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# $k$-PRODUCT CORDIAL BEHAVIOUR OF UNION OF GRAPHS

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**Abstract.** Let $f$ be a map from $V(G)$ to $\{0,1,...,k-1\}$ where $k$ is an integer, $1 \leq k \leq |V(G)|$. For each edge $uv$ assign the label $f(u)f(v)(mod\ k)$. $f$ is called a $k$-product cordial labeling if $\left|v_f(i) - v_f(j)\right| \leq 1$, and $\left|e_f(i) - e_f(j)\right| \leq 1$, $i,j \in \{0,1,...,k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with $x$ $(x = 0,1,...,k-1)$. In this paper, we investigate the $k$-product cordial behaviour of union of graphs.

*Key words and Phrases*: cordial labeling, product cordial labeling, $k$-product cordial labeling

## 1. INTRODUCTION

While studying graph theory, one that has gained a lot of popularity during the last 60 years is the concept of labelings of graphs due to its wide range of applications. Labeling is a function that allocates the elements of a graph to real numbers, usually positive integers. In 1967, Rosa [10] published a pioneering paper on graph labeling problems. Thereafter, many types of graph labeling techniques have been studied by several authors. Gallian [2] in his survey beautifully classified them into graceful labeling and harmonious labelings, variations of graceful

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labelings, variations of harmonious labelings, magic type labelings, anti-magic type labelings and miscellaneous labelings. Cordial labeling is a weaker version of graceful and harmonious labeling was introduced by Cahit [1]. Sundaram et al. [11] extended the concept of cordial labeling and introduced product cordial labeling. Many researchers have shown interest on this topic and established that several classes of graphs admit product cordial labeling [2].

Followed by this, Ponraj et al. [9] further extended the concept of product cordial labeling and introduced a new labeling called k-product cordial labeling. They proved that k-product cordial labeling of stars and bistars further they studied the 4-product cordial labeling behavior of paths, complete graphs and combs. Jeyanthi and Maheswari [8] proved that if $G_1$ is a 3-product cordial graph with $3m$ vertices and $3n$ edges and $G_2$ is any 3-product cordial graph, then $G_1 \cup G_2$ is also 3-product cordial graph. For further results on 3-product and 4-product cordial labeling an interested reader can refer to [2].

Inspired by the concept of k-product cordial labeling and the results in [9], we [4] showed that Napier bridge graphs when $k = 3 \; and \; 4$ admit k-product cordial labeling. In [5] we proved that fan and double fan graphs when $k = 4 \; and \; 5$ admit k-product cordial labeling. Also, we [6] established that cone and double cone graphs are 5-product cordial graph and double cone is not 4-product cordial graph. Further, we [7] investigated the k-product cordial behaviour of $G + \overline{K_t}$, where $G$ is a k-product cordial graph. In addition, we found an upper bound of the size of k-product cordial graphs. In this paper, we investigate the k-product cordial behaviour of union of graphs. We organize this paper as follows: In the next section, we give the definitions and notations which are useful for the present study. In section three, we establish the k-product cordial behaviour of union of graphs.

## 2. DEFINITIONS AND TERMINOLOGY

In this section, we give the definitions and terminology used in this paper. All graphs considered here are simple, finite and undirected. We follow the basic notations and terminology of graph theory as in [3]. The union $G_1 \cup G_2$ of two graphs $G_1$ and $G_2$ has the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. A fan graph $F_n$, is obtained by joining all the vertices of $P_n$ to a new vertex which is known as the center. Tadpole $T(n, m)$ is a graph in which path $P_m$ is attached to any one vertex of cycle $C_n$. The following definitions are useful for the present study.

**Definition 2.1.**[1] Let $f$ be a function from the vertices of $G$ to $\{0, 1\}$ and for each edge $xy$ assign the label $|f(x) - f(y)|$. $f$ is called a cordial labeling of $G$ if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1.

**Definition 2.2.**[11] Let $f$ be a function from $V(G)$ to $\{0, 1\}$. For each edge $uv$, assign the label $f(u)f(v)$. Then $f$ is called product cordial labeling if $|v_f(0) - v_f(1)| \leq$

1 and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denotes the number of vertices and edges respectively labeled with $i(i = 0, 1)$.

