(\int_0^{+\infty} r'^{i\tau} g(r')\frac{dr'}{r'}\Big)\Big(\int_0^{+\infty} y^{i\tau} g(y)\frac{dy}{y}\Big) = M(g)(\tau)M(f)(\tau).$$

(5) By a computation analogous to that of the convolution product we show that

$$M(fg) = M(f) *_{2\pi} M(g).$$

□

Lemma 2.3. *The Mellin transform of $f(\lambda x)$ is $\lambda^{-s}M[f(s), s]$*

Proof. The Mellin transform of $f(\lambda x)$ where $\lambda > 0$

$$M[f(\lambda s), s] = \int_0^{\infty} x^{s-1} f(\lambda x)dx$$

Let $t = \lambda x$, then $dt = \lambda dx$, $x = \frac{t}{\lambda}$,

$$\begin{aligned} M[f(\lambda s), s] &= \int_0^{\infty} \left(\frac{t}{\lambda}\right)^{s-1} f(t)\frac{dt}{\lambda} \\ &= \frac{1}{s}\int_0^{\infty} t^{s-1} f(t)dt = \lambda^{-s}M[f(s), s] \\ &= \lambda^{-s}M[f(s), s] \end{aligned}$$

□

Lemma 2.4. *The Mellin transform of $xf^{'}(x)$ is $-sM[f(s), s]$.*

Proof.

$$M[sf^{'}(s), s] = \int_0^{\infty} x^{s-1} x f^{'}(x)dx = \int_0^{\infty} x^s f^{'}(x)dx.$$

By integration by parts:
Let

$$u = x^s \Rightarrow u^{'} = sx^{s-1}$$
$$v^{'} = f^{'} \Rightarrow v = f(x)$$

then

$$\begin{aligned} M[sf^{'}(s), s] &= [x^s f(x)]_0^{\infty} - \int_0^{\infty} sx^{s-1} f(x)dx \\ &= -s\int_0^{\infty} x^{s-1} f(x)dx = -sM[f(s), s]. \end{aligned}$$

Provided that

$$\lim_{x\to\infty} x^s f(x) = 0 \;\; and \;\; \lim_{x\to 0^+} x^s f(x) = 0,$$

we deduce that

$$M[sf^{'}(s), s] = -sM[f(s), s]$$

□

Example 2.5. *Apply Mellin transform of* $x^2 f^{''}(x)$
Solution:

$$M[s^2 f^{''}(s), s] = \int_0^\infty x^{s-1} x^2 f^{''}(x)dx = \int_0^\infty x^{s+1} f^{''}(x)dx.$$

By integration by parts:
Let

$$u = x^{s+1} \Rightarrow u^{'} = (s+1)x^s$$
$$v^{'} = f^{''} \Rightarrow v = f^{'}(x),$$

then

$$M[s^2 f^{''}(s), s] = \Big[x^{s+1} f^{'}(x)\Big]_0^\infty - \int_0^\infty (s+1)x^s f^{'}(x)dx.$$

Provided that

$$\lim_{x\to\infty} x^{s+1} f^{'}(x) = 0 \;\; and \;\; \lim_{x\to 0^+} x^{s+1} f^{'}(x) = 0$$

$$\begin{aligned}
M[s^2 f^{''}(s), s] &= -(s+1)\int_0^\infty x^s f^{'}(x)dx \\
M[s^2 f^{''}(s), s] &= -(s+1)\int_0^\infty x^{s-1} x f^{'}(x)dx \\
&= -(s+1)M[sf^{'}(s), s] \\
&= -(s+1)(-s)M[f(s), s] \\
&= s(s+1)M[f(s), s],
\end{aligned}$$

then we deduce that

$$M[s^2 f^{''}(s), s] = s(s+1)M[f(s), s].$$

Provided that

$$\lim_{x\to 0^+} x^{s+1} f^{'}(x) = 0 \;,\; \lim_{x\to +\infty} x^{s+1} f^{'}(x) \neq 0$$

$$\lim_{x\to 0^+} x^s f(x) = 0 \;\; and \;\; \lim_{x\to\infty} x^s f^{(}x) = 0.$$

