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DEGREE SQUARE SUBTRACTION SPECTRA AND ENERGY

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Abstract. This paper defines and computes the degree square subtraction matrix, its characteristic polynomial and spectra in terms of the first Zagreb Index. Further, bounds for spectral radius and degree subtraction energy of graphs are also obtained.

Key words and Phrases: DSS(G), characteristic polynomial Of DSS(G), DSS-spectra, DSS-eigenvalues, DSSE(G).

1. INTRODUCTION

The study of spectral graph theory is concerned with the relationships between the spectra of certain matrices associated with a graph and the structural properties of that graph [7]. The ordinary energy of a graph G is closely related with the total π -electron energy of molecules [9, 10, 11]. Motivated by this many researchers introduced different matrices associated with the graph and studied their spectra and energies such as, Laplacian spectrum [15], Laplacian eigenvalue [8, 12], Laplacian energy [8, 12], distance matrix [2], distance spectra [2], distance energy [14] etc. Recently many other graph energy have been studied such as, degree subtraction energy [16], degree square sum energy [4], Degree Exponent Subtraction Energy [17], Degree sum adjacency polynomial [18] etc. Inspired by this present authors introduce degree square subtraction matrix and explore their characteristic polynomial, spectrum and energy.

Let a simple, finite, undirected, graph with n vertices and m edges be G. Let $V(G) = \{v_1, v_2, \cdots, v_j, \cdots, v_n\}$ is a vertex set and $d_j = deg_G(v_j)$ be the degree of a vertex v_j of G. Let $\lambda_1, \lambda_2, \cdots, \lambda_j, \cdots, \lambda_n$ be eigenvalues of adjacency matrix. Then the spectra of G [3, 13] is defined as

$$spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}.$$

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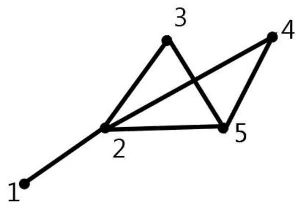
Gutman and Trinajestic introduced the Zagreb indices as $M_1(G) = \sum_{j=1}^{n} d_j^2$.[5, 1]. The degree square subtraction matrix of a graph G of order n which is denoted by $DSS(G)$ and defined as $DSS(G) = [dss_{jk}]$ where

$$dss_{jk} = \begin{cases} {d_j}^2 - {d_k}^2 & v_j \neq v_k \\ 0 & otherwise \end{cases} \tag{1}$$

If I is unit matrix of order n, then DSS-eigenvalues of G are the roots of the characteristic polynomial $\phi'(G, y) = 0$, and they are labeled as $y_1, y_2, \ldots, y_j, \ldots, y_n$. Since $DSS(G)$ is a skew symmetric matrix, its eigenvalues are either purely imaginary or zero. The Degree square subtraction energy of G is denoted by $DSSE(G)$ and defined as

$$DSSE(G) = \sum_{j=1}^{n} |y_j|. \tag{2}$$

2. Example



$$DSS(G) = \begin{bmatrix} 0 & -15 & -3 & -3 & -8 \\ 15 & 0 & 12 & 12 & 7 \\ 3 & -12 & 0 & 0 & -5 \\ 3 & -12 & 0 & 0 & -5 \\ 8 & -7 & 5 & 5 & 0 \end{bmatrix}$$

Graph and its DSS matrix

characteristic polynomial of above matrix is

$$\phi'(G, y) = y^5 + 694y^3$$

$$spec(DSS(G)) = \begin{pmatrix} 0 & 26.3439i & -26.3439i \\ 3 & 1 & 1 \end{pmatrix} \quad where \quad i = \sqrt{-1}.$$

$$DSSE(G) \approx 52.6878.$$

In this paper the present authors computes characteristic polynomial and spectra of degree square subtraction matrix in terms of the first Zagreb Index and explore its bounds for spectral radius and energy.

