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EXISTENCE AND UNIQUENESS OF HOMOCLINIC SOLUTION FOR SINGULAR BOUNDARY VALUE PROBLEM WITH DUFFING OSCILLATOR

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Abstract. In this paper, we consider a singular boundary value problem with the non-autonomous kind of Duffing equation. With the help of the Krein Rutman Theorem and by using fixed point arguments, we derive existence and uniqueness results of homoclinic solution. Finally, examples are given to illustrate our theoretical results.

Key words and Phrases: Duffing equation, positive solution, fixed point.

1. INTRODUCTION

This paper deals with the existence and uniqueness results of homoclinic solutions for the following singular boundary value problem with Duffing equation

$$\left\{ \begin{array}{l} -u''(t) + cu'(t) + u(t)(\lambda - q(t) .u^m(t)) = \phi(t) f(t, u(t)),\ t \in \mathbb{R}, \\ u(-\infty) = u(+\infty) = 0, \end{array} \right. \tag{1}$$

where $\lambda, c, > 0, m \geq -1$ and for $t > 0$, the function $x \mapsto f(t, x)$ is not defined at 0.

The existence of homoclinic solutions for Duffing equations attracted the attention of researchers from all over the world and as such have been extensively investigated in the literature, see ([2], [3], [5], [6], [7], [8], [9], [11], [13], [14], [15] and [17], and references therein. It can be considered as solutions having a finite limit to $\pm\infty$. The Duffing equation (or Duffing oscillator), named after Georg Duffing (1861–1944), is a non-linear second-order differential equation used to model certain damped and driven oscillators. The equation is given by $\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega t)$ where the (unknown) function $x(t)$ is the displacement at time t, the first derivative $\dot{x}$ is the velocity, and the second time $\ddot{x}$ derivative is acceleration.

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The Duffing system presents in the frequency response the jump resonance phenomenon that is a sort of frequency hysteresis behaviour, where δ controls the amount of damping, α controls the linear stiffness, β controls the amount of non-linearity in the restoring force, γ is the amplitude of the periodic driving force and ω is the angular frequency of the periodic driving force.

In [11], Duffing's equations with variable coefficients was studied in continuous and discrete cases and in the presence of both harmonic and nonharmonic external perturbations. In [17], the existence results for Duffing equations was established with a p-Laplacian operator.

Motivated by the above works, in this paper we study existence and uniqueness results of positive homoclinic solution for Duffing type problem with variable coefficient (1) posed in the real line.

Throughout the article, we assume that $f : \mathbb{R}\times(0,+\infty) \to \mathbb{R}^+$ is a continuous function, $\phi, q : \mathbb{R} \to \mathbb{R}^+$ are the measurable functions, where ϕ does not vanish identically on any subinterval of $\mathbb{R}$ such that

$$\int_{\mathbb{R}} q(s)\, e^{r_2|s|} ds < \infty, \quad \int_{\mathbb{R}} \max\left(e^{-r_1 t}, e^{-r_2 t}\right) \phi(t)\, dt < \infty, \tag{2}$$

and for all $r, R > 0$ with $r \leq R$, there exists a positive function $g_{r,R} : \mathbb{R} \to \mathbb{R}$ with

$$\int_{\mathbb{R}} \phi(t)\, g_{r,R}(t)\, dt < \infty,$$

such that

$$f\left(t, e^{r_2|t|}x\right) \leq g_{r,R}(t) \text{ for } (t,x) \in \mathbb{R} \times [\gamma(t)\, r, R] \tag{3}$$

where

$$\gamma(t) = \min(e^{2r_2 t}, e^{(r_1 - r_2)t}),$$

and r_1 and r_2 are solutions of caracteristic function $-X^2 + cX + \lambda = 0$ with

$$r_1 < 0 < r_2.$$

The rest of the paper is organized as follows. in Section 2, some preliminary materials to be used later are stated. In Section 3, we present and prove our main results consisting of existence and uniqueness results, where the singularity 0 is apparent in the first existence theorem and non-apparent in others. Finally, examples are given to illustrate our theoretical results.

2. Preliminaries

For sake of completeness let us recall some basic facts needed in this paper. Let E be a real Banach space equipped with its norme noted $\|.\|$. A nonempty closed convex subset P of E is said to be a cone if $P \cap (-P) = 0$ and $(tP) \subset P$ for all $t \geq 0$. It is well known that a cone P induces a partial order in the Banach space E. We write for all $x; y{:}\in E$; $x \leq y$ if $y - x \in P$.

The mapping $L : E \to E$ is said to be positive in P if $L(P) \subset P$, and compact if it is continuous and $L(B)$ is relatively compact in E for all bounded subset B of E. The real value

$$r(L) = \sup\{|\lambda| : \lambda \in Sp(L)\}$$

denotes the spectral radius of a linear and bounded operator L, where $Sp(L)$ is the spectrum of L, and we have

$$r(L) = \lim_{n\to\infty} \|L^n\|^{\frac{1}{n}}.$$

The main tool of this work is the following Guo-Krasnoselskii's version of expansion and compression of a cone principal in a Banach space [7] .

Theorem 2.1. *Let Ω_1, Ω_2 be open bounded subsets of E such that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. If $T : P \cap (\overline{\Omega_2} \backslash \Omega_1) \to P$ is a compact operator such that either:*

(1) $||Tu|| \leq ||u||$ for $u \in P \cap \partial\Omega_1$ and $||Tu|| \geq ||u||$ for $u \in P \cap \partial\Omega_2$, or
(2) $||Tu|| \geq ||u||$ for $u \in P \cap \partial\Omega_1$ and $||Tu|| \leq ||u||$ for $u \in P \cap \partial\Omega_2$,

Then T has a fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_2)$.

The following Krein Rutman Theorem has been established in [16]:

Theorem 2.2. *Let K be a cone in E and $L : E \to E$ be an linear, positive, and compact operator. Suppose that for some non-zero element $u \in K^*$, the following relation is satisfied:*

$$MLu \geq u, \text{ for some } M > 0.$$

Then L has a non-zero eigenvector $v \in K^*$:

$$Lv = \lambda v,$$

where the positive eigenvalue λ satisfies the inequality $\lambda \geq M^{-1}$.

