CONNECTIVITY INDICES OF COPRIME GRAPH OF GENERALIZED QUATERNION GROUP

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Abstract. Generalized quaternion group (Q_{4n}) is a group of order $4n$ that is generated by two elements x and y with the properties $x^{2n} = y^4 = e$ and $xy = yx^{-1}$. The coprime graph of Q_{4n}, denoted by $\Omega_{Q_{4n}}$, is a graph with the vertices are elements of Q_{4n} and the edges are formed by two elements that have coprime order. The first result of this paper presents that $\Omega_{Q_{4n}}$ is a tripartite graph for n is an odd prime and $\Omega_{Q_{4n}}$ is a star graph for n is a power of 2. The second one presents the connectivity indices of $\Omega_{Q_{4n}}$. Connectivity indices of a graph is a research area in mathematics that popularly applied in chemistry. There are six indices that are presented in this paper, those are first Zagreb index, second Zagreb index, Wiener index, hyper-Wiener index, Harary index, and Szeged index.

Key words and Phrases: Generalized quaternion group, Zagreb indices, Wiener indices, Harary index, Szeged index.

1. INTRODUCTION

Graph theory has been widely applied in many fields. One of them is in chemistry, which is related to connectivity indices. Connectivity indices are molecular descriptor which is computed based on the molecular graph of chemical compound. The molecular graph can be assumed as a graph. There are some kinds of connectivity indices that are interesting to be discussed. Such indices are hyper-Wiener, Harary, the first Zagreb, the second Zagreb, and Szeged. Some of these indices can be used to analyze the chemical properties of paraffines [9].

There are many kinds of research relating to graph and group theory since the properties of a group can be easily seen when a graph represents that group. There

are some previous results that have been discussed related to connectivity indices of a graph, especially in mathematics. Some of the results are determining connectivity indices of non-commuting graph of dihedral group (D_{2n}) [1] and generalized quaternion group (Q_{4n}) [8]. Other than the non-commuting graph, another kind of graph represents a group, namely a coprime graph. Recently, not many studies have learned connectivity indices of the graph associated with groups. Moreover, the quaternion group has similar properties to the dihedral group. Therefore we are interested in studying the connectivity indices of the coprime graph of Q_{4n}.

2. PRELIMINARIES

In this section we present some definitions that are needed in this study.

Definition 2.1. [11] *Let n be a natural number. The generalized quaternion group, denoted by Q_{4n}, is defined as*

$$Q_{4n} = <x, y | x^{2n} = y^4 = e, xy = yx^{-1}> = \{x^i y^j | 0 \leqslant i \leqslant 2n-1, \; j = 0, 1\}.$$

Hence the order of Q_{4n} is $4n$.

Definition 2.2. [5] *Let G be a finite group and $g \in G$. The order of g, denoted by $|g|$, is the smallest natural number n such that $g^n = e$, where e is an identity element of G.*

Definition 2.3. [7] *Let G be a finite group. The coprime graph of G, denoted by Ω_G, is a graph with vertex set $V(\Omega_G) = G$ and edge set $E(\Omega_G) = \{uv | \gcd(|u|, |v|) = 1\}$.*

Definition 2.4. [2] *Let k be a natural number, a graph Ω is a $k-$partite graph if its vertex set, $V(\Omega)$ can be partitioned into k subsets $V_1, V_2, ..., V_k$ such that every edge of Ω joins vertices in two different partite sets. A $2-$partite graph is called bipartite and $3-$partite graph is called tripartite.*

Definition 2.5. [2] *A complete $k-$partite graph Ω is a $k-$partite graph that two vertices are adjacent in Ω if and only if the vertices belong to different partite sets. If $|V_i| = n_i$ for $1 \leqslant i \leqslant k$, then Ω is denoted by $K_{n_1, n_2, ..., n_k}$. For complete bipartite graph $K_{1,n}$ is also called a star graph, denoted by S_n.*

Example 2.6. *Let $V_1 = \{v_1\}, V_2 = \{v_2, v_3\}$ and $V_3 = \{v_4, v_5, v_6\}$, then we have the complete tripartite graph $K_{1,2,3}$ as follows*

Graph $K_{1,2,3}$
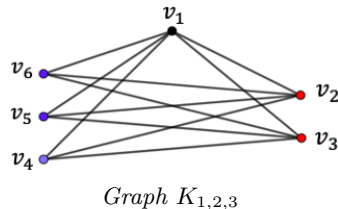
Definition 2.7. [3] *Let Ω be a simple connected graph. The first Zagreb index of Ω, denoted by $M_1(\Omega)$, is defined as*

$$M_1(\Omega) = \sum_{v \in V(\Omega)} (deg(v))^2$$

where $deg(v)$ is degree of vertex v, i.e. the number of edges that incident to v.