3. Spectra and Energy of $DSS(G)$

Lemma 3.1 (2). *If Q is a non-singular square matrix, then the following is valid*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |Q||M - NQ^{-1}P|.$$

Theorem 3.2. *Let G be a graph of order n with m edges and $M_1(G)$ is first Zagreb index, then the spectra of $DSS(G)$ is*

$$spec(G) = \begin{pmatrix} 0 & \sqrt{n\sum_{j=1}^{n} d_j^4 - (M_1(G))^2}\,i & -\sqrt{n\sum_{j=1}^{n} d_j^4 - (M_1(G))^2}\,i \\ n-2 & 1 & 1 \end{pmatrix}.$$

Proof. Let $\{v_j : 1 \leq j \leq n\}$ be vertex of G and let d_j be degree of a vertex v_j in G. Then the characteristic-polynomial of $DSS(G)$ is

$$\phi'(G:y) = det(yI \; - \; DSS(G))$$
$$= \begin{vmatrix} y & -d_1^2+d_2^2 & -d_1^2+d_3^2 & \cdots & -d_1^2+d_n^2 \\ -d_2^2+d_1^2 & y & -d_2^2+d_3^2 & \cdots & -d_2^2+d_n^2 \\ -d_3^2+d_1^2 & -d_3^2+d_2^2 & y & \cdots & -d_3^2+d_n^2 \\ \vdots & & & \vdots & \\ -d_n^2+d_1^2 & -d_n^2+d_2^2 & -d_n^2+d_3^2 & \cdots & y \end{vmatrix}. \tag{3}$$

To obtain the Eq.(4) from Eq.(3) subtract first row from all the succeeding rows

$$= \begin{vmatrix} y & -d_1^2+d_2^2 & -d_1^2+d_3^2 & \cdots & -d_1^2+d_n^2 \\ -d_2^2+d_1^2-y & y+d_1^2-d_2^2 & -d_2^2+d_1^2 & \cdots & -d_2^2+d_1^2 \\ -d_3^2+d_1^2-y & -d_3^2+d_1^2 & y+d_1^2-d_3^2 & \cdots & -d_3^2+d_1^2 \\ \vdots & & & \vdots & \\ -d_n^2+d_1^2-y & -d_n^2+d_1^2 & -d_n^2+d_1^2 & \cdots & y+d_1^2-d_n^2 \end{vmatrix}. \tag{4}$$

To obtain the Eq.(5) from Eq.(4) subtract first column from all the succeeding columns

$$= \begin{vmatrix} y & -d_1^2+d_2^2-y & -d_1^2+d_3^2-y & \cdots & -d_1^2+d_n^2-y \\ -d_2^2+d_1^2-y & 2y & y & \cdots & y \\ -d_3^2+d_1^2-y & y & 2y & \cdots & y \\ \vdots & & & \vdots & \\ -d_n^2+d_1^2-y & y & y & \cdots & 2y \end{vmatrix}. \tag{5}$$

To obtain the Eq.(6) from Eq.(5) subtract second column from all the succeeding columns

$$= \begin{vmatrix} y & -d_1^2+d_2^2-y & -d_2^2+d_3^2 & \cdots & -d_2^2+d_n^2-y \\ -d_2^2+d_1^2-y & 2y & -y & \cdots & -y \\ -d_3^2+d_1^2-y & y & y & \cdots & 0 \\ \vdots & & & \vdots & \\ -d_n^2+d_1^2-y & y & 0 & \cdots & y \end{vmatrix}. \tag{6}$$

To obtain the Eq.(7) from Eq.(6) add rows $3, 4, \cdots, n$ to the second row

$$= \begin{vmatrix} y & -d_1^2 + d_2^2 - y & -d_2^2 + d_3^2 & \cdots & -d_2^2 + d_n^2 - y \\ -M_1(G) + nd_1^2 - (n-1)y & ny & 0 & \cdots & 0 \\ -d_3^2 + d_1^2 - y & y & y & \cdots & 0 \\ \vdots & & & \vdots & \\ -d_n^2 + d_1^2 - y & y & 0 & \cdots & y \end{vmatrix}_{n \times n} \tag{7}$$

