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VERTEX EXPONENTS OF A CLASS OF TWO-COLORED HAMILTONIAN DIGRAPHS

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Abstract. A two-colored digraph $D^{(2)}$ is primitive provided there are nonnegative integers h and k such that for each pair of not necessarily distinct vertices u and v in $D^{(2)}$ there exists a (h,k)-walk in $D^{(2)}$ from u to v. The exponent of a primitive two-colored digraph $D^{(2)}$, $\exp(D^{(2)})$, is the smallest positive integer $h+k$ taken over all such nonnegative integers h and k. The exponent of a vertex v in $D^{(2)}$ is the smallest positive integer $s+t$ such that for each vertex u in $D^{(2)}$ there is an (s,t)-walk from v to u. We study the vertex exponents of primitive two-colored digraphs $L_n^{(2)}$ on $n \geq 5$ vertices whose underlying digraph is the Hamiltonian digraph consisting of the cycle $v_1 \to v_n \to v_{n-1} \to \cdots \to v_2 \to v_1$ and the arc $v_1 \to v_{n-2}$. For such two-colored digraph it is known that $2n^2-6n+2 \leq \exp(L_n^{(2)}) \leq (n^3-2n^2+1)/2$. We show that if $\exp(L_n^{(2)}) = (n^3-2n^2+1)/2$, then its vertex exponents lie on $[(n^3-2n^2-3n+4)/4, (n^3-2n^2+3n+6)/4]$ and if $\exp(L_n^{(2)}) = 2n^2-6n+2$, then its vertex exponents lie on $[n^2-4n+5, n^2-2n-1]$.

Key words: Two-colored digraph, primitive digraph, exponent, vertex exponent, hamiltonian digraph.

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Abstrak. Digraf dwiwarna $D^{(2)}$ adalah primitif dengan syarat terdapat bilangan bulat nonnegatif h dan k sehingga untuk setiap pasangan yang tidak perlu berbeda titik u dan v di $D^{(2)}$ terdapat sebuah jalan (h, k) di $D^{(2)}$ dari u ke v. Eksponen dari primitif digraf dwiwarna $D^{(2)}$, yang dinotasikan dengan $\exp(D^{(2)})$, adalah bilangan bulat positif terkecil $h + k$ dari semua jumlahan yang mungkin bilangan bulat nonnegatif h dan k . Eksponen dari sebuah titik v di $D^{(2)}$ adalah bilangan bulat positif terkecil $s + t$ sehingga untuk tiap titik u di $D^{(2)}$ terdapat sebuah jalan (s, t) dari v ke u. Pada paper ini akan dipelajari eksponen titik dari digraf dwiwarna primitif $L_n^{(2)}$ pada $n \geq 5$ titik dengan digraf dasar adalah digraf Hamilton yang memuat lingkaran $v_1 \to v_n \to v_{n-1} \to \cdots \to v_2 \to v_1$ dan busur $v_1 \to v_{n-2}$. Untuk digraf dwiwarna yang demikian telah diketahui bahwa $2n^2 - 6n + 2 \leq \exp(L_n^{(2)}) \leq (n^3 - 2n^2 + 1)/2$. Paper ini menunjukkan bahwa jika $\exp(L_n^{(2)}) = (n^3 - 2n^2 + 1)/2$, maka eksponen titiknya berada pada $[(n^3 - 2n^2 - 3n + 4)/4, (n^3 - 2n^2 + 3n + 6)/4]$ dan jika $\exp(L_n^{(2)}) = 2n^2 - 6n + 2$, maka eksponen titiknya berada pada $[n^2 - 4n + 5, n^2 - 2n - 1]$.

Kata kunci: Digraf dwiwarna, digraf primitif, eksponen, eksponen titik, digraf Hamilton.

1. Introduction

Given a vector $\mathbf{x}$ we use the notation $\mathbf{x} \geq 0$ to show that $\mathbf{x}$ is a nonnegative vector, that is, a vector each of whose entry is nonnegative. Thus for two vectors $\mathbf{x}$ and $\mathbf{y}$, the notion $\mathbf{x} \geq \mathbf{y}$ means that $\mathbf{x} - \mathbf{y} \geq 0$.

A digraph D is strongly connected provided for each pair of vertices u and v in D there is a uv-walk from u to v. A ministrong digraph is a strongly digraph such that removal of any single arc will result in a not strongly connected digraph. A strongly connected digraph D is *primitive* provided there exists a positive integer ℓ such that for every pair of not necessarily distinct vertices u and v in D there is a walk from u to v of length ℓ. The smallest of such positive integer ℓ is the *exponent* of D denoted by $\exp(D)$.

By a two-colored digraph $D^{(2)}$ we mean a digraph D such that each of it arcs is colored by either red or blue but not both colors. An (s, t)-walk in $D^{(2)}$ is a walk of length $s + t$ consisting of s red arcs and t blue arcs. For a walk w we denote $r(w)$ to be the number of red arcs in w and $b(w)$ to be the number of blue arcs in w. The length of w is $\ell(w) = r(w) + b(w)$ and the vector $\begin{bmatrix} r(w) \\ b(w) \end{bmatrix}$ is the composition of the walk w. A two-colored digraph $D^{(2)}$ is primitive provided there exist nonnegative integers h and k such that for each pair of vertices u and v in $D^{(2)}$ there is an (h, k)-walk from u to v. The smallest of such positive integer $h+k$ is the *exponent* of $D^{(2)}$ and is denoted by $\exp(D^{(2)})$. Researches on exponents of two-colored digraphs can be found in [2, 3, 5, 6] and [7].

Let $D^{(2)}$ be a strongly connected two-colored digraph and suppose that the set of all cycles in $D^{(2)}$ is $C = \{C_1, C_2, \ldots, C_q\}$. We define a cycle matrix of $D^{(2)}$

to be a 2 by q matrix

$$M = \begin{bmatrix} r(C_1) & r(C_2) & \cdots & r(C_q) \\ b(C_1) & b(C_2) & \cdots & b(C_q) \end{bmatrix},$$

that is M is a matrix such that its ith column is the composition of the ith cycle C_i, $i = 1, 2, \ldots, q$. If the rank of M is 1, the content of M is defined to be 0, and the content of M is defined to be the greatest common divisors of the 2 by 2 minors of M, otherwise. The following result, due to Fornasini and Valcher [1], gives an algebraic characterization of a primitive two-colored digraph.

Theorem 1.1. [1] *Let $D^{(2)}$ be a strongly connected two-colored digraph with at least one arc of each color. Let M be a cycle matrix of $D^{(2)}$. The two-colored digraph $D^{(2)}$ is primitive if and only if the content of M is* 1*.*

Let $D^{(2)}$ be a two-colored digraph on n vertices $v_1, v_2, \ldots, v_n$. Gao and Shao [4] define a more *local* concept of exponents of two-colored digraphs as follows. For any vertex v_k in $D^{(2)}, k = 1, 2, \ldots, n$, the exponent of the vertex v_k, denoted by $\gamma_{D^{(2)}}(v_k)$, is the smallest positive integer $p_1 + p_2$ such that for every vertex v in $D^{(2)}$ there is a (p_1, p_2)-walk from v_k to v. It is customary to order the vertices $v_1, v_2, \ldots v_n$ of $D^{(2)}$ such that $\gamma_{D^{(2)}}(v_1) \leq \gamma_{D^{(2)}}(v_2) \leq \cdots \leq \gamma_{D^{(2)}}(v_n)$. Gao and Shao [4] discuss the vertex exponents for primitive two-colored digraphs of Wielandt type, that is a Hamiltonial digraph consisting of the cycle $v_1 \to v_n \to v_{n-1} \to v_{n-2} \to \cdots \to v_2 \to v_1$ and the arcs $v_1 \to v_{n-1}$. Their results show that if the two-colored Wielandt digraph $W^{(2)}$ has only one blue arc $v_a \to v_{a-1}$, $a = 2, 3, \ldots, n-1$, then $\gamma_{D^{(2)}}(v_k) = n^2 - 2n + k - a + 1$. If the two-colored Wielandt digraph has two blue arcs then $\gamma_{D^{(2)}}(v_k) = n^2 - 2n + k$ or $\gamma_{D^{(2)}}(v_k) = n^2 - 2n + k + 1$. Suwilo [9] discusses the vertex exponents of two-colored ministrong digraphs $D^{(2)}$ on n vertices whose underlying digraph is the primitive extremal ministrong digraph D with $\exp(D) = n^2 - 4n + 6$.

We present formulae for vertex exponent of two-colored digraphs whose underlying digraph is the Hamiltonian digraph consisting of the cycle $v_1 \to v_n \to v_{n-1} \to v_{n-2} \to \cdots \to v_2 \to v_1$ and the arc $v_1 \to v_{n-2}$ where n is an odd integer with $n \geq 5$. In Section 2 we discuss previous result on exponent of two-colored Hamiltonian digraph. In Section 3 we present a way in setting up a lower and an upper bound for vertex exponents. We use these results in Section 4 to find vertex exponents for the class of two-colored Hamiltonian digraphs.