Definition 2.8. [3] *Let Ω be a simple connected graph. The second Zagreb index of Ω, denoted by $M_2(\Omega)$, is defined as*

$$M_2(\Omega) = \sum_{uv \in E(\Omega)} deg(u).deg(v)$$

where $deg(v)$ is degree of vertex v.

Definition 2.9. [4] *Let Ω be a simple connected graph. The Wiener index of Ω, denoted by $W(\Omega)$, is defined as*

$$W(\Omega) = \sum_{u,v \in V(\Omega)} d(u,v)$$

where $d(u,v)$ is the distance between vertex u and v, i.e. the number of edges in shortest path connecting u and v.

Definition 2.10. [10] *Let Ω be a simple connected graph. The hyper-Wiener index of Ω, denoted by $WW(\Omega)$, is defined as*

$$WW(\Omega) = \frac{1}{2} \left(W(\Omega) + \sum_{u,v \in V(\Omega)} (d(u,v))^2 \right)$$

where $d(u,v)$ is the distance between vertex u and v.

Definition 2.11. [10] *Let Ω be a simple connected graph. The Harary index of Ω, denoted by $H(\Omega)$, is defined as*

$$H(\Omega) = \sum_{u,v \in V(\Omega)} \frac{1}{d(u,v)}$$

where $d(u,v)$ is the distance between vertex u and v.

Definition 2.12. [6] *Let Ω be a simple connected graph and $e = uv$ be an edge of Ω. The Szeged index of Ω, denoted by $Sz(\Omega)$, is defined as*

$$Sz(\Omega) = \sum_{e \in E(\Omega)} |N_1(e|\Omega)|.|N_2(e|\Omega)|.$$

where $N_1(e|\Omega) = \{w \in V(\Omega)|d(w,u) < d(w,v)\}$ and

$$N_2(e|\Omega) = \{w \in V(\Omega)|d(w,v) < d(w,u)\}.$$

Example 2.13. *Let $K_{1,2,3}$ be a complete tripartite graph (Example 2.6). Then we have $M_1(K_{1,2,3}) = 84, M_2(K_{1,2,3}) = 157, W(K_{1,2,3}) = 19, WW(K_{1,2,3}) = 23, H(K_{1,2,3}) = 13$, and $Sz(K_{1,2,3}) = 37$.*

3. RESULTS AND DISCUSSIONS

This section consists of two subsections. The first subsection discuss about the coprime graph of Q_{4n} and the second one discuss about its connectivity indices.

3.1. **Coprime Graph of Generalized Quaternion Group.** In this subsection, we determine the coprime graph of Q_{4n}. Since the adjacency of the vertices depends on the order of elements of Q_{4n}, then firstly we determine the order of elements of Q_{4n} on the following lemma.

Lemma 3.1. *Let Q_{4n} be a generalized quaternion group. Then the order of its elements are showed as follows*

$$|x^i y^j| = \begin{cases} \frac{2n}{\gcd(i,2n)}, & \text{for } j = 0 \\ \\ 4, & \text{for } j = 1 \end{cases}$$

where $x^i y^j \neq e$ and $0 \leqslant i < 2n$.

Proof. Let $Q_{4n} = \{x^i y^j | 0 \leqslant i \leqslant 2n - 1, \ j = 0, 1\}$, then there are two cases.

Case 1. The order of elements $x^i y^j$ for $0 < i < 2n$ and $j = 0$.

According to Definition 2.1 we have $x^{2n} = e$, therefore $(x^i)^{\frac{2n}{\gcd(i,2n)}} = (x^{2n})^{\frac{i}{\gcd(i,2n)}} = e$. Suppose that there is $m \in \mathbb{N}$ and $(x^i)^m = x^{im} = e$, then $2n$ devides im and we have $\frac{2n}{\gcd(i,2n)}$ devides m which means $\frac{2n}{\gcd(i,2n)}$ is the smallest natural number satisfies $(x^i)^{\frac{2n}{\gcd(i,2n)}} = e$. Thus $|x^i| = \frac{2n}{\gcd(i,2n)}$.