$$LetM = \begin{bmatrix} y & -d_1^2 + d_2^2 - y \\ -M_1(G) + nd_1^2 - (n-1)y & ny \end{bmatrix}_{2 \times 2}$$

$$N = \begin{bmatrix} -d_2^2 + d_3^2 & \cdots & -d_2^2 + d_n^2 - y \\ 0 & \cdots & 0 \end{bmatrix}_{2 \times (n-2)}$$

$$P = \begin{bmatrix} -d_3^2 + d_1^2 - y & \\ \vdots & \vdots \\ -d_n^2 + d_1^2 - y & y \end{bmatrix}_{(n-2) \times 2}$$

$$Q = \begin{bmatrix} y & 0 & \cdots & 0 \\ 0 & y & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \cdots & y \end{bmatrix}_{(n-2) \times (n-2)}$$

Now By using Lemma (3.1)

$$\phi'(G : y) = y^{n-2} \left| M - N \tfrac{1}{y} I_{(n-2)} P \right|_{2 \times 2}$$

On simplification we get

$$= y^{n-2} \left| \begin{bmatrix} y & -d_1^2 + d_2^2 - y \\ -M_1(G) + nd_1^2 - (n-1)y & ny \end{bmatrix} - \tfrac{1}{y} \begin{bmatrix} K & L \\ 0 & 0 \end{bmatrix} \right|$$

Where

$$K = -\sum_{j=1}^{n} d_j^4 + M_1(G)(d_1^2 + d_2^2) - n(d_1 d_2)^2 - M_1(G)y + y(d_1^2 + d_2^2) + (n-2)d_2^2 y,$$

$$L = (M_1(G) - d_1^2 - d_2^2)y - (n-2)d_2^2 y,$$

$$\phi'(G : y) \quad = y^{n-2} \left(y^2 + n \textstyle\sum_{j=1}^{n} d_j^4 - (M_1(G))^2 \right). \tag{8}$$

From Eq.(8) we get the $DSS(G)$ eigenvalues are 0 ($(n-2)$ times) and

$$\pm \sqrt{n \sum_{j=1}^{n} d_j^4 - (M_1(G))^2} i.$$

□

Corollary 3.3. *If G is a regular graph, then the characteristic polynomial of degree square subtraction of G is*

$$\phi'(G:y) = y^n. \tag{9}$$

Proof. From Eq. (8) we get our required result. □

Corollary 3.4. *Let P_n be path with n vertices. Then*

$$spec(DSS(P_n)) = \begin{pmatrix} 0 & \sqrt{18n-36}i & -\sqrt{18n-36}i \\ n-2 & 1 & 1 \end{pmatrix}$$

where $n = 2, 3, \cdots, n$.

Proof. Using Eq. (8) characteristic polynomial of the matrix $DSS(P_n)$ is

$$\phi'(P_n : y) = y^{n-2}\left(y^2 + 18(n-2)\right).$$

On simplifying we get required spectra. □

Corollary 3.5. *Let $K_{m,n}$ be bipartite graph with $m+n$ vertices. Then*

$$spec(DSS(K_{m,n})) = \begin{pmatrix} 0 & \sqrt{mn}|m^2-n^2|i & -\sqrt{mn}|m^2-n^2|i \\ m+n-2 & 1 & 1 \end{pmatrix}$$

Proof. Using Eq. (8) characteristic polynomial of the matrix $DSS(K_{m,n})$ is

$$\phi'(K_{m,n} : y) = \left(y^2 + mn(m^2-n^2)^2\right).$$

On simplifying we get required spectra. □

Corollary 3.6. *Let $W_n = C_n + K_1$ be Wheel graph with $n+1$ vertices. Then*

$$spec(DSE(W_n) = \begin{pmatrix} 0 & \sqrt{n}\,|n^2-9|\,i & -\sqrt{n}\,|n^2-9|\,i \\ n-1 & 1 & 1 \end{pmatrix}$$

where $n = 3, 4, \cdots, n$.