2. **Two-colored Hamiltonian Digrahs**

It is a well known result, see [7], that the largest exponent of a primitive two-colored digraph lies on the interval $[(n^3 - 2n^2 + 1)/2, (3n^3 + 2n^2 - 2n)/2]$ when n is odd and lies on the interval $[(n^3 - 5n^2 + 7n - 2)/2, (3n^3 - 2n^2 - 2n)/2]$ when n is even. The left end of the first interval is obtained using two-colored digraphs consisting of two cycles whose underlying digraph is the primitive Hamiltonian digraph L_n on $n \geq 5$ vertices which consists of an n-cycle $v_1 \to v_n \to v_{n-1} \to v_{n-2} \to v_{n-3} \to \cdots \to v_2 \to v_1$ and the arc $v_1 \to v_{n-2}$ as shown in Figure 1. Notice that the digraph L_n consists of exactly two cycles, they are the n-cycle and the $(n-2)$-cycle $v_1 \to v_{n-2} \to v_{n-3} \to \cdots \to v_2 \to v_1$. Since L_n is primitive, it is necessary that n is odd. Let $L_n^{(2)}$ be a two-colored digraph with underlying digraph is L_n. Let M be the cycle matrix of $L_n^{(2)}$. By Theorem 1.1 the following lemma, see [7, 8] for proof, gives necessary and sufficient condition for $L_n^{(2)}$ to be a primitive two-colored digraph.

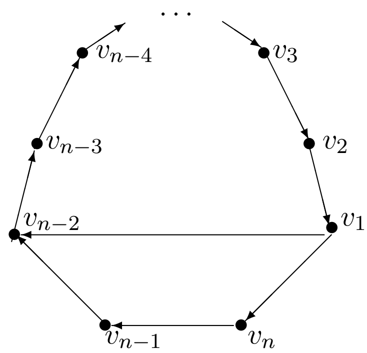

Figure 1. Digraph L_n

Lemma 2.1. [7, 8] *Let $L_n^{(2)}$ be a two-colored digraph with underlying digraph L_n. The digraph $L_n^{(2)}$ is primitive if and only if* $M = \begin{bmatrix} (n-1)/2 & (n+1)/2 \\ (n-3)/2 & (n-1)/2 \end{bmatrix}$.

The following theorem, due Suwilo [8] see also Shader and Suwilo [7], gives the lower and upper bound for exponent of class of two-colored digraphs whose underlying digraph is the digraph L_n.

Theorem 2.2. [7, 8] *Let $L_n^{(2)}$ be a two-colored digraph with underlying digraph L_n. Then* $2n^2 - 6n + 2 \leq \exp(L_n^{(2)}) \leq (n^3 - 2n^2 + 1)/2$.

Furthermore Suwilo [8] characterizes necessary and sufficient conditions for two-colored digraphs $L_n^{(2)}$ to have $\exp(L_n^{(2)}) = (n^3 - 2n^2 + 1)/2$ and to have $\exp(L_n^{(2)}) = 2n^2 - 6n + 2$, respectively.

Corollary 2.3. [8] *Let $L_n^{(2)}$ be a two-colored digraph with underlying digraph L_n. The* $\exp(L_n^{(2)}) = (n^3 - 2n^2 + 1)/2$ *if and only if $L_n^{(2)}$ has a red path of length* $(n+1)/2$ *and a blue path of length* $(n-1)/2$.

Corollary 2.4. [8] *Let $L_n^{(2)}$ be a two-colored digraph with underlying digraph L_n. The* $\exp(L_n^{(2)}) = 2n^2 - 6n + 2$ *if and only if $L_n^{(2)}$ has a unique* $(2,0)$*-path and this path lies on both cycles.*

Lemma 2.1 implies that for a two-colored digraph $L_n^{(2)}$ to be primitive, the n-cycles must contain exactly $(n+1)/2$ red arcs and the $(n-2)$-cycle must contain exactly $(n-1)/2$ red arcs. Corollary 2.3 implies that for the two-colored digraph $L_n^{(2)}$ to have exponent $(n^3 - 2n^2 + 1)/2$, the n-cycle must contain a red path of length $(n+1)/2$ and a blue path of length $(n-1)/2$. This implies there are four possible two-colored digraph $L_n^{(2)}$ with exponent $(n^3 - 2n^2 + 1)/2$. We characterize them as follows.

- The two-colored digraph $L_n^{(2)}$ is of Type I if the red arcs of $L_n^{(2)}$ are the arcs that lie on the path $v_n \to v_{n-1} \to v_{n-2} \to \cdots \to v_{(n-1)/2}$ of length $(n+1)/2$ plus the arc $v_1 \to v_{n-2}$.
- The two-colored digraph $L_n^{(2)}$ is of Type II if the red arcs of $L_n^{(2)}$ are the arcs that lie on the path $v_{(n-1)/2} \to v_{(n-3)/2} \to \cdots \to 1 \to v_n \to v_{n-1}$ of length $(n+1)/2$ plus the arc $v_1 \to v_{n-2}$.
- The two-colored digraph $L_n^{(2)}$ is of Type III if the red arcs of $L_n^{(2)}$ are the arcs that lie on the path $v_{(n+1)/2} \to v_{(n-1)/2} \to \cdots \to 2 \to 1 \to v_n$ of length $(n+1)/2$.
- The two-colored digraph $L_n^{(2)}$ is of Type IV if the red arcs of $L_n^{(2)}$ are the arcs that lie on the path $v_{n-1} \to v_{n-2} \to \cdots \to v_{(n-3)/2}$ of length $(n+1)/2$.

Considering Corollary 2.4, we have $\exp(L_n^{(2)}) = 2n^2 - 6n + 2$ if and only if the $(2,0)$-path in $L_n^{(2)}$ is the path $a \to a-1 \to a-2$ for some $3 \leq a \leq n-2$.

In Section 4 for two-colored digraphs $L_n^{(2)}$ with $\exp(L_n^{(2)}) = (n^3 - 2n^2 + 1)/2$ we show that the exponents of its vertices lie on $[(n^3 - 2n^2 - 3n + 4)/4, (n^3 - 2n^2 + 3n + 6)/4]$. For two-colored digraphs whose $\exp(L_n^{(2)}) = 2n^2 - 6n + 2$ we show that the exponents of its vertices lie on $[n^2 - 4n + 5, n^2 - 2n - 1]$.

3. Bounds for vertex exponents

In this section, a way in setting up an upper and a lower bound for vertex exponents of two-colored digraphs is discussed. We start with the lower bound of vertex exponent especially for primitive two-colored digraphs consisting of two cycles. We assume through out that the exponent of vertex v_k, $k = 1, 2, \ldots, n$ is obtained using (s,t)-walks.

Lemma 3.1. [9] *Let $D^{(2)}$ be a primitive two-colored digraph consisting of two cycles. Let v_k be a vertex in $D^{(2)}$ and suppose there is an (s,t)-walk from v_k to each vertex v_j in $D^{(2)}$ with $\begin{bmatrix} s \\ t \end{bmatrix} = M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ for some nonnegative integers q_1 and q_2. Then $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \geq M^{-1} \begin{bmatrix} r(p_{k,j}) \\ b(p_{k,j}) \end{bmatrix}$ for some path $p_{k,j}$ from v_k to v_j.*

Let v_k be a vertex in $D^{(2)}$. We note that from Lemma 3.1 for the vertex v_k and any vertex v_j in D we have

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \geq M^{-1} \begin{bmatrix} r(p_{k,j}) \\ b(p_{k,j}) \end{bmatrix} = \begin{bmatrix} b(C_2)r(p_{k,j}) - r(C_2)b(p_{k,j}) \\ r(C_1)b(p_{k,j}) - b(C_1)r(p_{k,j}) \end{bmatrix}.$$

If for some vertex v_j we have $b(C_2)r(p_{k,j}) - r(C_2)b(p_{k,j}) \geq 0$, then we define

$$u_0 = b(C_2)r(p_{k,j}) - r(C_2)b(p_{k,j}) \tag{1}$$

If for some vertex v_i we have $r(C_1)b(p_{k,i}) - b(C_1)r(p_{k,i}) \geq 0$, then we define

$$v_0 = r(C_1)b(p_{k,i}) - b(C_1)r(p_{k,i}) \tag{2}$$

By Lemma 3.1 we have $q_1 \geq u_0$ and $q_2 \geq v_0$. This implies

$$\begin{bmatrix} s \\ t \end{bmatrix} = M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \geq M \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

and hence

$$s + t \geq (r(C_1) + b(C_1))u_0 + (r(C_2) + b(C_2))v_0 = \ell(C_1)u_0 + \ell(C_2)v_0.$$

We have proved the following theorem.