Example 2.6. *Apply Mellin Transform of $xe^{-x} = g(x)$.*
Solution:

$$\begin{aligned} M[g(s), s] &= M[se^{-s}, s] = \int_0^\infty x^{s-1} x e^{-x} dx \\ M[g(s), s] &= \int_0^\infty x^s e^{-x} dx = \int_0^\infty x^{s+1-1} e^{-x} dx = \int_0^\infty x^s e^{-x} dx \\ M[se^{-s}, s] &= \Gamma(s+1) \;\; with \;\; Re(s+1) > 0. \end{aligned}$$

3. Problem Formulation:

Let

$$\begin{cases} \triangle u = 0 \quad in\ \mathbb{R} \times [0, \infty) \\ \phi(r, \theta) = 0 \;\; if \;\; \theta = 0\,, \;\; 0 < r < \infty \\ \phi(r, \pi) = g(r) \;\; if \;\; \theta = \pi\,, \;\; 0 < r < \infty \\ \phi(r, \theta) = \mathcal{O}(r^{\frac{1}{2}}) \;\; as \;\; r \to 0\,, \;\; 0 < r < \infty \\ \phi(r, \theta) = \mathcal{O}(r^{-\frac{1}{2}}) \;\; as \;\; r \to \infty\,, \;\; 0 < r < \infty. \end{cases} \tag{8}$$

Our goal is to determine the solution of this problem ($Laplace's \;\; Equation$) with singularity by using Mellin transform.

Solution:

Writing Laplace's equation in polar coordinates

$$r^2 \phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0$$

where $0 < r < \infty$ and $0 < \theta < \pi$.

We apply Mellin transform with respect the variable $r \in (0, \infty)$ to $r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0$ where $0 < r < \infty$ and $0 < \theta < \pi$.

By linearity we have:

$$M[s^2\phi_{ss} + r\phi_s + \phi_{\theta\theta}, s] = 0$$

$$M[s^2\phi_{ss}, s] + M[s\phi_s, s] + M[\phi_{\theta\theta}, s] = 0$$

$$s(s+1)M[\phi] - sM[\phi] + \frac{\partial^2}{\partial\theta^2} M[\phi] = 0.$$

Here:$\phi = \phi(s, \theta)$. Now,

$$\frac{\partial^2}{\partial\theta^2} M[\phi(s, \theta), s] + s^2 M[\phi(s, \theta), s] = 0,$$

we have a second order (ODE). Let $\Phi(\theta) = M[\phi(s, \theta), s]$, then we get,

$$\frac{d^2}{dt^2}\Phi(\theta) + s^2\Phi(\theta) = 0,$$

we use characteristic equation:

$$\alpha^2 + s^2 = 0 \Rightarrow \alpha^2 = -s^2 \Rightarrow \alpha = \pm is$$

then,

$$\Phi(s,\theta) = A\cos(s\theta) + B\sin(s\theta)$$

where $A, B \in \mathbb{R}$.
Using:

$$\phi(r,0) = 0 \Rightarrow M[\phi(s,0),s] = 0 \Rightarrow \Phi(s,0) = 0$$

therefore

$$\Phi(s,0) = A\cos(s.0) + B\sin(s.0) = A = 0,$$

then, we deduce:

$$\Phi(s,\theta) = B\sin(s.\theta)\,,\ B \in \mathbb{R}.$$

Using $\phi(r,\pi) = g(r)$, we apply Mellin transform to both sides we obtain

$$M[\phi(s,\pi),s] = M[g(s),s]$$

$$\Phi(s,\pi) = M[g(s),s]. \tag{9}$$

Since $\Phi(s,\theta) = B\sin(s\theta)$, evaluated at $\theta = \pi$ we obtain:

$$\Phi(s,\pi) = B\sin(s\pi) \tag{10}$$

(9) and (10) gives:

$$B = \frac{M[g(s),s]}{\sin(s\pi)}\ ,\ provided\ \ s \notin \mathbb{Z}.$$

Let

$$G(s) = M[g(s),s],$$

then

$$B = \frac{G(s)}{sin(s\pi)},$$

thus

$$\begin{aligned}\Phi(s,\theta) &= B\sin(s\theta)\\ &= \frac{G(s)}{\sin(s\pi)}\sin(s\theta)\end{aligned}$$

then we get,

$$\Phi(s,\theta) = \frac{\sin(s\theta)}{\sin(s\pi)}G(s).$$

Now let

$$g(r) = re^{-r} \Rightarrow G(s) = M[g(s),s] = \Gamma(s+1)$$

see example 2.6, then

$$\Phi(s,\theta) = \frac{\sin(s\theta)}{\sin(s\pi)}\Gamma(s+1)\,,\ 0 < \theta < \pi$$