Proof. Using Eq. (8) characteristic polynomial of the matrix $DSS(W_n)$ is

$$\phi'(W_n : y) = \left(y^2 + n(n^2-9)^2\right). \tag{10}$$

On simplifying we get required spectra. □

Corollary 3.7. *Let $DC_n = C_n + 2K_1$ be Double cone graph with $n+2$ vertices. Then*

$$spec(DSS(DC_n)) = \begin{pmatrix} 0 & \sqrt{2n}\,|n^2-16|\,i & -\sqrt{2n}\,|n^2-16|\,i \\ n & 1 & 1 \end{pmatrix}$$

where $n = 4, 5, \cdots, n$.

Proof. Using Eq. (8) characteristic polynomial of the matrix $DSS(DC_n)$ is

$$\phi'(DC_n : y) = \left(y^2 + 2n(n^2-16)^2\right). \tag{11}$$

On simplifying we get required spectra. □

Corollary 3.8. *Let $L_n = P_n \times P_2$ be Ladder graph with $2n$ vertices. Then*

$$spec(DSE(L_n)) = \begin{pmatrix} 0 & 10\sqrt{(2n-4)}\,i & -10\sqrt{(2n-4)}\,i \\ 2n-2 & 1 & 1 \end{pmatrix}$$

where $n = 2, 3, 4, \cdots, n$

Proof. Using Eq. (8) characteristic polynomial of the matrix $DSS(L_n)$ is

$$\phi'(L_n : y) = \left(y^2 + 200(n-2)^2\right). \tag{12}$$

On simplifying we get required spectra. □

Corollary 3.9. *Let $B_b = K_{1,b} \times P_2$ be book graph with $2b + 2$ vertices, Then*

$$spec(DSE(B_b)) = \begin{pmatrix} 0 & \sqrt{4b}|(b+1)^2 - 4|\,i & -\sqrt{4b}|(b+1)^2 - 4|\,i \\ 2b & 1 & 1 \end{pmatrix}$$

Proof. Using Eq. (8) characteristic polynomial of the matrix $DSS(B_n)$ is

$$\phi'(B_n : y) = \left(y^2 + 4b((b+1)^2 - 4)^2\right). \tag{13}$$

On simplifying we get required spectra. □

Theorem 3.10. *Let G be a graph of order n and d_j be degree of vertex v_j of G. Then the energy of degree square subtraction matrix of G is*

$$DSSE(G) = 2\sqrt{n\sum_{j=1}^{n} d_j^4 - (M_1(G))^2}. \tag{14}$$

Proof. It is very clear from Theorem(3.2) that the Eq.(14) is the energy of $DSS(G)$ □

On direct calculation the DSS energy for some standard graphs are listed in Table 1.

TABLE 1. $DSSE$ of some standard graphs

Graph	Energy	
P_n	$2\sqrt{2n-4}$,	$n \geq 2$
$K_{m,n}$	$2\sqrt{mn}\,\lvert m^2 - n^2\rvert$	
W_n	$2\sqrt{n}\,\lvert n^2 - 9\rvert$,	$n \geq 3$
DC_n	$2\sqrt{2n}\,\lvert n^2 - 16\rvert$,	$n \geq 4$
L_n	$20\sqrt{2(n-2)}$,	$n \geq 2$
B_b	$2\sqrt{4b}\,\lvert(b+1)^2 - 4\rvert$	
$W_{n,n}$	$2\sqrt{2n}\,\lvert 4n^2 - 9\rvert$,	$n \geq 2$