Theorem 3.2. *Let $D^{(2)}$ be a primitive two-colored digraph consisting of two cycles C_1 and C_2 and let v_k be a vertex in $D^{(2)}$. For some vertex v_i and v_j in $D^{(2)}$ define $u_0 = b(C_2)r(p_{k,j}) - r(C_2)b(p_{k,j})$ and $v_0 = r(C_1)b(p_{k,i}) - b(C_1)r(p_{k,i})$. Then $\begin{bmatrix} s \\ t \end{bmatrix} \geq M \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$ and hence $\gamma_{D^{(2)}}(v_k) \geq \ell(C_1)u_0 + \ell(C_2)v_0$.*

We now discuss a way in setting up an upper bound. First we consider upper bound for exponents of certain vertices two-colored digraph consisting two cycles.

Proposition 3.3. *Let $D^{(2)}$ be a primitive two-colored digraph consisting of two cycles C_1 and C_2. Suppose v_k be a vertex of $D^{(2)}$ that belongs to both cycles C_1 and C_2. If for each $i = 1, 2, \ldots, n$ and for some positive integers s and t, there is a path $p_{k,i}$ from v_k to v_i such that the system*

$$M\mathbf{x} + \begin{bmatrix} r(p_{k,i}) \\ b(p_{k,i}) \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} \tag{3}$$

has nonnegative integer solution, then $\gamma_{D^{(2)}}(v_k) \leq s + t$.

PROOF. Assume that the solution to the system (3) is $\mathbf{x} = (x_1, x_2)^T$. Since $D^{(2)}$ is primitive, then M is invertible and hence x_1 and x_2 cannot be both zero. We note that v_k belongs to both cycles and we consider three cases.

If $x_1, x_2 > 0$, then the walk that starts at v_k, moves x_1 and x_2 times around the cycles C_1 and C_2 respectively and back at v_k, and then moves to v_i along the path $p_{k,i}$ is an (s,t)-walk from v_k to v_i. If $x_1 = 0$ and $x_2 > 0$, then the walk that starts at v_k, moves x_2 times around the cycle C_2 and back at v_k, then moves to vertex v_i along the path $p_{k,i}$ is an (s,t)-walk from v_k to v_i. Similarly if $x_1 > 0$ and $x_2 = 0$, then then the walk that starts at v_k, moves x_1 times around the cycle C_1

and back at v_k, then moves to vertex v_i along the path $p_{k,i}$ is an (s,t)-walk from v_k to v_i. Therefore for every vertex $v_i, i = 1, 2, \ldots, n$ there is an (s,t)-walk from v_k to v_i. The definition of exponent of vertex v_k implies that $\gamma_{D^{(2)}}(v_k) \leq s + t$.

Proposition 3.4 gives an upper bound of a vertex exponent in term of the vertex exponent of a specified vertex. We define $d(v_k, v)$ to be the distance from v_k to v, that is the length of a shortest walk from v_k to v.

Proposition 3.4. [9] *Let $D^{(2)}$ be a primitive two-colored digraph on n vertices. Let v be a vertex in $D^{(2)}$ with exponent $\gamma_{D^{(2)}}(v)$. For any vertex $v_k, k = 1, 2, \ldots, n$ in $D^{(2)}$ we have $\gamma_{D^{(2)}}(v_k) \leq \gamma_{D^{(2)}}(v) + d(v_k, v)$*

4. **The vertex exponents**

In this section we discuss the vertex exponents of class of two-colored digraphs $L_n^{(2)}$ whose underlying digraph is the digraph L_n in Figure 1. We first discuss the case where $\exp(L_n^{(2)}) = (n^3 - 2n^2 + 1)/2$. Let v_k be a vertex in $L_n^{(2)}$ and suppose that the red path of length $(n+1)/2$ has $x*$ and $y*$ as its initial and terminal vertex, respectively. We use the the path $p_{k,y*}$ from v_k to $y*$ to determine the value of $u_0 = b(C_2)r(p_{k,y*}) - r(C_2)b(p_{k,y*})$ in equation (1). We use the path $p_{k,x*}$ from v_k to $x*$ in order to determine the value of $v_0 = r(C_1)b(p_{k,x*}) - b(C_1)r(p_{k,x*})$ in equation (2). We assume that $\gamma_{L_n^{(2)}}(v_k)$ is obtained using (s,t)-walks and we split our discussion into four parts depending on the type of the two-colored digraph $L_n^{(2)}$.

Lemma 4.1. *For the two-colored digraph $L_n^{(2)}$ of type I we have $\gamma_{L_n^{(2)}}(v_k) = (n^3 - 2n^2 - n + 2)/4 + k$ for all $k = 1, 2, \ldots, n$.*

Proof. We first show that $\gamma_{L_n^{(2)}}(v_k) \geq (n^3 - 2n^2 - n + 2)/4 + k$ for all $k = 1, 2, \ldots, n$. Since the red path of length $(n+1)/2$ in $L_n^{(2)}$ is the path $v_n \to v_{n-1} \to \cdots \to v_{(n-1)/2}$, we set $x^* = v_n$ and $y^* = v_{(n-1)/2}$. We split the proof into two cases.

Case 1: $1 \leq k \leq (n-1)/2$
Taking $y* = v_{(n-1)/2}$, we see that there are two paths from v_k to $v_{(n-1)/2}$. They are an $((n+1)/2, k)$-path and an $((n-1)/2, k-1)$-path. Using the $((n+1)/2, k)$-path $p_{k,(n-1)/2}$ from v_k to $v_{(n-1)/2}$ and the definition of u_0 in equation (1) we find

$$\begin{aligned} u_0 &= b(C_2)r(p_{k,(n-1)/2} - r(C_2)b(p_{k,(n-1)/2}) \\ &= \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) - \frac{n+1}{2}k = \frac{n^2-1}{4} - \frac{k(n+1)}{2}. \end{aligned} \quad (4)$$

Using the $((n-1)/2, k-1)$-path $p_{k,(n-1)/2}$ from v_k to $v_{(n-1)/2}$ and the definition of u_0 in equation (1) we have

$$\begin{aligned} u_0 &= b(C_2)r(p_{k,(n-1)/2} - r(C_2)b(p_{k,(n-1)/2}) \\ &= \left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}\right) - \frac{n+1}{2}(k-1) = \frac{n^2+3}{4} - \frac{k(n+1)}{2}. \end{aligned} \tag{5}$$

From equations (4) and (5) we conclude that $u_0 = (n^2-1)/4 - k(n+1)/2$.

Taking $x* = v_n$, there is a unique path $p_{k,n}$ from v_k to v_n which is a $(0,k)$-path. Using this path and the definition of v_0 in equation (2) we have

$$\begin{aligned} v_0 &= r(C_1)b(p_{k,n}) - b(C_1)r(p_{k,n}) \\ &= \frac{n-1}{2}k - \frac{n-3}{2}(0) = k(n-1)/2 \end{aligned}$$

By Theorem 3.2 we have

$$\begin{aligned} \begin{bmatrix} s \\ t \end{bmatrix} &\geq M \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = M \begin{bmatrix} (n^2-1)/4 - k(n+1)/2 \\ k(n-1)/2 \end{bmatrix} \\ &= \begin{bmatrix} (n^3-n^2-n+1)/8 \\ (n^3-3n^2-n+3+8k)/8 \end{bmatrix}. \end{aligned} \tag{6}$$

Therefore, we conclude that

$$\gamma_{L_n^{(2)}}(v_k) = s + t \geq (n^3 - 2n^2 - n + 2)/4 + k \tag{7}$$

for all $k = 1, 2, \ldots, (n-1)/2$.

Case 2: $(n+1)/2 \leq k \leq n$
Taking $y^* = v_{(n-1)/2}$, then there is a unique path $p_{k,(n-1)/2}$ from v_k to $v_{(n-1)/2}$ which is a $(k-(n-1)/2, 0)$-path. Using this path and considering the definition of u_0 in equation (1) we have $u_0 = k(n-1)/2 - (n-1)^2/4$. Taking $x^* = v_n$, there is a unique path $p_{k,n}$ from v_k to v_n. This path is a $(k-(n-1)/2, (n-1)/2)$-path. Using this path and the definition of v_0 in equation (2) we get $v_0 = (n-1)(2n-4)/4 - k(n-3)/2$. By Theorem 3.2 we get

$$\gamma_{L_n^{(2)}}(v_k) \geq \ell(C_1)u_0 + \ell(C_2)v_0 = (n^3 - 2n^2 - n + 2)/4 + k \tag{8}$$

for all $k = (n+1)/2, (n+3)/2, \ldots, n$.

Combining (7) and (8) we conclude that

$$\gamma_{L_n^{(2)}}(v_k) \geq (n^3 - 2n^2 - n + 2)/4 + k. \tag{9}$$

for all $k = 1, 2, \ldots, n$.