In what follows, we let E be a Banach space defined as

$$E = \left\{ u \in C(\mathbb{R}, \mathbb{R}) : \lim_{|t|\to\infty} e^{-r_2|t|} u(t) = 0 \right\}$$

equipped with the norm $\|\cdot\|$, where for $u \in E$ $\|u\| = \sup_{t\in\mathbb{R}} \left(e^{-r_2|t|} |u(t)|\right)$,

$$K = \{u \in E : u(t) \geq 0 \text{ for all } t \in \mathbb{R}\}$$

and

$$P = \{u \in K : u(t) \geq \widetilde{\gamma}(t) \|u\| \text{ for all } t \in \mathbb{R}\}$$

be the cone of E, with

$$\widetilde{\gamma}(t) = \min\left(e^{r_1 t}, e^{r_2 t}\right).$$

Lemma 2.3. [4] *A non empty subset M of E is relatively compact if the following conditions hold :*

(1) M is bounded in E,
(2) The set $\left\{u : e^{-r_2|t|}x(t), x \in M\right\}$ is locally equicontinuous on $[0, +\infty)$, and
(3) The set $\left\{u : e^{-r_2|t|}x(t), x \in M\right\}$ is equiconvergent at ∞

3. MAIN RESULTS

For $r > 0$, we consider the operator $T_r : P\backslash B(0,r) \to E$ defined by

$$T_r u(t) = \int_{-\infty}^{+\infty} G(t,s) F(s, u(s))\, ds$$

where $G, \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ defined by

$$G(t,s) = \frac{1}{r_2 - r_1} \begin{cases} \exp(r_1(t-s)) \text{ if } s \leq t, \\ \exp(r_2(t-s)) \text{ if } t \leq s \end{cases}$$

is the Green's function associated with the bvp (1) and

$$F(s,u) = q(s).u^{m+1} + \phi(s) f(s, u(s)), \ s \in \mathbb{R}.$$

Lemma 3.1. *Assume that Hypothesis (2) and (3) hold true and let $r > 0$. Then fixed points of T_r are positive solutions of bvp (1).*

Proof. Let $u \in P \setminus \{0\}$ be a fixed point of T_r , with $r = \|u\|$. For all $t \in \mathbb{R}$ we have

$$u(t) = \frac{1}{r_2 - r_1} \left(e^{r_1 t} \int_{-\infty}^{t} e^{-r_1 s} F(s, u(s)) ds + e^{r_2 t} \int_{t}^{+\infty} e^{-r_2 s} F(s, u(s)) ds \right),$$

$$u'(t) = \frac{r_1 e^{r_1 t}}{r_2 - r_1} \int_{-\infty}^{t} e^{-r_1 s} F(s, u(s)) ds + \frac{r_2 e^{r_2 t}}{r_2 - r_1} \int_{t}^{+\infty} e^{-r_2 s} F(s, u(s)) ds \text{ and}$$

$$u''(t) = \frac{(r_1)^2 e^{r_1 t}}{r_2 - r_1} \int_{-\infty}^{t} e^{-r_1 s} F(s, u(s)) ds + \frac{(r_2)^2 e^{r_2 t}}{r_2 - r_1} \int_{t}^{+\infty} e^{-r_2 s} F(s, u(s)) ds - F(t, u(t)).$$

Thus, we obtain

$$-u''(t) + cu'(t) + \lambda u(t) = \frac{-r_1^2 + cr_1 + \lambda}{r_2 - r_1} \int_{-\infty}^{t} G(t,s) F(s, u(s)) ds + \frac{-r_2^2 + cr_2 + \lambda}{r_2 - r_1} \int_{t}^{+\infty} G(t,s) F(s, u(s)) ds + F(t, u(t)) = F(t, u(t)).$$

Moreover, as $u(t) \in [\gamma(t) r, r]$, we have from (3) that $F(t, u(t)) \leq q(t) r^n + \phi(t) g_{r,r}(t) \in L^1(\mathbb{R})$ and so

$$\lim_{t \to +\infty} e^{r_2 t} \int_{t}^{+\infty} e^{-r_2 s} F(s, u(s)) ds \leq \lim_{t \to +\infty} \int_{t}^{+\infty} F(s, u(s)) ds = 0,$$

and

$$\lim_{t \to -\infty} e^{r_1 t} \int_{-\infty}^{t} e^{-r_2 s} F(s, u(s)) ds \leq \lim_{t \to -\infty} \int_{-\infty}^{t} F(s, u(s)) ds = 0,$$

leading to

$$\lim_{|t| \to +\infty} u(t) = 0,$$

completing the proof of the Lemma. □

Lemma 3.2. *The function G has the following properties:*

1: $0 < G(t,s) \leq \frac{1}{r_2 - r_1}$ for all $t, s \in \mathbb{R}$.

2: For all $t, \tau, s \in \mathbb{R}$

$$p(t)G(t,s) \geq \gamma(t)\, p(\tau)G(\tau,s).$$

where

$$\begin{aligned} &p(t) = e^{-r_2|t|},\\ &\gamma(t) = \min(e^{2r_2 t}, e^{(r_1-r_2)t}),\\ &\widetilde{\gamma}(t) = \frac{\gamma(t)}{p(t)} = \min\left(e^{r_1 t}, e^{r_2 t}\right) \end{aligned}$$

Proof. Assertions (1) is easy to prove, so we show assertion (2). For $t, \tau, s \in \mathbb{R}$,set $Q(t,\tau,s) = \frac{p(t)G(t,s)}{p(\tau)G(\tau,s)}$. We distingish four cases.
a/ $\tau,\, t \geq 0$, in this case we have

$$Q(t,s,\tau) = \begin{cases} \exp\left(-(r_2-r_1)\,t + (r_2-r_1)\,\tau\right) \geq e^{-(r_2-r_1)t} & \text{if } s \leq \tau \leq t\\ \exp\left(-(r_2-r_1)\,t + (r_2-r_1)\,s\right) \geq e^{-(r_2-r_1)t} & \text{if } \tau \leq s \leq t\\ 1 \geq e^{-(r_2-r_1)t} & \text{if } \tau \leq t \leq s\\ \exp\left(-(r_2-r_1)\,t + (r_2-r_1)\,\tau\right) \geq e^{-(r_2-r_1)t} & \text{if } s \leq t \leq \tau\\ \exp\left(-(r_2-r_1)\,s + (r_2-r_1)\,\tau\right) \geq e^{-(r_2-r_1)t} & \text{if } t \leq s \leq \tau\\ 1 \geq e^{-(r_2-r_1)t} & \text{if } t \leq \tau \leq s \end{cases} \geq \gamma(t)$$

b/ $\tau,\, t \leq 0$, in this case we have

$$Q(t,s,\tau) = \begin{cases} \exp\left((r_2+r_1)\,t - (r_2+r_1)\,\tau\right) \geq e^{(r_2+r_1)t} & \text{if } s \leq \tau \leq t\\ \exp\left((r_2+r_1)\,t - 2r_2\tau + (r_2-r_1)\,s\right) \geq e^{(r_2+r_1)t} & \text{if } \tau \leq s \leq t\\ \exp\left(2r_2\,(t-\tau)\right) \geq e^{2r_2 t} & \text{if } \tau \leq t \leq s\\ \exp\left((r_2+r_1)\,t - (r_2+r_1)\,\tau\right) \geq e^{(r_2+r_1)t} & \text{if } s \leq t \leq \tau\\ \exp\left(-(r_2-r_1)\,s + 2r_2 t - (r_2+r_1)\,\tau\right) \geq e^{2r_2 t} & \text{if } t \leq s \leq \tau\\ \exp\left(2r_2\,(t-\tau)\right) \geq e^{2r_2 t} & \text{if } t \leq \tau \leq s \end{cases} \geq \gamma(t)$$