Theorem 3.11. *Let G be a graph of order n with size m. let y_j where $j = 1, 2, \cdots, n$ are DSS-eigenvalues and let m_d and M_d be the minimum and maximum vertex degree of G respectively. Then*

$$0 \leq |y_j| \leq nM_d^2 - M_1(G).$$

Proof. Let $v_1, v_2, \cdots, v_n$ are the vertices of G and d_j be a degree of v_j. Since the sum of the elements of j^{th} row in $DSS(G)$ is $nd_j^2 - M_1(G)$. So It is well known that the eigenvalues of any matrix lie between the minimum row sum and maximum row sum. Hence

$$min\{nd_j^2 - M_1(G)\} \leq |y_j| \leq max\{nd_j^2 - M_1(G)\}.$$

Since we have pure imaginary eigenvalues so $|y_j|$ is either 0 or positive so we get the following equation

$$0 \leq |y_j| \leq max\{nd_j^2 - M_1(G)\}.$$

This implies

$$0 \leq |y_j| \leq nM_d^2 - M_1(G).$$

□

4. Bounds for spectral radius $|y_1|$ of DSS(G)

Theorem 4.1. *Let G be a graph of order n, and let $|y_1|$ and $|y_2|$ be the largest DSS-eigenvalues of G, then the following relations are true.*

$$(i) \qquad y_2 = -y_1 \tag{15}$$

$$(ii) \qquad y_1 = \pm\sqrt{Z}i \tag{16}$$

$$(iii) \qquad |y_1| = \sqrt{Z} \tag{17}$$

where $Z = \sum_{1 \leq j < k \leq n} \left({d_j}^2 - {d_k}^2\right)^2$

Proof. Since we have two nonzero eigenvalue y_1 and y_2 remaining eigen values are zero, and

$$\sum_{j=1}^{n} y_j = trace(DSS(G)) = 0$$

$$y_2 = -y_1 \tag{18}$$

$$\sum_{j=1}^{n} y_j^2 = trace(DSS(G)^2) = -2 \sum_{1 \leq j < k \leq n} \left({d_j}^2 - {d_k}^2\right)^2 = -2Z$$

$$y_1 = \pm\sqrt{Z}i$$

$$|y_1| = \sqrt{Z}$$

□

Lemma 4.2. *Let G be a graph of order n with size m_1 and let λ_j be real and non increasing adjacency eigenvalues of G such as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let Let G be a graph of order n with size m_2 be another graph and let $y_j, j = 1, 2, \cdots, n$ be DSS-eigenvalues of H in non increasing order and $|y_1|$ be the largest eigenvalue of $DSS(H)$. Then*

$$\sum_{j=1}^{n} \lambda_j |y_j| \quad \leq \quad \sqrt{4m_1 Z}, \tag{19}$$

where $Z = \sum_{1 \leq j < k \leq n} \left({d_j}^2 - {d_k}^2\right)^2$.

Proof. By Cauchy-Schwarz inequality[19], we get

$$\left(\sum_{j=1}^{n} (\lambda_j |y_j|)\right)^2 \quad \leq \quad \left(\sum_{j=1}^{n} {\lambda_j}^2\right)\left(\sum_{j=1}^{n} |y_j|^2\right) \tag{20}$$

Since $\sum_{j=1}^{n} |y_j|^2 = 2Z$ and $\sum_{j=1}^{n} {\lambda_j}^2 = 2m_1$.
On substituting and simplifying Eq.(20) we get the required result. □

Theorem 4.3. *Let G be a graph order n with size m. Let $d_j, j = 1, 2, \cdots, n$ be a degree of a vertex. Let the largest absolute DSS-eigenvalue be $|y_1|$. Then*

$$|y_1| \leq \begin{cases} \sqrt{\frac{2a}{a-1} Z} & 2 < a \leq n \\ \frac{\sqrt{2n(n-1)Z}}{n-2} & a = n \end{cases} \tag{21}$$

where $Z = \sum_{1 \leq j < k \leq n} \left({d_j}^2 - {d_k}^2\right)^2$.