We next show the upper bound, that is $\gamma_{L_n^{(2)}}(v_k) \leq (n^3 - 2n^2 - n + 2)/4 + k$ for all $k = 1, 2, \ldots, n$. We first show that $\gamma_{L_n^{(2)}}(v_1) = (n^3 - 2n^2 - n + 2)/4 + 1$ and then we use Proposition 3.4 in order to determine the upper bound for exponents of other vertices. From (9) it is known that $\gamma_{L_n^{(2)}}(v_1) \geq (n^3 - 2n^2 - n + 2)/4 + 1$. Thus, it remains to show that $\gamma_{L_n^{(2)}}(v_1) \leq (n^3 - 2n^2 - n + 2)/4 + 1$. By considering equation (6) we show that for each $i = 1, 2, \ldots, n$ there is a walk from v_1 to v_i with composition

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} (n^3 - n^2 - n + 1)/8 \\ (n^3 - 3n^2 - n + 11)/8 \end{bmatrix}. \tag{10}$$

Let $p_{1,i}$ be a path from v_1 to v_i, $i = 1, 2, \ldots, n$. Notice that since M is an invertible matrix, the system

$$M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} r(p_{1i}) \\ b(p_{1i}) \end{bmatrix} = \begin{bmatrix} (n^3 - n^2 - n + 1)/8 \\ (n^3 - 3n^2 - n + 11)/8 \end{bmatrix} \tag{11}$$

has solution the integer vector

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (n-3)(n+1)/4 + (n+1)b(p_{1i})/2 - (n-1)r(p_{1i})/2 \\ (n-1)/2 + (n-3)r(p_{1i})/2 - (n-1)b(p_{1i})/2 \end{bmatrix}.$$

If $i = 1$, we can choose $r(p_{1,1}) = b(p_{1,1}) = 0$ and hence we have that $x_1 = (n^2 - 2n - 3)/4 > 0$ and $x_2 = (n-1)/2 > 0$. If $i = n$, then using the $(0,1)$-path we have $x_1 = (n-3)(n+1)/4 + (n+1)/2 \geq 0$ and $x_2 = 0$. If $i = (n-1)/2$, then using the ((n+1)/2,1)-path $v_1 \to v_n \to v_{n-1} \to \cdots \to v_{(n-1)/2}$ we have $x_1 = 0$ and $x_2 = (n+1)(n-3)/4$. Notice that for each vertex $v_i, i \neq n, (n-1)/2$, there exists a path p_{1i} from v_1 to v_i with $0 \leq r(p_{1i}) \leq (n-3)/2$ and $0 \leq b(p_{1i}) \leq (n-3)/2$. More over if $b(p_{1i}) \geq 1$, then either $r(p_{1i}) = (n-1)/2$ or $r(p_{1i}) = 1$. These facts imply that $x_1 > 0$ and $x_2 > 0$. Hence we now conclude that the system (11) has a nonnegative integer solution. Proposition 3.3 implies that $\gamma_{L_n^{(2)}}(v_1) \leq (n^3 - 2n^2 - n + 2)/4 + 1$. Combining this with (9) we have $\gamma_{L_n^{(2)}}(v_1) = (n^3 - 2n^2 - n + 2)/4 + 1$. Since for every $k = 2, 3, \ldots, n$ we have $d(v_k, v_1) = k - 1$, Proposition 3.4 implies that

$$\gamma_{L_n^{(2)}}(v_k) \leq (n^3 - 2n^2 - n + 2)/4 + k. \tag{12}$$

for all $k = 1, 2, \ldots, n$.

Now using (9) and (12) we conclude that $\gamma_{L_n^{(2)}}(v_k) = (n^3 - 2n^2 - n + 2)/4 + k$ for all $k = 1, 2, \ldots, n$.

Lemma 4.2. *For the two-colored digraph $L_n^{(2)}$ of type II we have $\gamma_{L_n^{(2)}}(v_k) = (n^3 - 2n^2 - 3n + 4)/4 + k$ for all $k = 1, 2, \ldots, n$.*

Proof. We first show that $\gamma_{L_n^{(2)}}(v_k) \geq (n^3 - 2n^2 - 3n + 4)/4 + k$ for all $k = 1, 2, \ldots, n$. Since the red path of length $(n+1)/2$ in $L_n^{(2)}$ is the path $v_{(n-1)/2} \to v_{(n-3)/2} \to \cdots \to v_1 \to v_n \to v_{n-1}$, we set $x^* = v_{(n-1)/2}$ and $y^* = v_{n-1}$. We split the proof into three cases.

Case 1*:* $1 \leq k \leq (n-1)/2$
Taking $y^* = v_{n-1}$, then there is a unique path $p_{k,n-1}$ from v_k to v_{n-1}. This path is a $(k+1, 0)$-path. Using this path and the definition of u_0 in equation (1) we have

$$u_0 = (k+1)(n-1)/2 \tag{13}$$

Taking $x^* = v_{(n-1)/2}$, there are two paths $p_{k,(n-1)/2}$ from v_k to $v_{(n-1)/2}$. They are a $(k,(n-3)/2)$-path and a $(k+1,(n-1)/2)$-path. Using the $(k,(n-3)/2)$-path and the definition of v_0 in equation (2) we have $v_0 = (n-3)(n-1-2k)/4$. Using the $(k+1,(n-1)/2)$-path and the definition of v_0 in equation (2) we have $v_0 = (n-3)(n-1-2k)/4+1$. Hence, we conclude that

$$v_0 = (n-3)(n-1-2k)/4. \tag{14}$$

By Theorem 3.2, equations (13) and (14) we conclude that

$$\begin{bmatrix} s \\ t \end{bmatrix} \geq M \begin{bmatrix} (k+1)(n-1)/2 \\ (n-3)(n-1-2k)/4 \end{bmatrix} = \begin{bmatrix} (n^3-n^2-5n+5)/8+k \\ (n^3-3n^2-n+3)/8 \end{bmatrix}. \tag{15}$$

From (15) we conclude that

$$s+t = \gamma_{L_n^{(2)}}(v_k) \geq (n^3-2n^2-3n+4)/4+k \tag{16}$$

for all $k = 1,2,\ldots,(n-1)/2$.

Case 2: $(n+1)/2 \leq k \leq n-1$

Taking $y^* = v_{n-1}$, there is a unique path $p_{k,n-1}$ from v_k to v_{n-1} which is a $((n+1)/2, k-(n-1)/2)$-path. Using this path and the definition of u_0 in equation (1) we have

$$u_0 = (n^2-1)/2 - k(n+1)/2. \tag{17}$$

Taking $x^* = v_{(n-1)/2}$, there is a unique path $p_{k,(n-1)/2}$ from v_k to $v_{(n-1)/2}$. This path is a $(0, k-(n-1)/2)$-path. Using this path and the definition of v_0 in equation (2), we have

$$v_0 = k(n-1)/2 - (n-1)^2/4. \tag{18}$$

Equations (17), (18) and Theorem 3.2 imply that

$$\gamma_{L_n^{(2)}}(v_k) \geq \ell(C_1)u_0 + \ell(C_2)v_0 = (n^3-2n^2-3n+4)/4+k \tag{19}$$

for all $k = (n+1)/2, (n+3)/2, \ldots, n-1$.

Case 3: $k = n$

There is a $(1,0)$-path $p_{k,n-1}$ from v_k to v_{n-1}. Using this path and the definition of u_0 in equation (1) we have $u_0 = (n-1)/2$. There is a $(1,(n-1)/2)$-path $p_{k,(n-1)/2}$ from v_k to $v_{(n-1)/2}$. Using this path and the definition of v_0 in equation (2) we find that $v_0 = (n-1)^2/4 - (n-3)/2$. Theorem 3.2 implies that

$$\begin{aligned} \gamma_{L_n^{(2)}}(v_k) &\geq \ell(C_1)u_0 + \ell(C_2)v_0 = (n^3-2n^2-3n+4)/4+n \\ &= (n^3-2n^2-3n+4)/4+k \end{aligned} \tag{20}$$

for $k = n$.

Now from (16), (19), and (20) we conclude that

$$\gamma_{L_n^{(2)}}(v_k) \geq (n^3-2n^2-3n+4)/4+k \tag{21}$$

for all $k = 1,2,\ldots,n$.

We next show the upper bound, that is $\gamma_{L_n^{(2)}}(v_k) \leq (n^3-2n^2-3n+4)/4+k$ for all $k = 1,2,\ldots,n$. We first show that $\gamma_{L_n^{(2)}}(v_1) = (n^3-2n^2-3n+4)/4+1$ and then we use Proposition 3.4 in order to determine the upper bound for exponents of other vertices. From (16) it is known that $\gamma_{L_n^{(2)}}(v_1) \geq (n^3-2n^2-3n+4)/4+1$. Thus, it remains to show that $\gamma_{L_n^{(2)}}(v_1) \leq (n^3-2n^2-3n+4)/4+1$. By considering (16) we show that for each $i = 1,2,\ldots,n$ there is a walk from v_1 to v_i consisting of $(n^3-n^2-5n+13)/8$ red arcs and $(n^3-3n^2-n+3)/8$ blue arcs.