**Definition 2.3.**[9] Let $f$ be a map from $V(G)$ to $\{0, 1, ..., k-1\}$ where $k$ is an integer, $1 \leq k \leq |V(G)|$. For each edge $uv$ assign the label $f(u)f(v)(mod\ k)$. $f$ is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, ..., k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with $x$ $(x = 0, 1, ..., k-1)$.

## 3. MAIN RESULTS

**Theorem 3.1.** *If $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are k-product cordial graph such that (either $p_1$ or $p_2 \equiv 0(mod\ k)$) and (either $q_1$ or $q_2 \equiv 0(mod\ k)$), then $G_1 \cup G_2$ is also k-product cordial graph.*

*Proof.* Let $f$ be a k-product cordial labeling of $G_1$. Let $g$ be a k-product cordial labeling of $G_2$. We have either $p_1$ *or* $p_2 \equiv 0(mod\ k)$. Without loss of generality, we take $p_1 \equiv 0(mod\ k)$ which implies that $|v_f(i) - v_f(j)| = 0$ and $|v_g(i) - v_g(j)| \leq 1$. Also we have either $q_1$ *or* $q_2 \equiv 0(mod\ k)$. Without loss of generality, we take $q_2 \equiv 0(mod\ k)$ which implies that $|e_f(i) - e_f(j)| \leq 1$ and $|e_g(i) - e_g(j)| = 0$. Define a function $h : V(G_1 \cup G_2) \to \{0, 1, 2, ..., k-1\}$ by

$$h(v) = \begin{cases} f(v) & \text{if } v \in V(G_1) \\ g(v) & \text{if } v \in V(G_2). \end{cases}$$

Therefore, $v_h(i) = v_f(i) + v_g(i)$ and $e_h(i) = e_f(i) + e_g(i)$ for $0 \leq i \leq k-1$.
We have,

$$\begin{aligned} |v_h(i) - v_h(j)| &= |v_f(i) + v_g(i) - v_f(j) - v_g(j)| \\ &\leq |v_f(i) - v_f(j)| + |v_g(i) - v_g(j)| \\ &\leq 1 \\ |v_h(i) - v_h(j)| &\leq 1\ ;\ 0 \leq i, j \leq k-1. \end{aligned}$$

Also we have,

$$\begin{aligned} |e_h(i) - e_h(j)| &= |e_f(i) + e_g(i) - e_f(j) - e_g(j)| \\ &\leq |e_f(i) - e_f(j)| + |e_g(i) - e_g(j)| \\ &\leq 1 \\ |e_h(i) - e_h(j)| &\leq 1\ ;\ 0 \leq i, j \leq k-1. \end{aligned}$$

Therefore, $h$ is a k-product cordial labeling of $G_1 \cup G_2$. Hence, $G_1 \cup G_2$ is a k-product cordial graph. □

An example of 5-product cordial labeling of $P_{10} \cup F_8$ is shown in Figure 1.

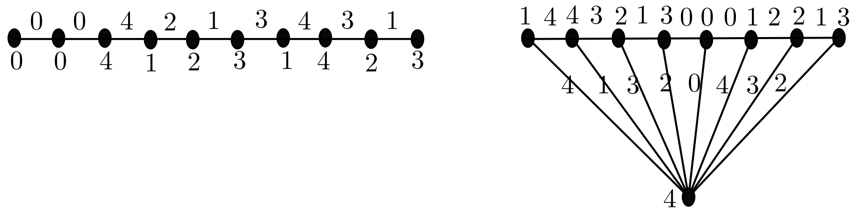


Figure 1. A 5-product cordial labeling of $P_{10} \cup F_8$

**Theorem 3.2.** *If $G_1(p_1, q_1)$, $G_2(p_2, q_2)$,..., $G_{n-1}(p_{n-1}, q_{n-1})$ are k-product cordial graph with $p_t \equiv 0(mod\ k)$ and $q_t \equiv 0(mod\ k)$ for $1 \leq t \leq n-1$ and $G_n(p_n, q_n)$ is any k-product cordial graph, then $G_1 \cup G_2 \cup G_3 \cup ... \cup G_n$ is also k-product cordial graph.*

*Proof.* Let $f_t$ be a k-product cordial labeling of $G_t(p_t, q_t)$ for $1 \leq t \leq n-1$. Since $p_t \equiv 0(mod\ k)$ and $q_t \equiv 0(mod\ k)$, $|v_{f_t}(i) - v_{f_t}(j)| = 0$ and $|e_{f_t}(i) - e_{f_t}(j)| = 0$ for $1 \leq t \leq n-1$. Let $f_n$ be a k-product cordial labeling of $G_n(p_n, q_n)$. Hence $|v_{f_n}(i) - v_{f_n}(j)| \leq 1$ and $|e_{f_n}(i) - e_{f_n}(j)| \leq 1$.