Case 2. The order of elements $x^i y^j$ for $0 \leqslant i < 2n$ and $j = 1$.

a. Let $m = 1$, clearly $(x^i y)^m = x^i y \neq e$.

b. Let $m = 2$, by induction we will show that

$$(x^i y)^2 = x^{i - \lfloor \frac{i}{2} \rfloor} y^2 x^{-\left(i - \lfloor \frac{i}{2} \rfloor\right)} \neq e.$$

For $i = 1$, we have $(xy)^2 = xy^2 x^{-1} = y^2 \neq e$

Assume that for $i = k$ we have $(x^k y)^2 = x^{k - \lfloor \frac{k}{2} \rfloor} y^2 x^{-\left(k - \lfloor \frac{k}{2} \rfloor\right)}$, then we have

$$\begin{aligned}
(x^{k+1} y)^2 &= (x^{k+1} y)(x^{k+1} y) \\
&= x(x^k y) x^k (xy) \\
&= x(x^k y)(x^k y) x^{-1} \\
&= xx^{\left(k - \lfloor \frac{k}{2} \rfloor\right)} y^2 x^{-\left(k - \lfloor \frac{k}{2} \rfloor\right)} x^{-1} \\
&= x^{\left(k+1 - \lfloor \frac{k+1}{2} \rfloor\right)} y^2 x^{-\left(k+1 - \lfloor \frac{k+1}{2} \rfloor\right)}.
\end{aligned}$$

c. Let $m = 3$, by induction we have

$$(x^i y)^3 = x^{\left(i - \lceil \frac{i}{2} \rceil\right)} y^3 x^{-\left(i - \lfloor \frac{i}{2} \rfloor\right)} \neq e.$$

The proof is similar to case 2b, hence it is omitted.

d. Let $m = 4$, by induction we have

$$(x^i y)^4 = x^{\left(i - \left\lceil \frac{i}{2} \right\rceil\right)} y^4 x^{-\left(i - \left\lceil \frac{i}{2} \right\rceil\right)} = x^{\left(i - \left\lceil \frac{i}{2} \right\rceil\right)} e x^{-\left(i - \left\lceil \frac{i}{2} \right\rceil\right)} = e.$$

The proof is similar to case 2b, hence it is omitted.

Thus $|x^i y| = 4$.   ∎

The next result is the shape of the coprime graph of generalized quaternion group that presented in the following theorem.

Theorem 3.2. *Let Q_{4n} be a generalized quaternion group and $\Omega_{Q_{4n}}$ be the coprime graph of generalized quaternion group. Then*

i. *$\Omega_{Q_{4n}}$ is a tripartite graph for n is an odd prime*
ii. *$\Omega_{Q_{4n}}$ is a star graph for n is a power of 2.*

PROOF. Since $|e| = 1$ and $|x^i y^j| \neq 1$ for $i, j \neq 0$, then clearly that vertex e is adjacent to any other vertices of $\Omega_{Q_{4n}}$.

i. Let $S = \{x^{2k} | 1 \leqslant k \leqslant n - 1\}$ and $T = \{x^i y, x^j | j \neq 2k, 1 \leqslant k \leqslant n - 1$ and $0 \leqslant i \leqslant 2n - 1\}$. Then the partition of $V(\Omega_{Q_{4n}})$ is $\{\{e\}, S, T\}$. According to Lemma 3.1, each vertex $x^{2k} \in S$ has the same order, i.e. n, which means any two vertices in S are not adjacent. For vertices in T, we divide into three groups, those are vertex x^n, vertex set x^{2k+1} for $1 \leqslant k \leqslant n - 1$, and vertex set $x^i y$ for $0 \leqslant i \leqslant 2n - 1$. Therefore from Lemma 3.1 we have $\gcd(|x^n|, |x^{2k+1}|) \neq 1, \gcd(|x^n|, |x^i y|) \neq 1$, and $\gcd(|x^{2k+1}|, |x^i y|) \neq 1$ which means any vertices in T is not adjacent to each other. Since n is an odd prime, then $\gcd(|x^{2k}|, |x^n|) = 1$. Hence S and T cannot be in the same partition. Thus $\Omega_{Q_{4n}}$ is a tripartite graph.

ii. Let $\{\{e\}, V(\Omega_{Q_{4n}}) - \{e\}\}$ be a partition of $V(\Omega_{Q_{4n}})$. Since n is a power of 2, then from Lemma 3.1 we have the order of non identity elements in Q_{4n} are not coprime which means the vertices in $V(\Omega_{Q_{4n}}) - \{e\}$ are not adjacent to each other. Since only vertex e that is adjacent to $4n - 1$ vertices on $\Omega_{Q_{4n}}$ then $\Omega_{Q_{4n}}$ is a complete bipartite graph $K_{1,4n-1}$, i.e. a star graph S_{4n-1}.