Poles in the complex s-plane.

$$\sin(s\pi) = 0 \Rightarrow s\pi \in \pi\mathbb{Z} \Rightarrow s \in \mathbb{Z}$$

so, the pole of $\Phi(s,\theta)$ are simple poles given by $s = k\ ,\ k \in \mathbb{Z}$.
Finally, we can deduce $\phi(r,\theta)$ by applying Mellin transform inverse.

4. Polar Coordinates

We recall here a simplified version of Kondratiev's results in our context and with our ratings. We always limit ourselves to the case of the dimension 2 and we work in $\Omega = \mathbb{R}_+^* \times (\theta_-, \theta_+)$.
We consider a differential operator of order 2 (so as not to weigh down this presentation).

$$L(x, \partial_x) = \sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha$$

with coefficients a_α of class $\mathcal{C}^\infty$, elliptical. The operators intervening in the boundary conditions are data by operators

$$B_\pm(x, \partial_x) = \sum_{|\beta| \leq m_\pm} \frac{b_\beta(x)}{|x|^{\mu_\beta}} \partial_x^\beta$$

with $\mu_\beta \leq m_\pm - |\beta|$ and the coefficients b_β are of class $\mathcal{C}^\infty$.
We associate with these operators the main parts in $x = 0$

$$\begin{aligned} L_0 &= \sum_{|\alpha|=2} a_\alpha(0) \partial_x^\alpha \quad textrmelliptical \\ B_{0,\pm} &= sum_{|\beta|+\mu^\beta = m_\pm} \frac{b_\beta(0)}{|x|^{m_\pm - |\beta|}} \partial_x^\beta \quad (\neq 0)\ ;. \end{aligned}$$

We make the hypothesis on the boundary problem

$$L(x, \partial_x)u = f \tag{11}$$
$$B_-(x, \partial_x)u\big|_{\theta=\theta_-} = g_- \tag{12}$$
$$B_+(x, \partial_x)u\big|_{\theta=\theta_+} = g_+ \tag{13}$$

checks Lopatinski conditions outside of $x = 0$, which ensures the ellipticity of the boundary problem outside the corner $x = 0$.
We then write the operators L_0, $B_{0,\ pm}$ in coordinates polar

$$L_0(x, \partial_x) = r^{-2} \tilde{L}_0(r\partial_r, \partial_\theta), \quad B_0(x, \partial_x) = r^{-m_\pm} \tilde{B}_0(r\ partial_r, \partial_\theta).$$

The system

$$\tilde{L}_0(iz, \partial_\theta)v = \varphi \tag{14}$$
$$\tilde{B}_{0,-}(iz, \partial_\theta)v(\theta_-) = \psi_- \tag{15}$$
$$\tilde{B}_{0,+}(iz,\ partial\ _{theta})v(\ theta_+) =\ psi_+ \tag{16}$$

then admits a resolvent $R(z)$ meromorphic on $\mathcal{C}$ according to the theory of analytical Fredholm (this condition is in general, in any dimension, still induced by ellipticity assumptions of this system).

In this more limited context, we specify the statements of Kondratiev's Theorems 1.2 and 2.2 see [3]. The first result concerns the case where the operators $L(x, \partial_x)$ and $B_\pm(x, \partial_x)$ are homogeneous operators :

$$L(x, \partial_x) = L_0(x, \partial_x) \quad \text{and} \quad B_\pm(x, \partial_x) = B_{0,\pm}(x, \partial_x)\ ;.$$

Theorem 4.1. *Suppose that* $u \in H^{k+2,k+2-\frac{\alpha}{2}}(\Omega)$ *is a solution of (11) (12) (13) with* $f \in H^{k_1,k_1-\frac{\alpha_1}{2}}(\Omega)$ *and* $g_\pm \in H^{k_1+2-m\ _{pm}-\frac{1}{2},k_1+2-m\ _{pm}-\frac{1}{2}-\frac{\alpha_1}{2}}(\ R_+^*)$ *with*

$$h_1 = \frac{2k_1 + 4 - 2 - \alpha_1}{2} > \frac{2k + 4 - 2 - \alpha}{2} = h, \quad k_1 \geq k.$$