Define a function $g : V(G_1 \cup G_2 \cup ... \cup G_n) \to \{0, 1, 2, ..., k-1\}$ by $g(v) = f_t(v)$ if $v \in V(G_t); 1 \leq t \leq n$.
Clearly, $v_g(i) = \sum_{t=1}^{n} v_{f_t}(i)$ and $e_g(i) = \sum_{t=1}^{n} e_{f_t}(i)$ for $0 \leq i \leq k-1$.
We have,

$$\begin{aligned}
|v_g(i) - v_g(j)| &= \left| \sum_{t=1}^{n} v_{f_t}(i) - \sum_{t=1}^{n} v_{f_t}(j) \right| \\
&\leq |v_{f_1}(i) - v_{f_1}(j)| + |v_{f_2}(i) - v_{f_2}(j)| + ... + |v_{f_n}(i) - v_{f_n}(j)| \\
&\leq 0 + 1 \\
|v_g(i) - v_g(j)| &\leq 1\ ;\ 0 \leq i, j \leq k-1.
\end{aligned}$$

Also we have,

$$\begin{aligned}
|e_g(i) - e_g(j)| &= \left| \sum_{t=1}^{n} e_{f_t}(i) - \sum_{t=1}^{n} e_{f_t}(j) \right| \\
&\leq |e_{f_1}(i) - e_{f_1}(j)| + |e_{f_2}(i) - e_{f_2}(j)| + ... + |e_{f_n}(i) - e_{f_n}(j)| \\
&\leq 0 + 1 \\
|e_g(i) - e_g(j)| &\leq 1\ ;\ 0 \leq i, j \leq k-1.
\end{aligned}$$

Therefore, $g$ is a k-product cordial labeling of $G_1 \cup G_2 \cup G_3 \cup ... \cup G_n$. Hence, $G_1 \cup G_2 \cup G_3 \cup ... \cup G_n$ is a k-product cordial graph. $\square$

An example of 3-product cordial labeling of $T(4, 2) \cup T(5, 7) \cup P_5$ is shown in Figure 2.

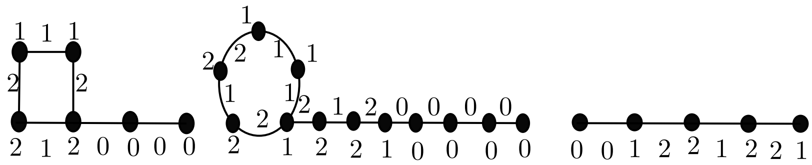

Figure 2. A 3-product cordial labeling of $T(4,2)\cup T(5,7)\cup P_5$

**Corollary 3.3.** *For any k-product cordial graph $G$ with $p\equiv 0(mod\ k)$ and $q\equiv 0(mod\ k)$, $mG$ is k-product cordial graph where $m$ is an integer.*

**Theorem 3.4.** *Let $f$ be a k-product cordial labeling of $G_1(p_1,q_1)$ with $p_1=nk$, $q_1=mk+a$ for $1\le a\le k-1$ and $g$ be a k-product cordial labeling of $G_2(p_2,q_2)$ with $q_2=lk+b$ for $1\le b\le k-a$. If $e_f(t)=m+1$ and $e_g(t)=l$, then $G_1\cup G_2$ is also a k-product cordial graph.*

*Proof.* Let $f$ be a k-product cordial labeling of $G_1(p_1,q_1)$. Since $p_1=nk$ and $q_1=mk+a$, $e_f(i)=m$ *or* $m+1$ for $0\le i\le k-1$. Hence, $|v_f(i)-v_f(j)|=0$ and $|e_f(i)-e_f(j)|\le 1$ for $0\le i,j\le k-1$. Let $g$ be a k-product cordial labeling of $G_2(p_2,q_2)$. Since $q_2=lk+b$; $1\le b\le k-a$, $e_g(i)=l$ *or* $l+1$ for $0\le i\le k-1$. Hence $|v_g(i)-v_g(j)|\le 1$ and $|e_g(i)-e_g(j)|\le 1$.
Define a function $h:V(G_1\cup G_2)\to\{0,1,2,...,k-1\}$ by

$$h(v)=\begin{cases} f(v) & \text{if } v\in V(G_1)\\ g(v) & \text{if } v\in V(G_2).\end{cases}$$