                                                  ∎

Example 3.3. *Let $n = 3$ and $n = 4 = 2^2$, then we have the coprime graph of Q_{12} and Q_{16} as follows:*
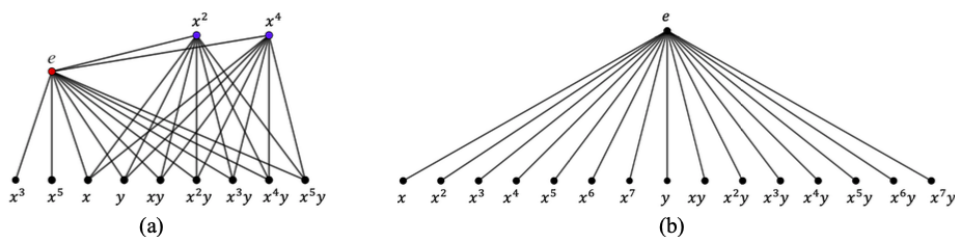
(a) Graph $\Omega_{Q_{12}}$ and (b) Graph $\Omega_{Q_{16}}$

For n is an odd prime, we redefine the vertex set and edge set of $\Omega_{Q_{4n}}$ to make easier in determining its connectivity indices.

Let $V(\Omega_{Q_{4n}}) = \{e\} \cup \{x^n\} \cup [x^{2k+1}] \cup [x^{2k}] \cup [x^i y]$, where

$$[x^{2k+1}] = \{x^{2k+1} | 0 \leqslant k \leqslant n-1 \text{ and } 2k+1 \neq n\}$$
$$[x^{2k}] = \{x^{2k} | 1 \leqslant k \leqslant n-1\}$$
$$[x^i y] = \{x^i y | 0 \leqslant i \leqslant 2n-1\}$$

and let $E(\Omega_{Q_{4n}}) = \{a\} \cup [b] \cup [c] \cup [d] \cup [f] \cup [g]$, where

$$a = ex^n$$
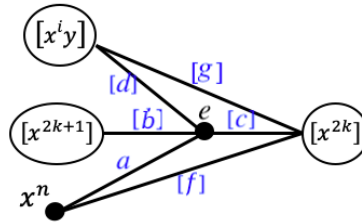$$[b] = \{b_k = ex^{2k+1} | 0 \leqslant k \leqslant n-1 \text{ and } 2k+1 \neq n\}$$
$$[c] = \{c_k = ex^{2k} | 1 \leqslant k \leqslant n-1\}$$
$$[d] = \{d_i = ex^i y | 0 \leqslant i \leqslant 2n-1\}$$
$$[f] = \{f_k = x^n x^{2k} | 1 \leqslant k \leqslant n-1\}$$
$$[g] = \{g_{ki} = x^{2k} x^i y | 1 \leqslant k \leqslant n-1, 0 \leqslant i \leqslant 2n-1\}$$

Based on the enumerate above, we can illustrate $\Omega_{Q_{4n}}$ as follows:

Graph $\Omega_{Q_{4n}}$ for n is an odd prime.

3.2. **Connectivity Indices.** The connectivity indices of $\Omega_{Q_{4n}}$ are determined on the following results.

Theorem 3.4. *Let $\Omega_{Q_{4n}}$ be the coprime graph of generalized quaternion group. The first Zagreb index of $\Omega_{Q_{4n}}$ is*

$$M_1(\Omega_{Q_{4n}}) = \begin{cases} 6n^3 + 21n^2 - 11n - 4, & \text{for } n \text{ is an odd prime} \\ 4n(4n-1), & \text{for } n \text{ is a power of 2.} \end{cases}$$

PROOF. Let n be an odd prime. Firstly we determine the degree of each vertex of $\Omega_{Q_{4n}}$ based on Figure 3.1.

  (i) Since vertex e is adjacent to any other vertices of $\Omega_{Q_{4n}}$, then $deg(e) = 4n-1$.
 (ii) Since vertex x^n is adjacent to all vertices in $[x^{2k}]$ and vertex e, then $deg(x^n) = n$.
(iii) Since each vertex in $[x^{2k+1}]$ is adjacent to only vertex e, then $deg(x^{2k+1}) = 1$.

(iv) Since each vertex in $[x^{2k}]$ is adjacent to all vertices in $[x^i y]$, vertex x^n and vertex e, then $deg(x^{2k}) = 2n + 2$.