Proof. Suppose further that the solver $R(z)$ of the problem (14) (15) (16) has no imaginary part pole h_1. So

$$u = \sum_j \sum_{\nu=0}^{\mu_j - 1} a_{j,\nu} r \psi_{\nu,j}(\theta) + w(x)$$

or

- the sum over j is a sum over the poles of $R(z)$ belonging at $\{h < \ Imz < h_1\}$ and μ_j being the multiplicity of the pole λ_j;
- the remainder $w(x)$ belongs to $H^{k_1+2,k_1+2-\frac{\alpha_1}{2}}$ with

$$\|w\|_{H^{k_1+2,k_1+2-\frac{\alpha_1}{2}}} \leq C\Big[\ \|u\|_{H^{k+2,k+2-\frac{\alpha}{2}}} + \|f\|_{H^{k_1,k_1-\frac{\alpha_1}{2}}} + \|g_\pm\|_{H^{k_1+2-m_\pm-\frac{1}{2},k_1+2-m_\pm-\frac{1}{2}-\frac{\alpha_1}{2}}}\ \Big]\ ;.$$

□

- In the article [?] by Kondratiev the statement is given with weighted spaces denoted $\overset{circ^m}{W}_{\alpha}$, $m \in \ N$, $\alpha \in \mathbb{R}$, (p 231). With our ratings, they coincide with $H^{m,m-\frac{\alpha}{2}}$.
- Note that for this statement the integer k_1, the real $-\alpha_1$ and therefore the real h_1 can be taken as large that we want. So if the data is arbitrarily regular (and cancel each other out to an arbitrary order in $x = 0$) we obtain a full asymptotic expansion of u in the neighborhood of $x = 0$ in calculating all the poles of $R(z)$ in $\{\Im z > h\}$ which contain the poles of Mu, and the corresponding residuals of Mu at these points.
- The proof of this result is the general version of the case particular that we presented in the previous paragraph. We we will repeat these calculations precisely in other cases, with even data having mesomorphic Mellin transforms, in which case the poles of Mu (and therefore
  the asymptotic expansion of u) are given by the poles of $R(z)$ and the poles of the transforms of Mellin's data.

The second statement concerns the entire problem.

Theorem 4.2. *Suppose that the solution u of (11) (12) (13) belongs to* $H^{k+2,k+2-\frac{\alpha}{2}}(\Omega)$ *with* $f \in H^{k,k-\frac{\alpha_1}{2}}(\Omega)$ *and* $g_\pm \in H^{k+2-m_\pm-\frac{1}{2},k+2-m_\pm-\frac{1}{2}-\frac{\alpha_1}{2}}(\ R_+^*)$, $\alpha - 2 \leq \alpha_1 < \alpha$

and $f, g_\pm$ to support included in $\{r \leq \rho_0\}$. Suppose further that the solver $R(z)$ of the problem (14) (15) (16) has no imaginary part pole $h_1 = \frac{-2-\alpha_1+2k+4}{2}$. So

$$u = \sum_j \sum_{\nu=0}^{\mu_j-1} a_{j,\nu} r^{-i\lambda_j} \ln^\nu(r)\ psi_{\nu,j}(\theta) + w(x)$$

or

- *the sum over j is a sum over the poles of $R(z)$ belonging at $\{h < \Im z < h_1\}$, $h = \frac{-2-\alpha+2k+4}{2}$ and μ_j being the multiplicity of the pole λ_j; item the remainder $w(x)$ belongs to $H^{k+2,k+2-\frac{\alpha_1}{2}}$ with*

$$\|w\|_{H^{k+2,k+2-\frac{\alpha_1}{2}}} \leq C\Big[\ \|u\|_{H^{k+2,k+2-\frac{\alpha}{2}}} + \|f\|_{H^{k,k-\frac{\alpha_1}{2}}} + \|g_\pm\|_{H^{k+2-m_\pm-\frac{1}{2},k+2-m_\pm-\frac{1}{2}-\frac{\alpha_1}{2}}}\ \Big]\ ;.$$