Let p_{1i} be a path form v_1 to v_i, $i = 1,2,\ldots,n$. Notice that the system

$$M\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix} r(p_{1i}) \\ b(p_{1i}) \end{bmatrix} = \begin{bmatrix} (n^3-n^2-5n+13)/8 \\ (n^3-3n^2-n+3)/8 \end{bmatrix} \tag{22}$$

has integer solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b(p_{1,i}) + (2+b(p_{1,i})-r(p_{1,i}))(n-1)/2 \\ (n-3)[n-3+2r(p_{1,i})]/4 - b(p_{1,i})(n-1)/2 \end{bmatrix}.$$

If $i = 1$, we choose $r(p_{1,1}) = b(p_{1,1}) = 0$. This implies $x_1 = n-1 > 0$ and $x_2 = (n-3)^2/4 > 0$. Since for every $i = 2,3\ldots,n$ there is a path p_{1i} from v_1 to v_i with $b(p_{1i})-r(p_{1i}) \geq -1$, we have $x_1 = b(p_{1,i})+[2+b(p_{1,i})-r(p_{1,i})](n-1)/2 \geq 0$. Notice also that for any $i = 2,3,\ldots,n$ there is a path p_{1i} from v_1 to v_i with $b(p_{1i}) \leq (n-3)/2$ and every such path p_{1i} has $r(p_{1i}) \geq 1$. Hence $x_2 = (n-3)[n-3+2r(p_{1,i})]/4 - b(p_{1,i})(n-1)/2 \geq 0$. Therefore the system (22) has a nonnegative integer solution. Since v_1 lies on both cycles, Proposition 3.3 implies that

$$\gamma_{L_n^{(2)}}(v_1) \leq (n^3-2n^2-3n+4)/4+1. \tag{23}$$

Now combining (16) and (23) we find that $\gamma_{L_n^{(2)}}(v_1) = (n^3-2n^2-3n+4)/4+1$. Since for any $k = 1,2,\ldots,n$ we have $d(v_k,v_1) = k-1$, by Proposition 3.4 we have

$$\gamma_{L_n^{(2)}}(v_k) \leq (n^3-2n^2-3n+4)/4+k \tag{24}$$

for all $k = 1,2,\ldots,n$.

Finally, combining (21) and (24) we conclude that $\gamma_{L_n^{(2)}}(v_k) = (n^3-2n^2-3n+4)/4+k$ for all $k = 1,2,\ldots,n$.

Lemma 4.3. *For the two-colored digraph $L_n^{(2)}$ of type III we have $\gamma_{L_n^{(2)}}(v_k) = (n^3-2n^2-3n)/4+k$ for all $k = 1,2,\ldots,n$.*

Proof. We first show that $\gamma_{L_n^{(2)}}(v_k) \geq (n^3-2n^2-3n)/4+k$ for all $k = 1,2,\ldots,n$. Since the red path of length $(n+1)/2$ is the path $v_{(n+1)/2} \to v_{(n-1)/2} \to \cdots \to v_1 \to v_n$, we set $x^* = v_{(n+1)/2}$ and $y^* = v_n$. We split the proof into two cases.

Case 1*:* $1 \leq k \leq (n+1)/2$
Considering the $(k,0)$-path from v_k to v_n and the definition of u_0 in equation (1) we have

$$u_0 = k(n-1)/2. \tag{25}$$

We note that there are two paths from v_k to $v_{(n+1)/2}$. They are a $(k-1,(n-3)/2)$-path and a $(k,(n-1)/2)$-path. Using the $(k-1,(n-3)/2)$-path and the definition of v_0 in equation (2) we have that $v_0=(n-3)(n-2k+1)/4$. Using the $(k,(n-1)/2)$-path and the definition of v_0 in equation (2) we have $v_0=(n-3)(n-2k+1)/4+1$. Hence, we conclude that

$$v_0=(n-3)(n-2k+1)/4. \tag{26}$$

Now by considering Theorem 3.2, equation (25) and equation (26) we have

$$\begin{bmatrix} s \\ t \end{bmatrix} \geq M \begin{bmatrix} (n-1)/2 \\ (n-3)(n-2k+1)/4 \end{bmatrix} = \begin{bmatrix} (n^3-n^2-5n-3)/8+k \\ (n^3-3n^2-n+3)/8 \end{bmatrix}. \tag{27}$$

Thus from (27) we conclude that

$$\gamma_{L_n^{(2)}}(v_k) \geq (n^3-2n^2-3n)/4+k \tag{28}$$

for all $k=1,2,\ldots,(n+1)/2$.

Case 2: $(n+3)/2 \leq k \leq n$
Considering the $((n+1)/2, k-(n+1)/2)$-path from v_k to v_n and the definition of u_0 in equation (1) we have $u_0=(n+1)(n-k)/2$. Considering the $(0,k-(n+1)/2)$-path from v_k to $v_{(n+1)/2}$ and the definition of v_0 in equation (2) we have $v_0=(n-1)(2k-n-1)/4$. By Theorem 3.2, we have

$$\gamma_{L_n^{(2)}}(v_k) \geq \ell(C_1)u_0+\ell(C_2)v_0=(n^2-2n^2-3n)/4+k \tag{29}$$

for all $k=(n+3)/2,(n+5)/2,\ldots,n$.

Now from (28) and (29) we conclude that

$$\gamma_{L_n^{(2)}}(v_k) \geq (n^3-2n^2-3n)/4+k \tag{30}$$

for all $k=1,2,\ldots,n$.

We next show that $\gamma_{L_n^{(2)}}(v_k) \leq (n^3-2n^2-3n)/4+k$ for all $k=1,2,\ldots,n$. For this purpose we first show that $\gamma_{L_n^{(2)}}(v_1)=(n^3-2n^2-3n)/4+1$ and then we use Proposition 3.4 in order to get the upper bounds for $\gamma_{L_n^{(2)}}(v_k)$ for $k=2,3,\ldots,n$. From (30) it is inferred that $\gamma_{L_n^{(2)}}(v_1) \geq (n^3-2n^2-3n)/4+1$. It remains to show that $\gamma_{L_n^{(2)}}(v_1) \leq (n^3-2n^2-3n)/4+1$.

By considering (27) we show that for each vertex $v_i, i=1,2,\ldots,n$, there is a walk from v_1 to v_i consisting of $(n^3-n^2-5n+5)/8$ red arcs and $(n^3-3n^2-n+3)/8$ blue arcs. For $i=1,2,\ldots,n$ let $p_{1,i}$ be a path from v_1 to v_i. Consider the system of equations

$$M\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} r(p_{1,i}) \\ b(p_{1,i}) \end{bmatrix} = \begin{bmatrix} (n^3-n^2-5n+5)/8 \\ (n^3-3n^2-n+3)/8 \end{bmatrix}. \tag{31}$$

The solution to the system (31) is the integer vector

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b(p_{1,i})+(1+b(p_{1,i})-r(p_{1,i}))(n-1)/2 \\ (n-3)(n-1)/4+r(p_{1,i})(n-3)/2-b(p_{1,i})(n-1)/2 \end{bmatrix}.$$

If $i = 1$, we can choose $r(p_{1,1}) = b(p_{1,1}) = 0$. This implies $x_1 = (n-1)/2 > 0$ and $x_2 = (n^2 - 4n + 3)/4 > 0$. Notice that for each $i = 2, 3, \dots, n$ there is a path p_{1i} from v_1 to v_i with $b(p_{1i}) \le (n-3)/2$. Hence, $x_2 \ge 0$. Moreover, there is a path from v_1 to v_i with $1 + b(p_{1i}) - r(p_{i1}) \ge 0$. Hence we have $x_1 \ge 0$. These imply that the system (31) has a nonnegative integer solution. Since the vertex v_1 lies on both cycles, by Proposition 3.3 we conclude that $\gamma_{L_n^{(2)}}(v_1) \le (n^3 - 2n^2 - 3n)/4 + 1$. Since for each vertex $v_k, k = 2, 3, \dots, n$, $d(v_k, v_1) = k - 1$, Proposition 3.4 guarantees that

$$\gamma_{L_n^{(2)}}(v_k) \le (n^3 - 2n^2 - 3n)/4 + k \tag{32}$$

for all $k = 1, 2, \dots, n$.

Now combining (30) and (32) we conclude that $\gamma_{L_n^{(2)}}(v_k) = (n^3 - 2n^2 - 3n)/4 + k$ for all $k = 1, 2, \dots, n$.