Therefore, $v_h(i)=v_f(i)+v_g(i)$ and $e_h(i)=e_f(i)+e_g(i)$ for $0\le i\le k-1$.
We have,

$$\begin{aligned}|v_h(i)-v_h(j)| &= |v_f(i)+v_g(i)-v_f(j)-v_g(j)|\\ &\le |v_f(i)-v_f(j)|+|v_g(i)-v_g(j)|\\ &\le 0+1\\ |v_h(i)-v_h(j)| &\le 1\ ;\ 0\le i,j\le k-1.\end{aligned}$$

Since $e_f(t)=m+1$ and $e_g(t)=l$, we have

$$\begin{aligned}|e_h(i)-e_h(j)| &= |e_f(i)+e_g(i)-e_f(j)-e_g(j)|\\ &= \begin{cases} m+1+l-(m+1+l) & \text{if } i=t, j=t\\ m+1+l-(m+l) & \text{if } i=t, j\ne t, e_g(j)=l\\ m+1+l-(m+1+l) & \text{if } i=t, j\ne t, e_g(j)=l+1\\ m+l-(m+1+l) & \text{if } i\ne t, j\ne t, e_g(i)=l, e_g(j)=l+1\\ m+1+l-(m+1+l) & \text{if } i\ne t, j\ne t, e_g(i)=e_g(j)=l+1\\ m+1+l-(m+1+l) & \text{if } i\ne t, j\ne t, e_g(i)=e_g(j)=l.\end{cases}\\ &\le 1\\ |e_h(i)-e_h(j)| &\le 1\ ;\ 0\le i,j\le k-1.\end{aligned}$$

Therefore, $h$ is a k-product cordial labeling of $G_1\cup G_2$. Hence, $G_1\cup G_2$ is a k-product cordial graph. □

An example of 5-product cordial labeling of $F_9\cup P_{12}$ is shown in Figure 3.

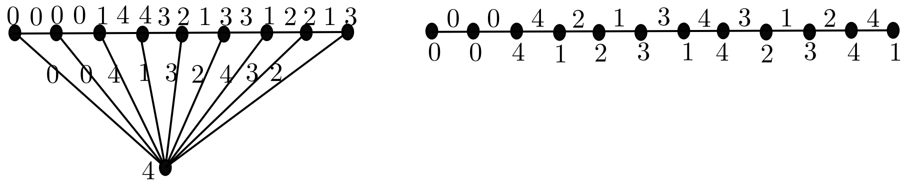


Figure 3. A 5-product cordial labeling of $F_9 \cup P_{12}$

**Corollary 3.5.** *Let $f$ be a k-product cordial labeling of $G_1(p_1, q_1)$ with $p_1 = nk+a$, $q_1 = mk$ for $1 \leq a \leq k-1$ and $g$ be a k-product cordial labeling of $G_2(p_2, q_2)$ with $p_2 = lk + b$ for $1 \leq b \leq k-a$. If $v_f(t) = n+1$ and $v_g(t) = l$, then $G_1 \cup G_2$ is also a k-product cordial graph.*

**Theorem 3.6.** *Let $f$ be a k-product cordial labeling of $G_1(p_1, q_1)$ with $p_1 = nk$, $q_1 = mk+a$ for $1 \leq a \leq k-1$ and $g$ be a k-product cordial labeling of $G_2(p_2, q_2)$ with $q_2 = lk+b$ for $k-a+1 \leq b \leq k-1$. If $e_f(t) = m$ and $e_g(t) = l+1$, then $G_1 \cup G_2$ is also a k-product cordial graph.*