Here we have taken $k_1 = k$ with the additional limitation $\alpha_1 \geq \alpha - 2$ on $-\alpha_1$. This is due to the terms lower order of $L(x, \partial_x)$ and $B_\pm(x, \partial_x)$, which must intervene if we want a complete asymptotic development of the solution and that the statement does not not take into account. What we will retain essentially from these statements is that a asymptotic development of the solution of (11) (12) (13) is determined by the poles of the solver $R(z)$ of the problem homogeneous (14) (15) (16) footnote and possibly the poles of Mellin transforms of the data. This is valid until any order for the homogeneous problem for data " regular " and up to a limited order if the operators $L(x, \partial_x)$, $B_\pm(x, \partial_x)$ contain terms lower order.

5. ASYMPTOTIC TYPES

In the previous section, we saw that at least first approximation, the asymptotic expansion of the solutions of (11) (12) (13) is determined by the study of the poles of the solver of the main homogeneous problem (14) (15) (16) (and possibly by the poles of Mellin transforms of the data). Before introducing definitions some remarks are in order:

- The final objective of our work is to improve the convergence methods of decomposition of domains in the neighborhoods of corners. As a result we only work with solutions truncated, $\chi(r)u$, with $\chi \in \mathcal{C}_0^\infty(\mathbb{R}_+)$, $\sup \chi \subset \{r \leq 1\}$ and $\chi \equiv 1$ in the neighborhood of $\{r = 0\}$, we can even put ourselves in the situation $u = \chi(r)u$. The elliptical problems that we consider are given by a variational formulation which ensure $u \in H^1(\Omega)$. In consequently, in dimension 2, we can already state that $M[u] = M[\chi u]$ is a holomorphic function in $\{\ Imz < 0\}$. Like the operators $L(x, \partial_x)$ and $B_+(x, \partial_x)$ are differentials the data $f\ inL^2(\Omega)$ has a holomorphic Mellin transform in $\{\Im z < -1\}$ and the data $g_\pm$ has also a holomorphic Mellin transform in a half-plane inferior.
- For the same reason as above we will limit the study to main homogeneous problem (14) (15) (16). Kondratiev's results assure us that we correctly describe the nature of the asymptotic development of the solutions if we do not is only interested in the first terms of asymptotic development. We thus come back to a problem with " easy " algebraic processing.

- To determine the main parts $L_0(x, \partial_x)$ and $B_{0,\pm}(x, \partial_x)$, just write the operators in polar coordinates, to consider ∂_r, $\frac{1}{r}\partial_\theta$ as operators of order -1, the multiplication by r^α as an operator of order α, the operators $r\partial_r$ and ∂_θ as operators of order 0 ... and keep only the terms of degree lowest homogeneity.
- C in accordance with the remarks in the previous section and the description of spaces $H^s(\Omega)$ and $H^s(\mathbb{R}_+^*)$ the data f, $g_\pm$ are not necessarily in spaces with weight, corresponding to holomorphic Mellin transforms as in the statements of Kondratiev. These Mellin transforms can have meromorphic extensions and asymptotic development of u involves both the poles of the resolvent $R(z)$ and the poles of the Mellin transforms of the data.
- Finally, the regularity of the rest does not matter. We do not wish only work on the first terms of development asymptotic, the remainder being considered negligible. For the traces on the right $\left\{\Im z = \gamma - \frac{n}{2}\right\}$ we will simply write that they are $L^2(\ R + i(\gamma - \frac{n}{2}))$ even if we actually have more consistency in many cases. We will we focus on the position of the poles of Mellin transforms and possibly on the calculation of the residues.

All these remarks lead to the following definitions which facilitate discussion.