(v) Since each vertex in $[x^i y]$ is adjacent to all vertices in $[x^{2k}]$ and vertex e, then $deg(x^i y) = n$.

Therefore

$$M_1(\Omega_{Q_{4n}}) = \sum_{v \in V(\Omega_{Q_{4n}})} (deg(v))^2$$

$$= (deg(e))^2 + (deg(x^n))^2 + \sum_{v \in [x^{2k+1}]} (deg(v))^2 + \sum_{v \in [x^{2k}]} (deg(v))^2 + \sum_{v \in [x^i y]} (deg(v))^2$$

$$= (4n-1)^2 + n^2 + (n-1).1^2 + (n-1)(2n+2)^2 + 2n.n^2$$

$$= 6n^3 + 21n^2 - 11n - 4.$$

Let n be a power of 2. Since $\Omega_{Q_{4n}}$ is a star graph S_{4n-1}, then $deg(e) = 4n-1$ and $deg(v) = 1$ for $v \in V(\Omega_{Q_{4n}}) - \{e\}$. Therefore

$$M_1(\Omega_{Q_{4n}}) = \sum_{v \in V(\Omega_{Q_{4n}})} (deg(v))^2$$

$$= (deg(e))^2 + \sum_{v \in V(\Omega_{Q_{4n}}) - \{e\}} (deg(v))^2$$

$$= (4n-1)^2 + (4n-1)$$

$$= 4n(4n-1).$$

∎

Theorem 3.5. *Let $\Omega_{Q_{4n}}$ be the coprime graph of generalized quaternion group. The second Zagreb index of $\Omega_{Q_{4n}}$ is*

$$M_2(\Omega_{Q_{4n}}) = \begin{cases} 4n^4 + 18n^3 - 16n + 3, & \text{for } n \text{ is an odd prime} \\ (4n-1)^2, & \text{for } n \text{ is a power of 2.} \end{cases}$$

PROOF. Let n be an odd prime. From Figure 3.1 and proof of Theorem 3.4, we have

$$M_2(\Omega_{Q_{4n}}) = \sum_{uv \in E(\Omega_{Q_{4n}})} deg(u)deg(v)$$

$$= deg(e)deg(x^n) + \sum_{uv \in [b]} deg(u)deg(v) + \sum_{uv \in [c]} deg(u)deg(v) + \sum_{uv \in [d]} deg(u)deg(v)$$

$$+ \sum_{uv \in [f]} deg(u)deg(v) + \sum_{uv \in [g]} deg(u)deg(v)$$

$$= (4n-1)n + (n-1)(4n-1) + (n-1)(4n-1)(2n+2) + 2n(4n-1)n$$

$$+ (n-1)n(2n+2) + (n-1)2n(2n+2)n$$

$$= 4n^4 + 18n^3 - 16n + 3.$$

Let n be a power of 2. Since $\Omega_{Q_{4n}}$ is a star graph S_{4n-1}, then $E(\Omega_{Q_{4n}}) = \{ev | v \in V(\Omega_{Q_{4n}}) - \{e\}\}$. Therefore

$$M_2(\Omega_{Q_{4n}}) = \sum_{ev \in E(\Omega_{Q_{4n}})} deg(e)deg(v)$$
$$= (4n-1).(4n-1).1$$
$$= (4n-1)^2.$$

∎

Theorem 3.6. *Let $\Omega_{Q_{4n}}$ be the coprime graph of generalized quaternion group. The Wiener index of $\Omega_{Q_{4n}}$ is*

$$W(\Omega_{Q_{4n}}) = \begin{cases} 14n^2 - 7n + 2, & \text{for } n \text{ is an odd prime} \\ (4n-1)^2, & \text{for } n \text{ is a power of 2.} \end{cases}$$

PROOF. Let n be an odd prime. From Figure 3.1 we can easily determine the distance between any two vertices on $\Omega_{Q_{4n}}$. Therefore

$$W(\Omega_{Q_{4n}}) = \sum_{u,v \in V(\Omega_{Q_{4n}})} d(u,v)$$

$$= \sum_{v \in V(\Omega_{Q_{4n}})-\{e\}} d(e,v) + \sum_{u \in [x^{2k}], v \in [x^i y]} d(u,v) + \sum_{u \in [x^{2k+1}], v \in \{x^n\} \cup [x^i y] \cup [x^{2k}]} d(u,v)$$

$$+ \sum_{v \in [x^{2k}]} d(x^n, v) + \sum_{v \in [x^i y]} d(x^n, v) + \sum_{u,v \in [x^{2k+1}]} d(u,v) + \sum_{u,v \in [x^{2k}]} d(u,v)$$