Lemma 4.4. *For the two-colored digraph $L_n^{(2)}$ of type IV we have $\gamma_{L_n^{(2)}}(v_k) = (n^3 - 2n^2 - n + 6)/4 + k$ for all $k = 1, 2, \dots, n$.*

Proof. Since the red path of length $(n+1)/2$ is the path $v_{n-1} \to v_{n-2} \to \cdots \to v_{(n-3)/2}$, we set $x^* = v_{n-1}$ and $y^* = v_{(n-3)/2}$. We split the proof into three cases depending on the position of v_k.

Case $1 : 1 \le k \le (n-3)/2$

There are two paths from v_k to $v_{(n-3)/2}$. They are a $((n-1)/2, k)$-path and a $((n+1)/2, k+1)$-path. Using the $((n-1)/2, k)$-path we find from the definition of u_0 in equation (1) that $u_0 = (n-1)^2/4 - k(n+1)/2$. Using the $((n+1)/2, k+1)$-path we find from the definition of u_0 in equation (1) that $u_0 = (n-1)^2/4 - k(n+1)/2 - 1$. Hence we choose

$$u_0 = (n-1)^2/4 - k(n+1)/2 - 1 = (n^2 - 1)/4 - (k+1)(n+1)/2. \tag{33}$$

We note that there is a unique path $p_{k,n-1}$ from v_k to v_{n-1} which is a $(0, k+1)$-path. Using this path we find from the definition of v_0 in equation (2) that

$$v_0 = (k+1)(n-1)/2 \tag{34}$$

Theorem 3.2, equation (33) and equation (34) imply that

$$\begin{bmatrix} s \\ t \end{bmatrix} \ge M \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} (n^3 - n^2 - n + 1)/8 \\ (n^3 - 3n^2 - n + 11)/8 + k \end{bmatrix}. \tag{35}$$

From (35) we conclude that

$$\gamma_{L_n^{(2)}}(v_k) \ge (n^3 - 2n^2 - n + 6)/4 + k \tag{36}$$

for all $k = 1, 2, \dots, (n-3)/2$.

Case $2 : (n-1)/2 \le k \le n-1$

There is a unique $p_{k,(n-3)/2}$-path from v_k to $v_{(n-3)/2}$ which is a $(k - (n-3)/2, 0)$-path. Using this path, from the definition of u_0 in equation (1) we find that $u_0 = k(n-1)/2 - (n-1)(n-3)/4$. There is a unique path $p_{k,n-1}$ from v_k to v_{n-1}

which is a $(k-(n-3)/2,(n-1)/2)$-path. Using this path, from the definition of v_0 in equation (2) we find that $v_0=(n-1)^2/4+(n-3)^2/4-k(n-3)/2$. Theorem 3.2 implies

$$\begin{aligned}\gamma_{L_n^{(2)}}(v_k) &\geq \ell(C_1)u_0+\ell(C_2)v_0\\ &= (n-2)(k(n-1)/2-(n-1)(n-3)/4)\\ &\quad + n[(n-1)^2/4+(n-3)^2/4-k(n-3)/2]\\ &= (n^3-2n^2-n+6)/4+k \qquad (37)\end{aligned}$$

for all $k=(n-1)/2,(n+1)/2,\ldots,n-1$.

Case 3: $k=n$

There is a unique path from v_k to $v_{(n-1)/2}$ which is a $((n+1)/2,1)$-path. Using this path we find from the definition of u_0 in equation (1) that $u_0=(n^2-1)/4-(n+1)/2$. There is a unique path $p_{k,n-1}$ from v_k to v_{n-1} which is a $(0,1)$-path. Using this path we find from the definition of v_0 in equation (2) that $v_0=(n-1)/2$. Theorem 3.2 implies

$$\begin{bmatrix} s\\ t\end{bmatrix} \geq M\begin{bmatrix} u_0\\ v_0\end{bmatrix} = \begin{bmatrix} (n^3-n^2-n+1)/8\\ (n^3-3n^2-n+11)/8\end{bmatrix},$$

and hence $\gamma_{L_n^{(2)}}(v_k)\geq (n^3-2n^2-5n+6)/4+n$. We note that for the $(0,1)$-path from v_n to v_{n-1}, the system

$$M\begin{bmatrix} x_1\\ x_2\end{bmatrix} + \begin{bmatrix} 0\\ 1\end{bmatrix} = \begin{bmatrix} (n^3-n^2-n+1)/8\\ (n^3-3n^2-n+11)/8\end{bmatrix}$$

has nonnegative integer solution $x_1=(n^2-1)/4$ and $x_2=0$. This implies there is no walk from v_n to v_{n-1} with composition $\begin{bmatrix} (n^3-n^2-n+1)/8\\ (n^3-3n^2-n+11)/8\end{bmatrix}$. Hence $\gamma_{L_n^{(2)}}(v_n) > (n^3-2n^2-5n+6)/4+n$. Notice that the shortest walk from v_n to v_{n-1} with at least $(n^3-n^2-n+1)/8$ red arcs and $(n^3-3n^2-n+11)/8$ blue arcs is the walk that starts at v_n, moves to v_{n-2} and then moves $(n^2-1)/4$ times around the cycle C_1 and back at v_{n-2}, finally moves to v_{n-1}. The composition of this walk is $\begin{bmatrix} (n^3-n^2+3n+5)/8\\ (n^3-3n^2+3n+7)/8\end{bmatrix}$. Thus we now have

$$\gamma_{L_n^{(2)}}(v_n)\geq (n^3-2n^2+3n+6)/4=(n^3-2n^2-n+6)/4+n. \qquad (38)$$

From (36), (37) and (38) we conclude that

$$\gamma_{L_n^{(2)}}(v_k)\geq (n^3-2n^2-n+6)/4+k \qquad (39)$$

for all $k=1,2,\ldots,n$.

We next show $\gamma_{L_n^{(2)}}(v_k)\leq (n^3-2n^2-n+6)+k$ by first showing that $\gamma_{L_n^{(2)}}(v_1)=(n^3-2n^2-n+6)/4+1$, and then use Proposition 3.4 to get upper bound for exponent of the vertex $v_k, k=2,3,\ldots,n$. From (36) we know that for $k=1$, $\gamma_{L_n^{(2)}}(v_1)\geq (n^3-2n^2-n+6)/4+1$ and from (35) we know that this bound is obtained by walks with composition $\begin{bmatrix} (n^3-n^2-n+1)/8 \\ (n^3-3n^2-n+19)/8 \end{bmatrix}$. It remains to show that $\gamma_{L_n^{(2)}}(v_1) \leq (n^3-2n^2-n+6)/4+1$. For each $i=1,2,\ldots,k$ we show that there is an (s,t)-walk $w_{1,i}$ from v_1 to v_i with composition

$$\begin{bmatrix} r(w_{1,i}) \\ b(w_{1,i}) \end{bmatrix} = \begin{bmatrix} (n^3-n^2-n+1)/8 \\ (n^3-3n^2-n+19)/8 \end{bmatrix}.$$

For any vertex $v_i, i=1,2,3,\ldots,n$ let p_{1i} be a path from v_1 to v_i. The system

$$M\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} r(p_{1i}) \\ b(p_{1i}) \end{bmatrix} = \begin{bmatrix} (n^3-n^2-n+1)/8 \\ (n^3-3n^2-n+19)/8 \end{bmatrix} \tag{40}$$

has integer solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r(p_{1,i}) + ((n-5)/2 + b(p_{1,i}) - r(p_{1,i}))(n+1)/2 \\ (2-b(p_{1,i}))(n-1)/2 + r(p_{1,i})(n-3)/2 \end{bmatrix}.$$

If $i=1$, we can choose $r(p_{1,1}) = b(p_{1,1}) = 0$. This implies $x_1 = (n^2-4n-5)/4 > 0$ and $x_2 = n-1 > 0$. We note that for any vertex $v_i, i=2,3,\ldots,n$ there is a path p_{1i} from v_1 to v_i with $2 \leq b(p_{1i}) \leq (n-3)/2$. Moreover, if $3 \leq b(p_{1i}) \leq (n-3)/2$, then $r(p_{1i}) = (n-1)/2$. Hence $x_2 \geq 0$. Notice also that for any vertex $v_i, i=2,3,\ldots,n$ we can find a path p_{1i} with $b(p_{1i}) - r(p_{1i}) \geq -(n-5)/2$. Hence $x_1 \geq 0$. Hence the system (40) has a nonnegative integer solution. Since the vertex v_1 lies on both cycles, Proposition 3.3 guarantees that $\gamma_{L_n^{(2)}}(v_1) \leq (n^3-2n^2-n+6)/4+1$. By considering equation (39) we conclude that $\gamma_{L_n^{(2)}}(v_1) = (n^3-2n^2-n+6)/4+1$. Since for each $k=2,3,\ldots,n$ we have $d(v_k,v_1) = k-1$, Proposition 3.4 implies that

$$\gamma_{L_n^{(2)}}(v_k) \leq (n^3-2n^2-n+6)/4+k \tag{41}$$

for $k=1,2,\ldots,n$.