Proof. Let $|y_1|, |y_2|, \cdots, |y_{n-a+1}|, |y_{n-a+2}|, \cdots, |y_n|$ be the non increasing absolute DSS-eigenvalues of G. Let $H = K_a \cup \overline{K_{n-a}}$. Then adjacency eigenvalues of H are $a - 1$, 0 $(n - a$ *times*$)$, and -1 $(a - 1$ *times*$)$. The size of H, $m_1 = \frac{a(a-1)}{2}$. By using Lemma(4.2), we get

$$\sum_{j=1}^{n} (\lambda_j |y_j|) \quad \leq \quad \sqrt{4m_1 Z}$$

$$(a-1)|y_1| - \sum_{j=n-a+2}^{n} (|y_j|) \quad \leq \quad \sqrt{2Za(a-1)}$$

$$|y_1| \quad \leq \quad \sqrt{\frac{2Za}{(a-1)}} + \frac{1}{a-1} \sum_{j=n-a+2}^{n} (|y_j|) \tag{22}$$

from Eq.(22) and Eq.(15) we get the required result (21). □

Remark. Since $DSS(G)$ is skew symmetric matrix therefore, $|y_1| = |y_2|$. Hence $|y_1|$, $|y_2|$ have the same upper bound.

Remark. The equality of (21) holds for regular graphs. As $Z = 0$ So $|y_1| = 0$.

Theorem 4.4. *Let G be a graph of order n with size m. Let the largest absolute DSS-eigenvalue be $|y_1|$. Then*

$$|y_1| \leq \begin{cases} \sqrt{\frac{2kaZ}{2(a-1)}} & 1 < k \leq n \\ \frac{\sqrt{2n(n-1)Z}}{n-2} & k = 1 \end{cases} \tag{23}$$

where $Z = \sum_{1\leq j<k\leq n} \left({d_j}^2 - {d_k}^2\right)^2$.

Proof. Let $|y_1|, |y_2|, \cdots, |y_k|, |y_{k+1}|, \cdots, |y_n|$ be the non increasing absolute DSS-eigenvalues of G. Let H be the union of k copies of complete graph K_a, that is $H = \cup_k K_a$ where $ka = n$. Then H-adjacency eigenvalues are $a-1$ (k $times$), -1 ($n-k$ $times$). The order and size of H are $n = ak$ and $\frac{ka(a-1)}{2}$ respectively. Using Lemma(4.2), we get

$$(a-1)\sum_{j=1}^{k} |y_j| - \sum_{j=k+1}^{n} |y_j| \leq \sqrt{\frac{4ka(a-1)Z}{2}} \tag{24}$$

from Eq.(24) and Eq.(15) we get our result (23). □

Theorem 4.5. *Let G be a graph of order n with size m. Let the largest absolute DSS-eigenvalue be $|y_1|$. Then*

$$|y_1| \leq \begin{cases} \sqrt{kZ} & 2 \leq k \leq n-2 \\ \sqrt{2kZ} & k = 1, n-1 \end{cases} \tag{25}$$

where $Z = \sum_{1\leq j<k\leq n} \left({d_j}^2 - {d_k}^2\right)^2$

Proof. Let $|y_1|, |y_2|, \cdots, |y_k|, |y_{k+1}|, \cdots, |y_n|$ be the non increasing absolute DSS-eigenvalues of G. Let H be the union of k copies of complete bipartite graph $K_{a,b}$, $H = \cup_k K_{a,b}$ where $n = ka$. Then adjacency eigenvalues of H are $\sqrt{ab}$ of multiplicity k, $zero$ of multiplicity $n-2k$ and $-\sqrt{ab}$ of multiplicity k. The size of H is kab. By using Lemma(4.2) we get

$$\sqrt{ab}\sum_{j=1}^{k} |y_j| - \sqrt{ab}\sum_{j=n-k+1}^{n} |y_j| \leq \sqrt{4kabZ}$$

$$\sqrt{ab}\sum_{j=1}^{k} |y_j| - \sqrt{ab}\sum_{j=1}^{k} |y_n - k + j| \leq \sqrt{4kabZ}$$

$$\sum_{j=1}^{k}(|y_j| - |y_{n-k+j}|) \leq \sqrt{4kZ} \tag{26}$$

from Eq.(26) and Eq.(15) we get our result Eq.(25). □

5. BOUNDS FOR DEGREE SQUARE SUBTRACTION ENERGY

Theorem 5.1. *Let G be a graph. Then*

$$DSSE(G) = 2\sqrt{Z}, \tag{27}$$

where $Z = \sum_{1\leq j<k\leq n} \left({d_j}^2 - {d_k}^2\right)^2$.