Definition 5.1. *We will call asymptotic type in Ω (resp. On R_+^*) a finite collection $\mathcal{T} = ((\lambda_j, \mu_j, \varphi_{j,\nu}(\theta),\ 0 \leq \nu \leq \mu_j - 1))_{j\in\{1,\dots,N\}}$ (resp. $\mathcal{T} = ((\lambda_j, \mu_j))_{j\in\{1,\ ldots,N\}}$) where λ_j are points of $\{\Im z \leq 0\}$, the integers μ_j are the multiplicities of the poles in λ_j and $\varphi_{j,\nu} \in L^2((\theta_-, \theta_+))$.*
We will note for $\gamma \geq 0$ $L^{2,\gamma}_{r\leq 1,\mathcal{T}}(\Omega)$ (resp. $L^{2,\gamma}_{r\leq 1,\mathcal{T}}(\mathbb{R}_+^*)$) the set of functions $u \in L^2(\Omega)$ (resp. $U \in L^2(\mathbb{R}_+^*)$) to support in $\{r \leq 1\}$ whose Mellin transform admits a meromorphic prolongation in $\{\ Imz <\ gamma - 1\}$ (resp $\left\{\Im z < \gamma - \frac{1}{2}\right\}$) with trace $L^2(\mathbb{R} + i(\gamma - 1); L^2(\theta_-, \theta_+))$ (resp. $L^2(\mathbb{R} + i(\gamma - \frac{1}{2}))$) and poles given by the asymptotic type $\mathcal{T}$.

We sometimes read in the literature the name of the group of singularities for the asymptotic type. This last term we seems more appropriate in the sense that it corresponds to development asymptotic data and solutions in the neighborhood of $r = 0$. In fact this notion contains the Taylor expansions of a function $\mathcal{C}^\infty$ at a point inside a open. Just replace the interval (θ_-, θ_+) by the circle S^1, $\lambda_j = ij$ and functions $\varphi_{\nu,j}$ equal to $\cos(j\theta), \sin(j\theta)$. A simple elimination of the poles of the Mellin transform already done, in several cases leads to

Proposition 5.2. *(1) If $u \in L^{2,\gamma}_{r\leq 1,\mathcal{T}}(\Omega)$, with none pole of $\mathcal{T}$ of imaginary part γ, then we have*

$$u(x) = 1_{[0,1]}(r) \sum_{0\leq \Im\lambda_j < \gamma - 1} \sum_{\nu=0}^{\mu_j - 1} c_{j,\nu} r^{-i\lambda_j} \ln^\nu(r) \varphi_{j,\nu}(\theta) + w(x)$$

with $w \in L^{2,\gamma}(\Omega)$ and where the $(\lambda_j, \mu_j, \phi_{j,\nu})$ are those given by $\mathcal{T}$.

(2) If $u \in L^{2,\gamma}_{r\leq 1,\mathcal{T}}(\mathbb{R}^\star_+)$, with none pole of $\mathcal{T}$ of imaginary part γ, then we have

$$u(r) = 1_{[0,1]}(r) \sum_{0\leq \Im\lambda_j<\gamma-\frac{1}{2}} \sum_{\nu=0}^{\mu_j-1} c_{j,\nu} r^{-i\lambda_j} \ln^\nu(r) + w(r)$$

with $w \in L^{2,\gamma}(\mathbb{R}^\star_+)$ and where the $(\lambda_j, \mu_j)$are those given by $\mathcal{T}$.

The resolution after Mellin transform of the homogeneous problem leads to also at the

Proposition 5.3. *If $u \in H^1(\Omega)$, $\sup u \subset \{r\ leq 1\}$ solves the homogeneous problem (14) (15) (16) with $f \in L^{2,\gamma}_{r\leq 1,\mathcal{T}}(\Omega)$, $\gamma \leq 2$, $g_\pm \in L^{2,\gamma+2-m_\pm-\frac{1}{2}}_{r\leq 1,\mathcal{T}_\pm}(\mathbb{R}^*_+)$, and if the poles of $\mathcal{T}$ translated by $2i\gamma$, the poles of $\mathcal{T}_\pm$ translated from $m_\pm i$ and the poles of the resolving $R(z)$ does not meet $\{\Im z = \gamma + 1\}$ then $u \in L^{2,\gamma+2}_{r\leq 1,\mathcal{T}_\infty}(\Omega)$ where the type asymptotic $\mathcal{T}_\infty$ is deduced by union (by adding the multiplicities optionally) of $\mathcal{T}$, $\mathcal{T}_\pm$ and poles of $R(z)$.*

Proof. From the hypotheses of the proposition 5.3 and of the definition 5.1 we have

- $f \in L^{2,\gamma}_{r\leq 1,\mathcal{T}}(\Omega)$, so $M(f)$ admits a mesomorphic prolongation in $\Im(z) < \gamma - 1$, moreover $f \in L^2$,
- of the same if $g_\pm \in L^{2,\gamma+2-m_\pm-\frac{1}{2}}_{r\leq 1,\mathcal{T}_\pm}(\mathbb{R}^*_+)$ then $M(g_\pm)$ admits a mesomorphic continuation in $\Im(z) < \gamma + 2 - m_\pm - \frac{1}{2} - \frac{1}{2} = \Im(z) < \gamma + 1 - m_\pm$, in addition $g_\pm \in L^2(\mathbb{R}^*_+)$.