$$+ \sum_{u,v \in [x^i y]} d(u,v)$$

$$= (4n-1).1 + (n-1).2n.1 + (n-1).3n.2 + (n-1).1 + 2n.2$$

$$+ \binom{n-1}{2}.2 + \binom{n-1}{2}.2 + \binom{2n}{2}.2$$

$$= 14n^2 - 7n + 2.$$

Let n be a power of 2. Since $\Omega_{Q_{4n}}$ is a star graph S_{4n-1}, then the distance between vertex e and any other vertices on $\Omega_{Q_{4n}}$ is one and the distance is two for any two vertices in $V(\Omega_{Q_{4n}}) - \{e\}$. Therefore

$$W(\Omega_{Q_{4n}}) = \sum_{v \in V(\Omega_{Q_{4n}})-\{e\}} d(e,v) + \sum_{u,v \in V(\Omega_{Q_{4n}})-\{e\}} d(u,v)$$

$$= (4n-1).1 + \binom{4n-1}{2}.2$$

$$= (4n-1)^2.$$

∎

Theorem 3.7. *Let* $\Omega_{Q_{4n}}$ *be the coprime graph of generalized quaternion group. The hyper-Wiener index of* $\Omega_{Q_{4n}}$ *is*

$$WW(\Omega_{Q_{4n}}) = \begin{cases} 20n^2 - 12n + 4, & \text{for } n \text{ is an odd prime} \\ 24n^2 - 14n + 2, & \text{for } n \text{ is a power of 2.} \end{cases}$$

PROOF. Let n be an odd prime. Firstly we determine the sum of square of the distance between any two vertices of $\Omega_{Q_{4n}}$ as follows

$$\sum_{u,v \in V(\Omega_{Q_{4n}})} (d(u,v))^2 = \sum_{v \in V(\Omega_{Q_{4n}})-\{e\}} (d(e,v))^2 + \sum_{v \in [x^{2k}]} (d(x^n,v))^2 + \sum_{v \in [x^i y]} (d(x^n,v))^2$$

$$+ \sum_{u \in [x^{2k}], v \in [x^i y]} (d(u,v))^2 + \sum_{u \in [x^{2k+1}], v \in \{x^n\} \cup [x^i y] \cup [x^{2k}]} (d(u,v))^2$$

$$+ \sum_{u,v \in [x^{2k+1}]} (d(u,v))^2 + \sum_{u,v \in [x^{2k}]} (d(u,v))^2 + \sum_{u,v \in [x^i y]} (d(u,v))^2$$

$$= (4n-1).1^2 + (n-1).1^2 + 2n.2^2 + (n-1).2n.1^2 + (n-1).3n.2^2$$

$$+ \binom{n-1}{2}.2^2 + \binom{n-1}{2}.2^2 + \binom{2n}{2}.2^2$$

$$= 26n^2 - 17n + 6.$$

Referring Theorem 3.6 we have

$$WW(\Omega_{Q_{4n}}) = \frac{1}{2}\left(W(\Omega_{Q_{4n}}) + \sum_{u,v \in V(\Omega_{Q_{4n}})} (d(u,v))^2 \right)$$

$$= \frac{1}{2}\left((14n^2 - 7n + 2) + (26n^2 - 17n + 6) \right)$$

$$= 20n^2 - 12n + 4.$$

Let n be a power of 2. According to Theorem 3.6 and its proof, we have

$$WW(\Omega_{Q_{4n}}) = \frac{1}{2}\left(W(\Omega_{Q_{4n}}) + \sum_{u,v \in V(\Omega_{Q_{4n}})} (d(u,v))^2 \right)$$

$$= \frac{1}{2}\left((4n-1)^2 + (4n-1).1^2 + \binom{4n-1}{2}.2^2 \right)$$

$$= 24n^2 - 14n + 2.$$

∎

Theorem 3.8. *Let* $\Omega_{Q_{4n}}$ *be the coprime graph of generalized quaternion group. The Harary index of* $\Omega_{Q_{4n}}$ *is*

$$H(\Omega_{Q_{4n}}) = \begin{cases} \frac{1}{2}(10n^2 + n - 2), & \text{for } n \text{ is an odd prime} \\ \\ \frac{1}{2}(4n-1)(2n+1), & \text{for } n \text{ is a power of 2.} \end{cases}$$