c/ $\tau \leq 0,\, t \geq 0$, in this case we have

$$Q(t,s,\tau) = \begin{cases} \exp\left(-(r_2-r_1)\,t - (r_2+r_1)\,\tau\right) \geq e^{-(r_2-r_1)t} & \text{if } s \leq \tau \leq t\\ \exp\left(-(r_2-r_1)\,t - 2r_2\tau + (r_2-r_1)\,s\right) \geq e^{-(r_2-r_1)t} & \text{if } \tau \leq s \leq t\\ \exp\left(-2r_2\tau\right) \geq e^{-(r_2-r_1)t} & \text{if } \tau \leq t \leq s \end{cases} \geq \gamma(t)$$

d/ $\tau \geq 0,\, t \leq 0$, in this case we have

$$Q(t,s,\tau) = \begin{cases} \exp\left((r_2+r_1)\,t + (r_2-r_1)\,\tau\right) \geq e^{(r_2+r_1)t} & \text{if } s \leq t \leq \tau\\ \exp\left(-(r_2-r_1)\,s + 2r_2 t + (r_2-r_1)\,\tau\right) \geq e^{2r_2 t} & \text{if } t \leq s \leq \tau\\ \exp\left(2r_2 t\right) & \text{if } t \leq \tau \leq s \end{cases} \geq \gamma(t).$$

Consequently, $\frac{p(t)G(t,s)}{p(\tau)G(\tau,s)} \geq \gamma(t)$. $\square$

Lemma 3.3. *Let $r > 0$ and assume that Hypothesis (2) and (3) hold true. Then the operator $T_r : P \backslash B(0,r) \to P$ is compact.*

Proof. For $R > 0$, let $\Omega_{r,R} = P \cap \bar{B}(0,R) \backslash B(0,r)$ be a bounded subset of $P \backslash B(0,r)$.
1. We show that the set $M_{r,R} = T_r(\Omega_{r,R})$ is a subset of.E. Let $\psi_{r,R}$ be a function deined by

$$\psi_{r,R}(t) = q(t).e^{r_2|t|}R^{m+1} + \phi(t)\, g_{r,R}(t) \in L^1(\mathbb{R}).$$

We see from continuity of the Green function G that $M_{r,R} \subset C(\mathbb{R},\mathbb{R})$. Moreover for $u \in \Omega_{r,R}$ and $t \in \mathbb{R}$
we have

$$r\gamma(t) \leq p(t)u(t) \leq R.$$

Then

$$\begin{aligned} T_r(u)(t) &= \int_{\mathbb{R}} G(t,s)F(s,u(s))ds \\ &\leq \frac{1}{r_1 - r_2}\int_{\mathbb{R}} \psi_{r,R}(s)\,ds < \infty \end{aligned}$$

leading to

$$\lim_{|t|\to\infty} e^{-r_2|t|}\left|T_r(u)(t)\right| = 0.$$

2. We show that $M_{r,R}$ is relatively compact.
In first, we show that the set $M_{r,R}$ is bounded. Let $u \in \Omega_{r,R}$ and $g_{r,R}$ the function given in (3).
We have

$$\begin{aligned} e^{-r_2|t|}\left|T_r(u)(t)\right| &\leq \left|T_r(u)(t)\right| \leq \int_{\mathbb{R}} G(t,s)\psi_{r,R}(s)ds \\ &\leq \frac{1}{r_2 - r_1}\int_{\mathbb{R}} \psi_{r,R}(s)ds < \infty \end{aligned}$$

proving the boundeness of $M_{r,R}$.
Let $t_1, t_2 \in [\eta, \zeta] \subset \mathbb{R}$, for all $u \in \Omega$ we have

$$\begin{aligned} |p(t_2)T_r u(t_2) - p(t_1)T_r u(t_1)| \leq{}& |p_1(t_2) - p_1(t_1)| \textstyle\int_{-\infty}^{\zeta} e^{-r_1 s}\psi_{r,R}(s)ds \\ &+ |p_2(t_2) - p_2(t_1)| \textstyle\int_{\eta}^{+\infty} e^{-r_2 s}\psi_{r,R}(s)ds \\ &+ C_{\eta,\zeta}\textstyle\int_{t_1}^{t_2} \psi_{r,R}(s)ds \end{aligned}$$

where for $i = 1, 2,\ p_i(t) = e^{-r_2|t| + r_i t}$ and $C_{\eta,\zeta} = 2\sup_{t,s\in[\eta,\zeta]} .p(t)G(t,s)$.

Because that p_1, p_2 and $t \to \int_0^t \psi_{r,R}(s)ds$ are uniformly continuous on compact intervals, the above estimates prove that $M_{r,R}$ is equicontinuous on compact intervals.
Finally, let $u \in \Omega_{r,R}$. For $t \in \mathbb{R}$

$$e^{-r_2|t|}\left|T_r(u)(t)\right| \leq \left(\frac{1}{r_2 - r_1}\int_{\mathbb{R}} \psi_{r,R}(s)\,ds\right) e^{-r_2|t|},$$

with
$$\lim_{|t|\to\infty} e^{-r_2|t|}\left(\frac{1}{r_2-r_1}\int_{\mathbb{R}}\psi_{r,R}(s)\,ds\right)=0,$$
so, the equiconvergence of $M_{r,R}$ holds. By Lemma (2.3), we deduce that $M_{r,R}$ is relatively compact.
3. We show that T_r is continuous in $\Omega_{r,R}$.
Let $(u_n)_n$ be a sequence in $\Omega_{r,R}$ such that
$$\lim_{n\to\infty} u_n = u \in \Omega.$$
$$e^{-r_2|t|}\left|T_r(u_n)(t)-T_r(u)(t)\right| \le \frac{1}{r_2-r_1}\int_{\mathbb{R}}\left|F(s,u_n(s))-F(s,u(s))\right|ds,$$
then
$$\|T_r(u_n)-T_r(u)\| \le \frac{1}{r_2-r_1}\int_{\mathbb{R}}\left|F(s,u_n(s))-F(s,u(s))\right|ds.$$