Now we know that

$$Mu(z) = R(z)\Big[M(r^2 f)(z), M(r^{m_\pm} g_\pm)(z)\Big]$$

furthermore $M(r^2 f)(z) = M(f)(z - 2i)\mathrm{et}M(r^{m_\pm} g_\pm)(z) = M(g_\pm)(z - im_\pm)$.

So the poles of $M(f)$ (resp. $M(g)$) are translated by $2i$ (resp. $M_\pm i$), and therefore $M(f)(.-2i)$ admits a mesomorphic prolongation in $\{\Im(z) < \gamma+1\}$, moreover $M(g_\pm)(.-im_\pm)$ admits a mesomorphic prolongation in $\Im(z) < \gamma + 1 - m_\pm + m_\pm = \Im(z) < \gamma + 1$. If we further assume that the poles of the resolvent $R(z)$ do not meet $\Im(z) = \gamma + 1$ then $M(u)$ is a mesomorphic function in $\Im(z) < \gamma + 1$, so $u \in L^{2,\gamma+2}_{r\leq 1,\mathcal{T}_\infty}(\Omega)$.

So the asymptotic type $\mathcal{T}_\infty$ is deduced by union (by adding the multiplicities optionally) of $mathcalT$, $\mathcal{T}_\pm$ and poles of $R(z)$.

□

According to the proposition 5.3, the asymptotic expansion of u, involves the poles of the resolvent and the poles of the data $(f, g_{\ pm})$, precise calculations will be made completely in special cases.

6. Conclusion

We work with differential operators with real coefficients. It is clear that in this case the asymptotic expansions of the solutions must be real for real data. This requires that the poles associated with the resolvent $R(z)$ of the homogeneous problem must have symmetries about the axis $i\mathbb{R}$ so that the first terms of the asymptotic expansion are real. We will in fact check in all cases that we interest that the poles are exactly on the imaginary axis. The problem we are looking at comes from a formulation variational which ensures the membership in H^1 of the solutions for the full domain and each subdomain (We will take good care to ensure this condition in terms of the choice of conditions interface). This eliminates the possibility of having a multiple pole at $z = 0$. We know so that the solutions we look at in the full field or in each subdomain admits an underline simple pole. From these two remarks we deduce that we must concentrate on the treatment of poles of the form $z = 0$ with $t > 0$.

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REFERENCES

[1] Altoum, S.H., Solution of Second Order Ordinary Differential Equation Associated with Toeplitz and Stiffness Matrices, *American Journal of Applied Sciences*, 2018, 416-422.

[2] Altoum, S.H., Analytical and Three Numerical approach to Solve Second Order ODEs, *International Journal of Advanced Scientific and Technical Research*, (January 2018), DOI: 10.26808/rs.st.i8v4.06.

[3] Baker, I.N., and Rippon, P.J., Convergence of infinite exponentials, *Annales Academiac Scentiarium Fennicae. Mathematika, Series AI*, **8** (1983), 179-186.

[4] Euler, L., De formulis exponentialibus replicatus, *Acta Academiac Petropolitenae*, **1** (2018), 38-60

[5] Folland, G.B., *Real Analysis: Modern Techniques and Their Applications*, 2nd ed. Wiley Interscience, 1999.

[6] Feynman, R.P., *Surely You're Joking, Mr. Feynman*, pp. 71-72, Bantam Books, 1986.

[7] Kuo, H.-H., *White noise distribution theory*, CRC press, Boca Raton, 1996.

[8] Rudin, W., *Principles of Mathematical Analysis*, United States, McGraw Hill, 1976.

[9] Williams, D., *Probability with Martingales*, Cambridge University Press, 1991.