Proof. Since the Harary index is the summation of inverse of distances between any two vertices of $\Omega_{Q_{4n}}$, then we can determine it based on the proof of Theorem 3.6. Therefore, for n is an odd prime we have

$$
H(\Omega_{Q_{4n}}) = \sum_{u,v \in V(\Omega_{Q_{4n}})} \frac{1}{d(u,v)}
$$

$$
= \sum_{v \in V(\Omega_{Q_{4n}})-\{e\}} \frac{1}{d(e,v)} + \sum_{u \in [x^{2k}], v \in [x^i y]} \frac{1}{d(u,v)} + \sum_{u \in [x^{2k+1}], v \in \{x^n\} \cup [x^i y] \cup [x^{2k}]} \frac{1}{d(u,v)}
$$

$$
+ \sum_{v \in [x^{2k}]} \frac{1}{d(x^n,v)} + \sum_{v \in [x^i y]} \frac{1}{d(x^n,v)} + \sum_{u,v \in [x^{2k+1}]} \frac{1}{d(u,v)} + \sum_{u,v \in [x^{2k}]} \frac{1}{d(u,v)} + \sum_{u,v \in [x^i y]} \frac{1}{d(u,v)}
$$

$$
= (4n-1).1 + (n-1).2n.1 + (n-1).3n.\frac{1}{2} + (n-1).1 + 2n.\frac{1}{2}
$$

$$
+ \binom{n-1}{2}.\frac{1}{2} + \binom{n-1}{2}.\frac{1}{2} + \binom{2n}{2}.\frac{1}{2}
$$

$$
= \frac{1}{2}(10n^2 + n - 2).
$$

For n is a power of 2 we have

$$
H(\Omega_{Q_{4n}}) = \sum_{v \in V(\Omega_{Q_{4n}})-\{e\}} \frac{1}{d(e,v)} + \sum_{u,v \in V(\Omega_{Q_{4n}})-\{e\}} \frac{1}{d(u,v)}
$$

$$
= (4n-1).1 + \binom{4n-1}{2}.\frac{1}{2}
$$

$$
= \frac{1}{2}(4n-1)(2n+1).
$$

∎

Theorem 3.9. *Let $\Omega_{Q_{4n}}$ be the coprime graph of generalized quaternion group. The Szeged index of $\Omega_{Q_{4n}}$ is*

$$
Sz(\Omega_{Q_{4n}}) = \begin{cases} 4n^4 - 4n^3 + 3n^2 + 1, & \text{for } n \text{ is an odd prime} \\ (4n-1)(4n-2), & \text{for } n \text{ is a power of 2.} \end{cases}
$$

Proof. Let n be an odd prime. From Figure 3.1 we can determine the vertices of $\Omega_{Q_{4n}}$ which are closer to one of two vertices that are adjacent as follows:

(i) Edge $a = ex^n$.

$$
N_1(a|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}}) | d(u,e) < d(u,x^n)\} = [x^{2k+1}] \cup [x^i y]
$$

$$
N_2(a|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}}) | d(u,x^n) < d(u,e)\} = \{x^n\}.
$$

Thus $|N_1(a|\Omega_{Q_{4n}})| = (n-1) + 2n = 3n - 1$ and $|N_2(a|\Omega_{Q_{4n}})| = 1$.

(ii) Edge $b_k = ex^{2k+1}$, for $k \in \{0, 1, 2, ..., n-1$ and $2k+1 \neq n\}$.

$$
N_1(b_k|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}}) | d(u,e) < d(u,x^{2k+1})\} = \{x^n\} \cup [x^{2k}] \cup [x^i y]
$$

$$
N_2(b_k|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}}) | d(u,x^{2k+1}) < d(u,e)\} = \{x^{2k+1}\}.
$$

Thus $|N_1(b_k|\Omega_{Q_{4n}})| = 1 + (n-1) + 2n = 3n$ and $|N_2(b_k|\Omega_{Q_{4n}})| = 1$.

(iii) Edge $c_k = ex^{2k}$, for $k \in \{1, 2, 3, ..., n-1\}$.

$$N_1(c_k|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}})|d(u,e) < d(u,x^{2k})\} = [x^{2k+1}]$$
$$N_2(c_k|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}})|d(u,x^{2k}) < d(u,e) = \{x^{2k}\}.$$

Thus $|N_1(c_k|\Omega_{Q_{4n}})| = n-1$ and $|N_2(c_k|\Omega_{Q_{4n}})| = 1$.