By continuity of F in $\mathbb{R}^+\backslash\{0\}$ we have
$$\lim_{n\to\infty}\left|F(s,u_n(s))-F(s,u(s))\right| = 0 \ a.e. \text{ in } \mathbb{R}.$$
Moreover, we have
$$\begin{aligned}\left|F(s,u_n(s))-F(s,u(s))\right| &\le F(s,u_n(s))+F(s,u(s))\\ &\le 2\psi_{r,R}(s)\in L^1(\mathbb{R})\end{aligned}$$
then the Lebesgue dominated convergence theorem garantees that
$$\lim_{n\to\infty}\|T_r(u_n)-T_r(u)\| = 0$$
which shows the continuity of T_r.
4. Finally, we prove that $T_r\left(P\cap\bar{B}(0,R)\backslash B(0,r)\right)\subset P$.
Set $v=T_r u$, $u\in\Omega_{r,R}$, and let $t\in\mathbb{R}$. Assertion 2 of Lemma (3.2) gives
$$v(t)\ge\int_{\mathbb{R}}\frac{\gamma(t)}{p(t)}p(\tau)\,G(\tau,s)F(s,u(s))ds=\widetilde{\gamma}(t)\,p(\tau)\,v(\tau)$$
this is for all $\tau\in\mathbb{R}$, then
$$v(t)\ge\widetilde{\gamma}(t)\,\|v\|$$
proving our claim. □

For $\theta>0$, set

$$f^0=\overline{\lim_{x\to0^+}}\left(\sup_{t\in\mathbb{R}}\frac{f(t,e^{r_2|t|}x)}{x}\right),\qquad f^\infty=\overline{\lim_{x\to+\infty}}\left(\sup_{t\in\mathbb{R}}\frac{f(t,e^{r_2|t|}x)}{x}\right)$$
$$f_0(\theta)=\underline{\lim_{x\to0^+}}\left(\inf_{t\in[-\theta,\theta]}\frac{f(t,e^{r_2|t|}x)}{x}\right),\quad f_\infty(\theta)=\underline{\lim_{x\to+\infty}}\left(\inf_{t\in[-\theta,\theta]}\frac{f(t,e^{r_2|t|}x)}{x}\right).$$

In the main results, we use the following notations

$$\begin{aligned} \lambda(\theta) &= \left(\sup_{t\in\mathbb{R}} \left\{ e^{-r_2|t|} \int_{-\theta}^{+\theta} G(t,s)\phi(s)\gamma(s)\,ds \right\}\right)^{-1} \\ \mu &= \left(\sup_{t\in\mathbb{R}} \left\{ e^{-r_2|t|} \int_{\mathbb{R}} G(t,s)\left[q(s)e^{r_2|s|}+\phi(s)\right] ds \right\}\right)^{-1} \end{aligned} \tag{4}$$

3.1. **Existence results.**

Theorem 3.4. *Assume that Hypothesis (2) and (3) hold true.*
If $m>1$ and there exist $\theta>0$ such that

$$\mu^{-1} f^0 < 1 < \lambda^{-1}(\theta)\, f_\infty(\theta) \tag{5}$$

then bvp (1) admits at least one positive solution.

Proof. Let $\theta, \epsilon > 0$ such that $(f_\infty(\theta) - \epsilon) > \lambda(\theta)$. There exists $R_0 > 0$ such that for all $(t,x) \in [-\theta,\theta] \times [R_0, +\infty[$

$$f(t, e^{r_2|t|}x) \geq (f_\infty(\theta) - \epsilon)\, x > \lambda(\theta)\, x.$$

Let $R = \frac{R_0}{\delta}$ with $\delta = \min\{\gamma(t), t \in [-\theta,\theta]\}$. Let $u \in \partial B(0,R) \cap P$ and let v be a function defined in $\mathbb{R}$ by

$$v(t) = \int_{\mathbb{R}} G(t,s)F(s,u(s))ds.$$

For all $t \in [-\theta, \theta]$

$$e^{-r_2|t|}u(t) \geq R_0,$$

and so for $t \in \mathbb{R}$

$$\begin{aligned} e^{-r_2|t|}v(t) &\geq e^{-r_2|t|}\int_{\mathbb{R}} G(t,s)\phi(s)\, f(s,u(s))ds \\ &\geq e^{-r_2|t|}\int_{-\theta}^{+\theta} G(t,s)\phi(s)\,\lambda(\theta)\, e^{-r_2|s|}u(s)ds \\ &\geq \|u\|\,\lambda(\theta)\, e^{-r_2|t|}\int_{-\theta}^{+\theta} G(t,s)\phi(s)\, e^{-r_2|s|}\widetilde{\gamma}(s)\,ds. \end{aligned}$$

then

$$\|v\| = \sup_{t\in\mathbb{R}} \left\{ e^{-r_2|t|}\,|Tu(t)| \right\} \geq \|u\|\,\lambda(\theta)\,\lambda^{-1}(\theta) = \|u\|.$$

Now, suppose that $\mu^{-1}f^0 < 1$ and let $\epsilon > 0$ be such that $f^0 + \epsilon < \mu$, then there exists $r > 0$ such that for all $(t,x) \in \mathbb{R} \times (0,r]$

$$f(t, e^{r_2|t|}x) \leq \left(f^0 + \epsilon\right)x \text{ and } x^{m+1} \leq \left(f^0 + \epsilon\right)x$$

Let $u \in \partial B(0,r) \cap P$ and let w be a function defined in $\mathbb{R}$ by

$$w(t) = \int_{\mathbb{R}} G(t,s)F(s,u(s))ds.$$

. For $t \in \mathbb{R}$

$$f(t,u(t)) = f(t, e^{r_2|t|}x(t))$$

with

$$0 < x(t) = e^{-r_2|t|}u(t) \leq r.$$

Then

$$\begin{aligned} e^{-r_2|t|}w(t) &= e^{-r_2|t|}\int_{\mathbb{R}} G(t,s)\left[q\left(s\right)u^{m+1}\left(s\right)+\phi\left(s\right)f(s,u(s))\right]ds \\ &\leq e^{-r_2|t|}\int_{\mathbb{R}} G(t,s)\left(f^0+\epsilon\right)\left[q(s)u(s)+\phi\left(s\right)e^{-r_2|s|}u(s)\right]ds \\ &\leq \|u\|\left(f^0+\epsilon\right).e^{-r_2|t|}\int_{\mathbb{R}} G(t,s)\left[q(s)e^{r_2|s|}+\phi\left(s\right)\right]ds \end{aligned}$$

leading to

$$\|w\| = \sup_{t\in\mathbb{R}}\left\{e^{-r_2|t|}Tu(t)\right\} \leq \|u\|\left(f^0+\epsilon\right).\mu^{-1} \leq \|u\|.$$

Thus, for all $u \in \partial B\left(0,R\right)\cap P$

$$\|T_r u\| \geq \|u\|$$

and for all $u \in \partial B\left(0,r\right)\cap P$

$$\|T_r u\| \leq \|u\|.$$

We deduce from assertion 1 of Theorem (2.1), that T_r admits a fixed point $u \in P$ with $r \leq \|u\| \leq R$ which is a positive solution of bvp (1). $\square$