Combining (39) and (41) we conclude that $\gamma(v_k) = (n^3-2n^2-n+6)+k$ for all $k=1,2,\ldots,n$.

Theorem 4.5. *Let $L_n^{(2)}$ be a primitive two-colored digraph on $n \geq 5$ vertices whose underlying digraph is the digraph L_n in Figure 1. If $\exp(L_n^{(2)}) = (n^3-2n^2+1)/2$, then $(n^3-2n^2-3n+4)/4 \leq \gamma_{L_n^{(2)}}(v_k) \leq (n^3-2n^2+3n+6)/4$ for all $k=1,2,\ldots,n$*

Proof. By Lemma 4.1 through Lemma 4.4 for each $k=1,2,\ldots,n$ we have that $(n^3-2n^2-3n)/4+k \leq \gamma_{L_n^{(2)}}(v_k) \leq (n^3-2n^2-n+6)/4+k$. This implies for any $k=1,2,\ldots,n$ we have $(n^3-2n^2-3n+4)/4 \leq \gamma_{L_n^{(2)}}(v_k) \leq (n^3-2n^2+3n+6)/4$.

We now discuss vertex exponents for the two-colored digraphs $L_n^{(2)}$ whose exponents is $2n^2-6n+2$.

Theorem 4.6. *Let $L_n^{(2)}$ be a primitive two-colored digraph on $n \geq 5$ vertices whose underlying digraph is the digraph L_n in Figure 1. If $\exp(L_n^{(2)}) = 2n^2-6n+2$, then for any vertex $v_k, k=1,2,\ldots,n$ we have $(n^2-4n+5) \leq \gamma_{L_n^{(2)}}(v_k) \leq (n^2-2n-1)$.*

PROOF. Since $\exp(L_n^{(2)}) = 2n^2-6n+2$, Corollary 2.4 implies that $L_n^{(2)}$ has a unique $(2,0)$-path that lies on both cycles. This implies the $(2,0)$-path of $L_n^{(2)}$ is of the form $a \to a-1 \to a-2$ for some $3 \le a \le n-2$. We show that $\gamma_{L_n^{(2)}}(v_k) = n^2-3n+k+2-a$ for all $k = 1, 2, \dots, n$. We first show that $\gamma_{D^{(2)}}(v_k) \ge n^2 - 3n + k + 2 - a$ for all $k = 1, 2, \dots, n$. We use path from v_k to v_{a-2} to determine the value of the quantity u_0 in equation (1) and we use path from v_k to v_a to determine the value of the quantity v_0 in equation (2). We split the proof into three cases depending on the position of the vertex v_k.

Case $1 : 1 \le k \le a-2$
We note that there are two paths from v_k to v_{a-2}. They are a $((n-1)/2 - \lfloor (a-2-k)/2 \rfloor, (n-3)/2 - \lceil (a-2-k)/2 \rceil)$-path and a $((n+1)/2 - \lfloor (a-2-k)/2 \rfloor, (n-1)/2 - \lceil (a-2-k)/2 \rceil)$-path. Using the first path and the definition of u_0 in equation (1) we have $u_0 = 1 + \frac{n+1}{2}\left\lceil \frac{a-2-k}{2} \right\rceil - \frac{n-1}{2}\left\lfloor \frac{a-2-k}{2} \right\rfloor$. Using the second path and the definition of u_0 in equation (1) we have $u_0 = \frac{n+1}{2}\left\lceil \frac{a-2-k}{2} \right\rceil - \frac{n-1}{2}\left\lfloor \frac{a-2-k}{2} \right\rfloor$. Hence we conclude that

$$u_0 = (n+1)\lceil (a-2-k)/2 \rceil /2 - (n-1)\lfloor (a-2-k)/2 \rfloor /2. \tag{42}$$

There are two paths from v_k to v_a. They are a $((n-5)/2 - \lfloor (a-2-k)/2 \rfloor, (n-3)/2 - \lceil (a-2-k) \rceil)$-path and a $((n-3)/2 - \lfloor (a-2-k)/2 \rfloor, (n-1)/2 - \lceil (a-2-k) \rceil)$-path. Using the first path and the definition of v_0 in equation (2) we have $v_0 = n - 3 - \frac{n-1}{2}\left\lceil \frac{a-2-k}{2} \right\rceil + \frac{n-3}{2}\left\lfloor \frac{a-2-k}{2} \right\rfloor$. Using the second path and the definition of v_0 in equation (2) we have $v_0 = n - 2 - \frac{n-1}{2}\left\lceil \frac{a-2-k}{2} \right\rceil + \frac{n-3}{2}\left\lfloor \frac{a-2-k}{2} \right\rfloor$. Hence we conclude that

$$v_0 = n - 3 - (n-1)\lceil (a-2-k)/2 \rceil /2 + (n-3)\lfloor (a-2-k)/2 \rfloor. \tag{43}$$

Now Theorem 3.2, equation (42) and equation (43) imply that

$$\begin{aligned}
\begin{bmatrix} s \\ t \end{bmatrix} &\ge M \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \\
&= \begin{bmatrix} (n-1)/2 & (n+1)/2 \\ (n-3)/2 & (n-1)/2 \end{bmatrix} \begin{bmatrix} \frac{n+1}{2}\left\lceil \frac{a-2-k}{2} \right\rceil - \frac{n-1}{2}\left\lfloor \frac{a-2-k}{2} \right\rfloor \\ n - 3 - \frac{n-1}{2}\left\lceil \frac{a-2-k}{2} \right\rceil + \frac{n-3}{2}\left\lfloor \frac{a-2-k}{2} \right\rfloor \end{bmatrix} \\
&= \begin{bmatrix} (n^2-2n-3)/2 - \lfloor (a-k-2)/2 \rfloor \\ (n^3-4n+3)/2 - \lceil (a-2-k)/2 \rceil \end{bmatrix}
\end{aligned} \tag{44}$$

Hence we now have

$$\begin{aligned}
\gamma_{D^{(2)}}(v_k) &\ge n^2 - 3n - (\lfloor (a-2-k)/2 \rfloor + \lceil (a-2-k)/2 \rceil) \\
&= n^2 - 3n + k + 2 - a
\end{aligned} \tag{45}$$

for $k = 1, 2, \dots, a-2$.

Case $2 : k = a-1, a$
There is a unique path from v_k to v_{a-2} and it is a $(k-a+2, 0)$-path. Using this path and the definition of u_0 in equation (1) we have that

$$u_0 = (n-1)(k-a+2)/2. \tag{46}$$

There are two paths from v_k to v_a. They are a $(k-a+2+(n-5)/2,(n-3)/2)$-path and a $(k+2-a+(n-3)/2,(n-1)/2)$-path. Using the first path and the definition of v_0 in equation (2) we have that $v_0=n-3-(n-3)(k+2-a)/2$. Using the second path and the definition of v_0 in equation (2) we have that $v_0=n-2-(n-3)(k+2-a)/2$. Hence we conclude that

$$v_0 = n-2-(n-3)(k+2-a)/2. \tag{47}$$

Now Theorem 3.2, equation (46) and equation (47) imply that

$$\begin{aligned}\gamma_{D^{(2)}}(v_k) &\geq \ell(C_1)u_0+\ell(C_2)v_0\\ &= (n-2)(n-1)(k-a+2)/2+n[(n-3)-(n-3)(k-a+2)/2]\\ &= n^2-3n+k+2-a \qquad (48)\end{aligned}$$

for all $k=a-1,a$.

Case 3 : $a+1\leq k\leq n$

There is a unique path from v_k to v_{a-2} which is $(\lfloor (k-a)/2\rfloor+2,\lceil (k-a)/2\rceil)$-path. Using this path and the definition of u_0 in equation (1) we have $u_0=\left(\frac{n-1}{2}\right)\left(\left\lfloor\frac{k-a}{2}\right\rfloor+2\right)-\frac{n+1}{2}\left\lceil\frac{k-a}{2}\right\rceil$. There is a unique path from v_k to v_a which is a $(\lfloor (k-a)/2\rfloor,\lceil (k-a)/2\rceil)$-path. Using this path and the definition of v_0 in equation (2) we have that $v_0=\frac{n-1}{2}\left\lfloor\frac{k-a}{2}\right\rfloor-\frac{n-3}{2}\left\lceil\frac{k-a}{2}\right\rceil$. By Theorem 3.2 we have

$$\begin{aligned}\gamma_{L_n^{(2)}}(v_k) &\geq \ell(C_1)u_0+\ell(C_2)v_0\\ &= (n-2)\left(\frac{n-1}{2}\left(\left\lfloor\frac{k-a}{2}\right\rfloor+2\right)-\frac{n+1}{2}\left\lceil\frac{k-a}{2}\right\rceil\right)\\ &\quad +n\left(\frac{n-1}{2}\left\lfloor\frac{k-a}{2}\right\rfloor-\frac{n+1}{2}\left\lceil\frac{k-a}{2}\right\rceil\right)\end{aligned}$$

Hence

$$\begin{aligned} &= n^3-3n+2+\lfloor (k-a)/2\rfloor+\lceil (k-a)/2\rceil\\ &= n^2-3n+k+2-a \qquad (49)\end{aligned}$$

for $a+1\leq k\leq n$.