*Proof.* Let $f$ be a k-product cordial labeling of $G_1(p_1, q_1)$. Since $p_1 = nk$ and $q_1 = mk + a$, $e_f(i) = m \text{ or } m+1$ for $0 \leq i \leq k-1$. Hence, $|v_f(i) - v_f(j)| = 0$ and $|e_f(i) - e_f(j)| \leq 1$ for $0 \leq i, j \leq k-1$. Let $g$ be a k-product cordial labeling of $G_2(p_2, q_2)$. Since $q_2 = lk + b$; $k - a + 1 \leq b \leq k-1$, $e_g(i) = l \text{ or } l+1$ for $0 \leq i \leq k-1$. Hence $|v_g(i) - v_g(j)| \leq 1$ and $|e_g(i) - e_g(j)| \leq 1$.
Define a function $h : V(G_1 \cup G_2) \to \{0, 1, 2, ..., k-1\}$ by

$$h(v) = \begin{cases} f(v) & \text{if } v \in V(G_1) \\ g(v) & \text{if } v \in V(G_2). \end{cases}$$

Therefore, $v_h(i) = v_f(i) + v_g(i)$ and $e_h(i) = e_f(i) + e_g(i)$ for $0 \leq i \leq k-1$.
We have,

$$\begin{aligned} |v_h(i) - v_h(j)| &= |v_f(i) + v_g(i) - v_f(j) - v_g(j)| \\ &\leq |v_f(i) - v_f(j)| + |v_g(i) - v_g(j)| \\ &\leq 0 + 1 \\ |v_h(i) - v_h(j)| &\leq 1 \ ; \ 0 \leq i, j \leq k-1. \end{aligned}$$

Since $e_f(t) = m$ and $e_g(t) = l+1$, we have

$$\begin{aligned} |e_h(i) - e_h(j)| &= |e_f(i) + e_g(i) - e_f(j) - e_g(j)| \\ &= \begin{cases} m+1+l-(m+1+l) & \text{if } i = t, j = t \\ m+1+l-(m+1+l) & \text{if } i = t, j \neq t, e_g(j) = l \\ m+1+l-(m+1+l+1) & \text{if } i = t, j \neq t, e_g(j) = l+1 \\ m+1+l-(m+1+l+1) & \text{if } i \neq t, j \neq t, e_g(i) = l, e_g(j) = l+1 \\ m+1+l+1-(m+1+l+1) & \text{if } i \neq t, j \neq t, e_g(i) = e_g(j) = l+1 \\ m+1+l-(m+1+l) & \text{if } i \neq t, j \neq t, e_g(i) = e_g(j) = l. \end{cases} \\ &\leq 1 \\ |e_h(i) - e_h(j)| &\leq 1 \ ; \ 0 \leq i, j \leq k-1. \end{aligned}$$

Therefore, $h$ is a k-product cordial labeling of $G_1 \cup G_2$. Hence, $G_1 \cup G_2$ is a k-product cordial graph. □

An example of 5-product cordial labeling of $F_9 \cup P_{15}$ is shown in Figure 4.

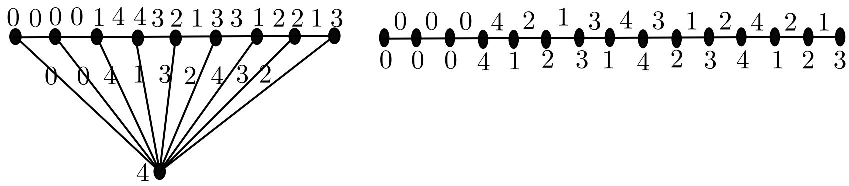

Figure 4. A 5-product cordial labeling of $F_9 \cup P_{15}$

**Corollary 3.7.** *Let $f$ be a $k$-product cordial labeling of $G_1(p_1, q_1)$ with $p_1 = nk + a$, $q_1 = mk$ for $1 \leq a \leq k-1$ and $g$ be a $k$-product cordial labeling of $G_2(p_2, q_2)$ with $p_2 = lk + b$ for $k - a + 1 \leq b \leq k-1$. If $v_f(t) = n$ and $v_g(t) = l + 1$, then $G_1 \cup G_2$ is also a $k$-product cordial graph.*

We conclude this paper with the following remark.

**Remark 3.8.** *If $G_1$ and $G_2$ are $k$-product cordial graphs then the disjoint union of $G_1$ and $G_2$ need not be a $k$-product cordial graph.*

For example, $C_6$ is a 4-product cordial graph [9] but the disjoint union of two copies of $C_6$ is not a 4-product cordial graph.

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