Now we consider the following condition

$$\left\{\begin{array}{c} \text{there exists } \theta > 0 \text{ such that} \\ \mu^{-1}f^{\infty} < 1 < \lambda^{-1}\left(\theta\right)f_0\left(\theta\right) \leq \infty \end{array}\right. \tag{6}$$

Theorem 3.5. *Assume that Hypothesis (2), (3) and (6) hold true.*
If $-1 \leq m < 0$, then bvp (1) admits at least one positive solution

Proof. Suppose that $\lambda^{-1}\left(\theta\right)f_0\left(\theta\right) > 1$ for some $\theta > 0$. We show that there exist $\pi > \lambda\left(\theta\right)$ and $r' > 0$ such that

$$f(t, e^{r_2|t|}x) \geq \pi.x \text{ for all } (t,x) \in [-\theta,\theta]\times(0,r'].$$

We distinguish two cases.
Case 1. If $f_0\left(\theta\right) < \infty$, then there exists $\epsilon > 0$ such that

$$f_0\left(\theta\right) - \epsilon > \lambda\left(\theta\right)$$

and so, there exists $r' > 0$ such that

$$f(t, e^{r_2|t|}x) \geq \pi.x \text{ for all } (t,x) \in [-\theta,\theta]\times(0,r']$$

where

$$\pi = f_0\left(\theta\right) - \epsilon.$$

Case 2. If $f_0\left(\theta\right) = \infty$, then for every $\epsilon_0 > \lambda\left(\theta\right)$, there exists $r' > 0$ such that

$$f(t, e^{r_2|t|}x) \geq \epsilon_0.x \text{ for all } (t,x) \in [-\theta,\theta]\times(0,r'].$$

Now, let $u \in \partial B(0, r') \cap P$ and let v be a function defined in $\mathbb{R}$ by

$$v(t) = \int_{\mathbb{R}} G(t,s)F(s,u(s))ds.$$

For $t \in \mathbb{R}$

$$\begin{aligned}
e^{-r_2|t|}v(t) &\geq e^{-r_2|t|}\int_{\mathbb{R}} G(t,s)\phi(s) f(s,u(s))ds \\
&\geq e^{-r_2|t|}\int_{-\theta}^{\theta} G(t,s)\phi(s)\pi.e^{-r_2|s|}u(s)ds \\
&\geq \|u\|\pi.e^{-r_2|t|}\int_{-\theta}^{\theta} G(t,s)\phi(s) e^{-r_2|s|}\widetilde{\gamma}(s)ds \\
&\geq \|u\|\pi.e^{-r_2|t|}\int_{-\theta}^{\theta} G(t,s)\phi(s)\gamma(s)ds.
\end{aligned}$$

then

$$\|v\| = \sup_{t\in\mathbb{R}}\left\{e^{-r_2|t|}|v(t)|\right\} \geq \|u\|(f_0(\theta) - \epsilon)\lambda^{-1}(\theta) \geq \|u\|.$$

Now, suppose that $\mu^{-1}f^{\infty} < 1$ and let $\epsilon_1 > 0$ be such that

$$f^{\infty} + \epsilon_1 < \mu.$$

Then there exists $R_1 > 0$ such that for all $(t,x) \in \mathbb{R} \times [R_1, +\infty)$,

$$f(t, e^{r_2|t|}x) \leq (f^{\infty} + \epsilon_1)x \text{ and } x^{m+1} \leq (f^{\infty} + \epsilon_1)x$$

and by the condition (3), there exists a positive constant $c_1 > 0$ such that

$$f(t, e^{r_2|t|}x) \leq (f^{\infty} + \epsilon_1)x + c_1 \text{ for all } (t,x) \in \mathbb{R} \times [\gamma(t)R_1, +\infty)$$

where

$$c_1 = \sup\{g_{R_1,R_1}(t),\ t \in \mathbb{R}\}.$$

We show that there exists $R' > R = \max\{R_1, r'\}$ such that for all $u \in \partial B(0, R') \cap P$

$$\left\|\int_{\mathbb{R}} G(t,s)F(s,u(s))ds\right\| \leq \|u\|.$$

In the contrary, assume that for all $n \geq [R]+1$, there exist $u_n \in \partial B(0,n) \cap P$ and $t_n \in \mathbb{R}$ such that

$$\|u_n\| < e^{-r_2|t_n|}\int_{\mathbb{R}} G(t_n,s)F(s,u_n(s))ds.$$

As $e^{-r_2|t|}u_n(t) \geq e^{-r_2|t|}.\widetilde{\gamma}(t).\|u_n\| \geq \gamma(t)R_1$, for $t \in \mathbb{R}$, then

$$f(t,u_n(t)) \leq (f^{\infty} + \epsilon_1)e^{-r_2|s|}u_n(t) + c_1 \text{ for all } t \in \mathbb{R}$$

and so

$$\begin{aligned}
\|u_n\| &< e^{-r_2|t_n|}\int_{\mathbb{R}} G(t_n,s)F(s,u_n(s))ds \\
&\le e^{-r_2|t_n|}\int_{\mathbb{R}} G(t_n,s)\left(f^{\infty}+\epsilon_1\right)\Big[\phi\left(s\right)e^{-r_2|s|}u_n\left(s\right)+q(s)u_n\left(s\right)\Big]ds \\
&\quad +e^{-r_2|t_n|}\int_{\mathbb{R}} G(t_n,s)\phi\left(s\right)c_1 ds \\
&\le \|u_n\|\left(f^{\infty}+\epsilon_1\right)\sup_{t\in\mathbb{R}}\Big\{e^{-r_2|t_n|}\int_{\mathbb{R}} G(t,s)\left[\phi\left(s\right)\right. \\
&\quad \left. +e^{r_2|s|}q(s)\right]ds\Big\}+e^{-r_2|t_n|}\int_{\mathbb{R}} G(t_n,s)\phi\left(s\right)c_1 ds,
\end{aligned}$$

leading to the following contradiction

$$1 \le \lim_{n\to\infty}\mu^{-1}\left(f^{\infty}+\epsilon_1\right)+\frac{e^{-r_2|t_n|}}{\|u_n\|}\int_{\mathbb{R}} G(t_n,s)\phi\left(s\right)c_1 ds=\left(f^{\infty}+\epsilon_1\right)\mu^{-1}<1.$$

Thus, for all $u\in\partial B\left(0,R'\right)\cap P$

$$\|T_{r'}u\|\le\|u\|$$

and for all $u\in\partial B\left(0,r'\right)\cap P$

$$\|T_{r'}u\|\ge\|u\|.$$

We deduce from assertion 2 of Theorem (2.1) that $T_{r'}$ admits a fixed point $u\in P$ with $r'\le\|u\|\le R'$ which is a positive solution of bvp (1). □