From equation (45), equation (48) and equation (49) we conclude that

$$\gamma_{L_n^{(2}}(v_k)\geq n^3-3n+k+2-a \tag{50}$$

for all $k=1,2,\ldots,n$.

We now show that $\gamma_{L_n^{(2)}}(v_k)\leq n^3-3n+k+2-a$ for $k=1,2,\ldots,n$. We first show that $\gamma_{L_n^{(2)}}(v_1)\leq n^3-3n+3-a$ and then we use Proposition 3.4 to show that $\gamma_{L_n^{(2)}}\leq n^2-3n+k+2-a$ for $k=2,3,\ldots,n$. By considering equation (44) it suffices to show that for each vertex $v_i, i=1,2,\ldots,n$ there is a walk $w_{1,i}$ from v_1 to v_i with composition

$$\begin{bmatrix} r(w_{1,i})\\ b(w_{1,i})\end{bmatrix}=\begin{bmatrix}(n^2-2n-3)/2-\lfloor (a-3)/2\rfloor\\ (n^3-4n+3)/2-\lceil (a-3)/2\rceil\end{bmatrix}. \tag{51}$$

For each $i = 1, 2, \ldots, n$, let $p_{1,i}$ be a path from v_1 to v_i. We note that the solution to the system

$$M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} r(p_{1,i}) \\ b(p_{1,i}) \end{bmatrix} = \begin{bmatrix} (n^2 - 2n - 3)/2 - \lfloor (a-3)/2 \rfloor \\ (n^3 - 4n + 3)/2 - \lceil (a-3)/2 \rceil \end{bmatrix} \tag{52}$$

is the integer vector

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \lceil (a-3)/2 \rceil (n+1)/2 - \lfloor (a-3)/2 \rfloor (n-1)/2 \\ n - 3 + \lfloor (a-3)/2 \rfloor (n-3)/2 - \lceil (a-3)/2 \rceil (n-1)/2 \end{bmatrix} \\ &\quad + \begin{bmatrix} b(p_{1,i}) + (b(p_{1,i}) - r(p_{1,i}))(n-1)/2 \\ (r(p_{1,i}) - b(p_{1,i}))(n-3)/2 - b(p_{1,i}) \end{bmatrix}. \end{aligned} \tag{53}$$

We show that $x_1 \geq 0$ and $x_2 \geq 0$. We consider two cases when a is even and a is odd.

If a is even, then $\lceil (a-3)/2 \rceil = (a-2)/2$ and $\lfloor (a-3)/2 \rfloor = (a-4)/2$. This implies

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (n-1)/2 + (a-2)/2 + b(p_{1,i}) - [r(p_{1,i}) - b(p_{1,i})](n-1)/2 \\ (n-a-1)/2 + [r(p_{1,i}) - b(p_{1,i})](n-3)/2 - b(p_{1,i}) \end{bmatrix}.$$

Since a is even, we have that $0 \leq r(p_{1,i}) - b(p_{1,i}) \leq 2$. If $r(p_{1,i}) - b(p_{1,i}) = 0$, then $b(p_{1,i}) \leq (n-1-a)/2$. This implies $x_1 > 0$ and $x_2 \geq 0$. If $r(p_{1,i}) - b(p_{1,i}) = 1$, there is a path $p_{1,i}$ with $b(p_{1,i}) \leq (n-3)/2$. This implies $x_1 > 0$ and $x_2 > 0$. If $r(p_{1,i}) - b(p_{1,i}) = 2$, then $b(p_{1,i}) \geq (n+1-a)/2$. This implies $x_1 \geq 0$ and $x_2 > 0$. Therefore for each vertex $v_i, i = 1, 2, \ldots, n$, there is a path $p_{1,i}$ from v_1 to v_i such that the system (52) has nonnegative integer solution $x_1 \geq 0$ and $x_2 \geq 0$.

If a is odd, then $\lceil (a-3)/2 \rceil = \lfloor (a-3)/2 \rfloor = (a-3)/2$. This implies

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (a-3)/2 + b(p_{1,i}) - [r(p_{1,i}) - b(p_{1,i})](n-1)/2 \\ n - 3 - (a-3)/2 + [r(p_{1,i}) - b(p_{1,i})](n-3)/2 - b(p_{1,i}) \end{bmatrix}.$$

Since a is odd, for each vertex $v_i, i = 1, 2, \ldots, n$ there is a path $p_{1,i}$ with $-1 \leq r(p_{1,i}) - b(p_{1,i}) \leq 1$. If $r(p_{1,i}) - b(p_{1,i}) = -1$, there is a path $p_{1,i}$ with $b(p_{1,i}) \leq (n-a)/2$. This implies $x_1 > 0$ and $x_2 \geq 0$. If $r(p_{1,i}) - b(p_{1,i}) = 0$, there is a path $p_{1,i}$ with $b(p_{1,i}) \leq (n-3)/2$. This implies $x_1 > 0$ and $x_2 > 0$. Finally if $r(p_{1,i}) - b(p_{1,i}) = 1$, there is a path $p_{1,i}$ with $b(p_{1,i}) \geq (n-a+2)/2$. This implies $x_1 \geq 0$ and $x_2 > 0$. Therefore for each vertex $v_i, i = 1, 2, \ldots, n$, there is a path $p_{1,i}$ from v_1 to v_i such that the system (52) has nonnegative integer solution $x_1 \geq 0$ and $x_2 \geq 0$.

Since the system (52) has a nonnegative integer solution and the vertex v_1 belongs to both cycles, Proposition 3.3 guarantees that $\gamma_{L_n^{(2)}}(v_1) \leq n^2 - 3n + 3 - a$. Combining this with equation (45) we conclude that $\gamma_{L_n^{(2)}}(v_1) = n^2 - 3n + 3 - a$. Since for $k = 2, 3, \ldots, n$ we have $d(v_k, d_1) = k - 1$, Proposition 3.4 implies that

$$\gamma_{L_n^{(2)}}(v_k) \leq n^2 - 3n + k + 2 - a \tag{54}$$

for $k = 1, 2, \ldots, n$.

Finally combining equation (50) and equation (54) we conclude that $\gamma_{L_n^{(2)}}(v_k) = n^2 - 3n + k + 2 - a$ for $k = 1, 2, \ldots, n$. We note that $3 \leq a \leq n - 2$. This implies

$n^2 - 4n + 4 + k \leq \gamma_{L_n^{(2)}}(v_k) \leq n^2 - 3n + k - 1$. Therefore for any $k = 1, 2, \ldots, n$ we have $n^2 - 4n + 5 \leq \gamma_{L_n^{(2)}}(v_k) \leq n^2 - 2n - 1$.

References

[1] Fornasini, E. and Valcher, M. E., "Primitivity positive matrix pairs: algebraic characterization graph theoritic description and 2D systems interpretations", *SIAM J. Matrix Anal. Appl.* 19 (1998), 71–88

[2] Gao, Y. and Shao, Y., "Exponents of two-colored digraphs with cycles", *Linear Algebra Appl.* 407 (2005), 263–276

[3] Gao, Y. and Shao, Y., "Exponents of a class two-colored digraphs", *Linear and Multilinear Algebra* **53 (3)** (2005), 175–188

[4] Gao, Y. and Shao, Y., "Generalized exponents of primitive two-colored digraphs", *Linear Algebra Appl.* **430** (2009), 1550–1565

[5] Lee, S. G. and Yang, J. M., "Bound for 2-exponents of primitive extremal ministrong digraphs", *Commun. Korean. Math. Soc.* **20** no. 1 (2005), 51–62

[6] Huang, F. and Liu, B., "Exponents of a class of two-colored digraphs with two cycles", *Linear Algebra and Appl. 429* (2008), 658–672

[7] Shader, B. L. and Suwilo, S., "Exponents of nonnegative matrix pairs", *Linear Algebra Appl.* 263 (2003), 275–293

[8] Suwilo, S., *On 2-exponents of 2-digraphs*, Ph.D. Dissertation, University of Wyoming, 2001.

[9] Suwilo, S., "Vertex Exponents of two-colored primitive extremal ministrong digraphs", *Global Journal of Technology and Optimization*, Vol 2 No 2 (2011), 166–174