(iv) Edge $d_i = ex^i y$, for $i \in \{0, 1, 2, ..., 2n-1\}$.

$$N_1(d_i|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}})|d(u,e) < d(u,x^i y)\} = \{x^n\} \cup [x^{2k+1}]$$
$$N_2(d_i|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}})|d(u,x^i y) < d(u,e) = \{x^i y\}.$$

Thus $|N_1(d_i|\Omega_{Q_{4n}})| = 1 + (n-1) = n$ and $|N_2(d_i|\Omega_{Q_{4n}})| = 1$.

(v) Edge $f_k = x^n x^{2k}$, for $k \in \{1, 2, 3, ..., n-1\}$.

$$N_1(f_k|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}})|d(u,x^n) < d(u,x^{2k})\} = \{x^{2j}|j \neq k\} \cup \{x^n\}$$
$$N_2(f_k|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}})|d(u,x^{2k}) < d(u,x^n) = \{x^{2k}\} \cup [x^i y].$$

Thus $|N_1(f_k|\Omega_{Q_{4n}})| = (n-1-1) + 1 = n-1$ and $|N_2(f_k|\Omega_{Q_{4n}})| = 1 + 2n$.

(vi) Edge $g_{ki} = x^{2k} x^i y$, for $k \in \{1, 2, 3, ..., n-1\}$ and $i \in \{0, 1, 2, ..., 2n-1\}$.

$$N_1(g_{ki}|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}})|d(u,x^{2k}) < d(u,x^i y)\} = \{x^j y|j \neq i\} \cup \{x^n\} \cup \{x^{2k}\}$$
$$N_2(g_{ki}|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}})|d(u,x^i y) < d(u,x^{2k}) = \{x^{2j}|j \neq k\} \cup \{x^i y\}.$$

Thus $|N_1(g_{ki}|\Omega_{Q_{4n}})| = 1 + 1 + (2n-1) = 2n+1$ and $|N_2(g_{ki}|\Omega_{Q_{4n}})| = (n-1-1) + 1 = n-1$.

Therefore

$$Sz(\Omega_{Q_{4n}}) = \sum_{x \in E(\Omega_{Q_{4n}})} |N_1(x|\Omega_{Q_{4n}})|.|N_1(x|\Omega_{Q_{4n}})|$$

$$= |N_1(a|\Omega_{Q_{4n}})|.|N_2(a|\Omega_{Q_{4n}})| + \sum_{b_k \in [b]} |N_1(b_k|\Omega_{Q_{4n}})|.|N_2(b_k|\Omega_{Q_{4n}})|$$

$$+ \sum_{c_k \in [c]} |N_1(c_k|\Omega_{Q_{4n}})|.|N_2(c_k|\Omega_{Q_{4n}})| + \sum_{d_i \in [d]} |N_1(d_i|\Omega_{Q_{4n}})|.|N_2(d_i|\Omega_{Q_{4n}})|$$

$$+ \sum_{f_k \in [f]} |N_1(f_k|\Omega_{Q_{4n}})|.|N_2(f_k|\Omega_{Q_{4n}})| + \sum_{g_{ki} \in [g]} |N_1(g_{ki}|\Omega_{Q_{4n}})|.|N_2(g_{ki}|\Omega_{Q_{4n}})|$$

$$= 1.(3n-1).1 + (n-1).3n.1 + (n-1)(n-1).1 + 2n.n.1$$
$$+ (n-1).(n-1)(1+2n) + (n-1).2n.(2n+1).(n-1)$$
$$= 4n^4 - 4n^3 + 3n^2 + 1.$$

Let n be a power of 2. Let $V(\Omega_{Q_{4n}}) = \{e, v_1, v_2, ..., v_{4n-1}\}$ and $E(\Omega_{Q_{4n}}) = \{e_i = ev_i | 1 \leqslant i \leqslant 4n-1\}$. Hence

$$N_1(e_i|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}})|d(u,e) < d(u,v_i)\} = \{v_j|j \neq i\}$$
$$N_2(e_i|\Omega_{Q_{4n}}) = \{u \in V(\Omega_{Q_{4n}})|d(u,v_i) < d(u,e) = \{v_i\}.$$

Thus $|N_1(e_i|\Omega_{Q_{4n}})| = (4n-1) - 1 = 4n - 2$ and $|N_2(e_i|\Omega_{Q_{4n}})| = 1$. Therefore

$$Sz(\Omega_{Q_{4n}}) = \sum_{i=1}^{4n-1} |N_1(e_i|\Omega_{Q_{4n}})|.|N_2(e_i|\Omega_{Q_{4n}})|$$
$$= (4n-1).(4n-2).$$

∎

4. Conclusion

In this study, we have found the shape of coprime graph of generalized quaternion group and determined its six connectivity indices.

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