3.2. **Uniqueness results.** By Lemma (3.3), for each $\theta,c>0$ the linear mappings L, $L_\theta : E\to E$ defined as

$$L^{\theta}u(t)=\int_{-\theta}^{\theta} G(t,s)\phi\left(s\right)e^{-r_2|s|}u(s)ds$$

and

$$L_c u(t)=\int_{-\infty}^{+\infty} G(t,s)\left[\phi\left(s\right)+c^{-1}e^{r_2|s|}q(s)\right]e^{-r_2|s|}u(s)ds$$

are compact and

$$L_c\left(K\backslash\{0\}\right)\subset P\backslash\{0\}. \tag{7}$$

Lemma 3.6. *For all $c>0$, L_c admits an eigenpair (λ,v) such that $\lambda>0$ and $v\in P\backslash\{0\}$.*

Proof. Let $c>0$ and $u=\widetilde{\gamma}\in P\backslash\{0\}$. For some $\theta>0$, we have

$$L_c u(t)\ge\widetilde{\gamma}(t)\left\|L_c u\right\|\ge\widetilde{\gamma}(t)\left\|u\right\|\sup_{t\epsilon\mathbb{R}}\left\{e^{-r_2|t|}\int_{-\theta}^{\theta} G(t,s)\phi\left(s\right)\gamma(s)ds\right\}.$$

As

$$\|\widetilde{\gamma}\|=1,$$

then$_c$

$$\lambda(\theta).L_c u \geq u.$$

Then we deduce from Theorem (2.2) that

$$\lambda \geq \lambda^{-1}(\theta) > 0,$$

where

$$L_c v = \lambda v,\ \ v \in P\backslash\{0\}.$$

□

Remark 3.7. *It follows from the above lemma that $r(L_c) > 0$, for all $c > 0$.*

Theorem 3.8. *Assume that Hypothesis (2) and (3) hold true.*
If $m = 0$ and there exist $r, \theta > 0$ and $c \geq 0$ with

$$c < \frac{1}{r(L_c)},$$

such that for all $(t, x) \in [-\theta, \theta] \times [\gamma(t) r, +\infty)$

$$f(t, e^{r_2|t|}x) \geq \lambda(\theta).x, \tag{8}$$

and for all $(t, x, y) \in \mathbb{R} \times [\gamma(t) r, +\infty)^2$

$$\left| f\left(t, e^{r_2|t|}x\right) - f\left(t, e^{r_2|t|}y\right)\right| \leq c|x - y|. \tag{9}$$

Then bvp (1) has a unique positive solution in $P\backslash B(0, r)$.

Proof. The case $c = 0$ is obvious, so we suppose that $c > 0$:

Uniqueness. If $u_1,\ u_2 \in P\backslash B(0, r)$ are two solutions of (1) with $u_1 \neq u_2$, then $u_1,\ u_2$ are fixed points of T_r .
For all $t \in \mathbb{R}$, $i \in \{1, 2\}$

$$e^{-r_2|s|}u_i(t) \geq \gamma(t).r$$

and so

$$\begin{aligned} |u_1 - u_2| &= |T_{r,R}u_1 - T_{r,R}u_2| \leq \int_{-\infty}^{+\infty} G(t, s)\,[q(s)\,|u_1 - u_2| \\ &\quad +\phi(s)\,|f(s, u_1) - f(s, u_2)|]\,ds \\ &\leq c.\int_{-\infty}^{+\infty} G(t, s)\left[q(s)c^{-1}e^{r_2|s|} + \phi(s)\right]e^{-r_2|s|}\,|u_1 - u_2|\,ds \leq c.L_c(|u_1 - u_2|) \end{aligned}$$

where

$$L_c(u)(t) = \int_{-\infty}^{+\infty} G(t, s)\left[q(s)c^{-1}e^{r_2|s|} + \phi(s)\right]e^{-r_2|s|}u(s)\,ds.$$

By using the condition (9) we obtain

$$|u_1 - u_2| \leq cL_c(|u_1 - u_2|).$$

Then

$$L_c|u_1 - u_2| \leq cL_c^2|u_1 - u_2|.$$

Set $w = L_c |u_1 - u_2|$. As $|u_1 - u_2| \in K \backslash \{0\}$, then by (7) this leads to $w \in P \backslash \{0\}$. Then we have

$$w \leq cL_c w, \ w \in P \backslash \{0\},$$

and by Theorem (2.2) we deduce that L_c admits a positive eigenvalue λ_1 such that

$$\lambda_1 \geq c^{-1}$$

this leads the contradiction

$$r(L_c) \geq \lambda_1 \geq c^{-1} > r(L_c).$$

The uniqueness is proved.

Existence. Let $v \in P \cap \partial B(0, r)$ and consider the sequence $(u_n)_n$ defined by

$$\begin{cases} u_{n+1} = T_r u_n \\ u_0 = v. \end{cases}$$

First, we show that $(u_n)_n \subset P \backslash B(0, r)$.
Let $u \in P \backslash B(0, r)$. Since $e^{-r_2|t|} u \geq \gamma(t) r$ for all $t \in \mathbb{R}$, the condition (8) leads

$$f(t, u) \geq \lambda(\theta) e^{-r_2|t|} u.$$

Therefore

$$T_r u \geq \lambda(\theta) L^\theta \left(e^{-r_2|s|} u \right)$$

where

$$\begin{aligned} L^\theta \left(e^{-r_2|s|} u \right)(t) &= \int_{-\theta}^{+\theta} G(t, s) \phi(s) e^{-r_2|s|} u(s) \, ds \\ &\geq \|u\| \int_{-\theta}^{+\theta} G(t, s) \phi(s) \gamma(s) \, ds. \end{aligned}$$

Then

$$\|T_r u\| \geq \|u\| = r.$$

Then $T_r (P \backslash B(0, r)) \subset P \backslash B(0, r)$, which means that $(u_n)_n \subset P \backslash B(0, r)$.
Now, by (9) we have for all $n \geq 1$

$$\begin{aligned} |u_{n+1} - u_n| &= |T_r u_n - T_r u_{n-1}| \leq \int_{-\infty}^{+\infty} G(t, s) |F(s, u_n) - F(s, u_{n-1})| \, ds \\ &\leq cL_c |u_n - u_{n-1}|. \end{aligned}$$

Then, for all $n \geq 0$

$$|u_{n+1} - u_n| \leq c^n L_c^n |u_1 - u_0|.$$

Therefore, for $m > n \geq 1$,

$$\begin{aligned} |u_m - u_n| &\leq |u_m - u_{m-1}| + |u_{m-1} - u_{m-2}| + ... + |u_{n+1} - u_n| \\ &\leq c^{m-1} L_c^{m-1} |u_1 - u_0| + c^{m-2} L_c^{m-2} |u_1 - u_0| + ... + c^n L_c^n |u_1 - u_0| \end{aligned}$$

then

$$\begin{aligned} \|u_m - u_n\| &\leq c^{m-1} \left\| L_c^{m-1} w \right\| + c^{m-2} \left\| L_c^{m-2} w \right\| + ... + c^n \|L_c^n w\| \\ &= S_{m-1} - S_{n-1}, \end{aligned}$$

where

$$S_n = \sum_{n=0}^{n=+\infty} c^n \left\| L_c^n w \right\|, \text{ with } w = \left| u_1 - u_0 \right|.$$

Since $c < (r(L))^{-1}$, we have that

$$\lim_{n\to\infty} \sqrt[n]{c^n \left\| L_c^n w \right\|} \leq c \lim_{n\to\infty} \sqrt[n]{\left\| L_c^n \right\|} = c.r(L_c) < 1,$$

then $(S_n)_n$ converges and

$$\lim_{n\to\infty} \left\| u_m - u_n \right\| = \lim_{n\to\infty} S_{m-1} - S_{n-1} = 0.$$

Therefore, the sequence $(u_n)_n$ is also a cauchy sequence and the completeness of E leads to $\lim_{n\to\infty} u_n = u \in P$, with

$$\|u\| \geq r > 0.$$

At the end, passing to the limit in $u_{n+1} = T_r u_n$, and by continuity of T_r in $P\backslash B(0,r)$, we obtain $u = T_r u$, and u is the unique fixed point of T_r, which is the unique positive homoclinic solution of bvp (1) in $P\backslash B(0,r)$.

□

Theorem 3.9. *Assume that Hypothesis (2), (3) and (6) hold true.*
If $-1 < m \leq 0$ and there exist $r > 0$ and a positive function $h : \mathbb{R} \to \mathbb{R}^+$ such that

$$\sup_{t\geq 0} \left(e^{-r_2|t|} \int_{-\infty}^{+\infty} G(t,s) \left[(m+1)\, q(s).e^{r_2|s|} \left(\gamma(s).r\right)^m + h(s).\phi(s) \right] ds \right) < 1 \tag{10}$$

and such that for all $(t,x,y) \in \mathbb{R} \times [\gamma(t)\, r, +\infty)^2$

$$\left| f\left(t, e^{r_2|t|}.x\right) - f\left(t, e^{r_2|t|}.y\right) \right| \leq h(t).\left| x - y \right|.$$

Then bvp (1) has a unique positive solution in $P\backslash B(0,r)$.

Proof. The case $c = 0$ is obvious, so we suppose that $c > 0$:

Existence. By using theorem (3.5), we deduce from Hypothesis (2), (3) and (6) the existence of solution.

Uniqueness. If u_1, $u_2 \in P\backslash B(0,r)$ are two solutions of (1) with $u_1 \neq u_2$, then u_1, u_2 are fixed points of T_r .
For all $t \in \mathbb{R}$, $i \in \{1,2\}$

$$e^{-r_2|s|} u_i(t) \geq \gamma(t).r$$

and so

$$\begin{aligned} |u_1 - u_2| &= \left| T_{r,R} u_1 - T_{r,R} u_2 \right| \leq \int_{-\infty}^{+\infty} G(t,s) \left[q(s) \left| u_1^{m+1} - u_2^{m+1} \right| \right. \\ &\quad \left. + \phi(s) \left| f(s,u_1) - f(s,u_2) \right| \right] ds \\ &\leq \|u_1 - u_2\| \int_{-\infty}^{+\infty} G(t,s) \left[(m+1)\, q(s).e^{r_2|s|} \left(\gamma(s).r\right)^m + h(s).\phi(s) \right] ds \end{aligned}$$

this leads the following contradiction

$$\begin{aligned}\|u_1-u_2\| &\leq \sup_{t\geq 0}\left(e^{-r_2|t|}\int_{-\infty}^{+\infty} G(t,s)\left[(m+1)\,q(s).e^{r_2|s|}\left(\gamma(s).r\right)^m\right.\right.\\ &\quad \left.+h(s).\phi(s)\right]ds\Big)\,\|u_1-u_2\|\\ &< \|u_1-u_2\|,\end{aligned}$$

ending the proof of our claim. □

Example 3.10. *We consider the following bvp*

$$\begin{cases} -u''(t)+u'(t)+u(t)\left(2-q(t).u^m(t)\right)=\phi(t)\,f(t,u(t)), \ t\in\mathbb{R},\\ u(-\infty)=u(+\infty)=0,\end{cases}\tag{11}$$

where

$$q(t)=e^{-3|t|}=\phi(t).$$

We have $r_1=-1<0<r_2=2$, $\gamma(t)=\min\left\{e^{4t},e^{-3t}\right\}$ *and* $\tilde{\gamma}(t)=\min\left\{e^{-t},e^{2t}\right\}$. *Moreover, we have*

$$\max\left\{e^t,e^{-2t}\right\}\phi(t)=\begin{cases} e^t.e^{-3t}=e^{-2t} & t>0\\ e^{-2t}.e^{3t}=e^t & t<0\end{cases}\in L^1(\mathbb{R}),\ e^{2|t|}.e^{-3|t|}=e^{-|t|}\in L^1(\mathbb{R}).$$

Let $f:\mathbb{R}^+\times(0,+\infty)\to\mathbb{R}^+$ *be a function defined by*

$$f(t,x)=\frac{e^{-2\alpha|t|}.\gamma(t).x^\alpha}{\arctan\left(\frac{x}{\tilde{\gamma}(t)}\right)},\alpha\geq 0.$$

For $(t,x)\in\mathbb{R}\times[\gamma(t).r,R]$, $r<R$, *we have*

$$f\left(t,e^{2|t|}x\right)=\left(e^{2|t|}\right)^{-\alpha}.\gamma(t)\frac{\left(e^{2|t|}x\right)^\alpha}{\arctan\left(\frac{e^{2|t|}x}{\tilde{\gamma}(t)}\right)}\leq g_{r,R}(t)=\frac{(R)^\alpha}{\arctan(r)}\gamma(t)$$

and

$$\phi(t).g_{r,R}(t)=\frac{(R)^\alpha}{\arctan(r)}\phi(t)\,\gamma(t)\in L^1(\mathbb{R})$$

since

$$e^{-3|t|}.\gamma(t)=\begin{cases} e^{-6t} & t>0\\ e^{7t} & t<0.\end{cases}\in L^1(\mathbb{R}).$$

So, the hypothesis (2) and (3) are satisfied.

$$f\left(t,e^{2|t|}x\right)=\gamma(t)\frac{x^\alpha}{\arctan\left(\frac{x}{\gamma(t)}\right)}.$$

Moreover, for all $x>0$, $\theta>0$

$$\sup_{t\geq 0}\left\{f\left(t,e^{2|t|}x\right)\right\}=f(0,x)=\frac{x^\alpha}{\arctan(x)}\ \text{and}\ \inf_{|t|\leq\theta}\left\{f\left(t,e^{2|t|}x\right)\right\}=e^{-4\theta}\frac{x^\alpha}{\arctan(x.e^{4\theta})}.$$

We deduce from theorems (3.4) and (3.5), that if one of the following conditions holds

1. $m>1$ *and* $\alpha>2$ *(f is super-linear), or*

2. $-1 \leq m < 0$ *and* $\alpha < 2$ *(*f *is sub-linear),*
then bvp (11) admits at least one positive solution.

Example 3.11. *We consider the following bvp*

$$\begin{cases} -u''(t) + u'(t) + u(t)\left(2 - q(t).u^m(t)\right) = \phi(t) f(t, u(t)),\ t \in \mathbb{R}, \\ u(-\infty) = u(+\infty) = 0, \end{cases} \tag{12}$$

where $-\frac{1}{4} < m < 0$ *and*

$$q(t) = e^{-3|t|} = \phi(t)$$

are the functions given in example (3.10). Let $f : \mathbb{R}^+ \times (0, +\infty) \to \mathbb{R}^+$ *be a function defined by*

$$f(t, x) = e^{-2\alpha|t|}.\gamma(t).x^\alpha, -\frac{3}{4} < \alpha < 0.$$

Then conditions (2), (3) and (6) are verified.
For $(t, x, y) \in \mathbb{R} \times [\gamma(t).r, +\infty)^2$*,* $r > 1$*, we have*

$$\left| f\left(t, e^{2|t|}x\right) - f\left(t, e^{2|t|}y\right) \right| = \gamma(t).|x^\alpha - y^\alpha| \leq h(t)|x - y|$$

where

$$h(t) = |\alpha| \gamma(t).(\gamma(t).r)^{\alpha-1} = |\alpha|.(\gamma(t))^\alpha.(r)^{\alpha-1}.$$

We have

$$h(t).\phi(t) \in L^1(\mathbb{R}) \text{ and } q(t).e^{r_2|t|}(\gamma(t))^m \in L^1(\mathbb{R})$$

because $\alpha > -\frac{3}{4}$ *and* $m > -\frac{1}{4}$*. Set*

$$\Lambda(t) = e^{-2|t|} \int_{-\infty}^{+\infty} G(t, s) \left[(m+1) q(s).e^{r_2|s|}(\gamma(s).r)^m + h(s).\phi(s)\right] ds,\ t > 0.$$

We have

$$\begin{aligned} \Lambda(t) &\leq \frac{1}{3} \int_{-\infty}^{+\infty} \left[(m+1) q(s).e^{r_2|s|}(\gamma(s).r)^m + h(s).\phi(s)\right] ds \\ &\leq \frac{r^m}{3} \int_{-\infty}^{+\infty} \left[(m+1) q(s).e^{r_2|s|}(\gamma(s))^m + (|\alpha|.(\gamma(t))^\alpha).\phi(s)\right] ds. \end{aligned}$$

Then for

$$r > \max\left\{1, \frac{1}{3}\sqrt[|m|]{\int_{-\infty}^{+\infty} \left[(m+1) q(s).e^{r_2|s|}(\gamma(s))^m + (|\alpha|.(\gamma(t))^\alpha).\phi(s)\right] ds}\right\},$$

the condition (10) of theorem (3.9) holds, so, bvp (12) has a unique positive solution in $P \backslash B(0, r)$. □

REFERENCES

[1] Agarwal, R.P. and O'Regan, D., *Infinite interval problems for differential, difference and integral equations*, Kluwer Academic Publisher, Dordrecht, 2001.

[2] Avramescu, C. and Vladimirescu, C., "Existence of solutions to second order ordinary differential equations having finite limits at $\pm\infty$", *Electronic Journal of Differen-tial Equations*, Vol. 2004(2004), No. 18, pp. 1.12.

[3] Bonheure, D. and Torres, P.J., "Bounded and homoclinic-like solutions of a second-order singular differential equation", *Bull. Lond. Math. Soc.* 44, 47–54 (2012).

[4] Corduneanu, C., *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.

[5] Erbe, L.H. and Mathsen, R.M., "Positive solutions for singular nonlinear boundary value problems", *Nonlin. Anal.* 46 979–86, 2001.

[6] Eze, E.O., Ujumadu, R.N., Ezugorie, G.S., and Uchenna, E.O., "Existence of Homoclinic Orbits and Conditions for the Onset of Chaotic Behavior in a Perturbed Double-Well Oscillator", *International Journal of Mathematical Analysis*, Vol. 14, 2020, no. 7, 329 – 343.

[7] Eze, S.C., "Analysis of fractional Duffing oscillator", *Revista Mexicana de Fısica* 66 (2) 187–191 (2020).

[8] Minhós, F. and Carrasco, H., "Existence of homoclinic solutions for nonlinear second order discontinuous problems", *Mediterranean Journal of Mathematics* 13(6)(2016).

[9] Alessio, F., Caldiroli, P., and Montecchiari, P., "On The Existence of Homoclinic Orbits for the Asymptotically Periodic Duffing Equation", *Topological Methods in Nonlinear Analysis Journal of the Juliusz Schauder Center*, Vol. 12, 1998, 275–292.

[10] Guo, D. and Lakshmikantaham, V., *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[11] Jiménez, S., González, J.A., and Vázquez, L., "Fractional Duffing's equation and geometrical resonance", *International Journal of Bifurcation and Chaos*, 23 (2013), 1350089-13.

[12] Murray, J.D., *Mathematical biology, An introduction*, Springer-Verlag, 2001.

[13] Torres, P.J., "Existence and Stability of Periodic Solutions of a Duffing Equation by Using a New Maximum Principle", *Mediterranean Journal of Mathematics* 1 (2004), 479–486.

[14] Torres, P.J., "Guided waves in a multi-layered optical structure", *Nonlinearity* 19 (2006) 2103–2113.

[15] Cheng, Z. and Yuan, Q., "Damped superlinear Duffing equation with strong singularity of repulsive type", *Journal of Fixed Point Theory and Applications* vol. 22, Article number: 37 (2020).

[16] Zeidler, E., *Nonlinear Functional Analysis and its applications , Vol. I: Fixed point theorems*, Springer-Verlag, New-York 1986.

[17] Zima, M., "On positive solutions of boundary value problems on the half-line", *J. Math. Anal. Appl.* 259 (2001) 127–136.