Proof. By Cauchy-Schwarz inequality[19] we get

$$\begin{aligned}\left(\sum_{j=1}^{n} |y_j|\right)^2 &\geq \sum_{j=1}^{n} |y_j|^2\\ DSSE(G) &= \sum_{j=1}^{n} |y_j|\\ DSSE(G) &= 2|y_1|\\ DSSE(G) &= 2\sqrt{Z}.\end{aligned}$$

□

Theorem 5.2. *Let a graph with order n be G. Then*

$$DSSE(G) \quad \leq 2M_1(G)\sqrt{n-1} \tag{28}$$

Proof. Since

$$\sum_{j=1}^{n} d_i^4 \leq \left(\sum_{j=1}^{n} d_i^2\right)^2 \tag{29}$$

From Eq.(14) and Eq.(29) we get Eq.(28). □

Theorem 5.3. *Let G be a connected graph with order n and size m. Let g be a girth of G and Δ and δ are maximum and minimum degree of G. Then*

$$\begin{aligned}
(i) \quad DSSE(G) &\leq 2m(\frac{2m}{n-1} + n - 2)\sqrt{n-1} && (30)\\
(ii) \quad DSSE(G) &\leq 2m(m+1)\sqrt{n-1} && (31)\\
(iii) \quad DSSE(G) &\leq 2n(2m-n+1)\sqrt{n-1} && (32)\\
(iv) \quad DSSE(G) &\leq (4mn - 2n(n-1)\delta + 4m(\delta-1))\sqrt{n-1} && (33)\\
(v) \quad DSSE(G) &\leq 2m(n+\Delta-1)\sqrt{n-1} && (34)\\
(vi) \quad DSSE(G) &\leq (4m(\delta+\Delta) - n\delta\Delta)\sqrt{n-1} && (35)\\
(vii) \quad DSSE(G) &\leq (2mn+6t)\sqrt{n-1} && (36)\\
(viii) \quad DSSE(G) &\leq (2m(L+2)+3t)\sqrt{n-1} && (37)\\
(ix) \quad DSSE(G) &\leq 2m^2\sqrt{n-1} && (38)
\end{aligned}$$

Proof. In [5] given that

$$M_1(G) \quad \leq \quad m(\frac{2m}{n-1}+n-2) \tag{39}$$

. The equality holds if and only if G is star or a complete graph or a complete graph with one isolated vertex. From Eq.(39) and Eq.(28) we get the result Eq.(30),

$$M_1(G) \quad \leq \quad m(m+1) \tag{40}$$

equality holds for $n > 3$ if and only if $G \cong K_3$ or $G \cong K_{1,n-1}$. From Eq.(40) and Eq.(28) we get the result Eq.(31),

$$M_1(G) \quad \leq \quad n(2m-n+1) \tag{41}$$

equality holds if and only if $G \cong K_n$ or $G \cong K_{n-1}$ or $G \cong mK_2$. From Eq.(41) and Eq.(28) we get the result Eq.(32)

$$, M_1(G) \quad \leq \quad 2mn - n(n-1)\delta + 2m(\delta-1) \tag{42}$$

equality holds if and only if G is a star or a regular graph. From Eq.(42) and Eq.(28) we get the result Eq.(33),

$$M_1(G) \quad \leq \quad m(n+\Delta-1) \tag{43}$$

equality holds if and only if G is complete or complete bipartite graph. From Eq.(43) and Eq.(28) we get the result Eq.(34),

$$M_1(G) \quad \leq \quad 2m(\delta+\Delta) - n\delta\Delta \tag{44}$$

equality holds if and only if G is bidegreed graph. From Eq.(44) and Eq.(28) we get the result Eq.(35),

$$M_1(G) \quad \leq \quad mn + 3t \tag{45}$$

equality holds if and only if G is complete or complete bipartite graph. From Eq.(45) and Eq.(28) we get the result Eq.(36),

$$M_1(G) \quad \leq \quad m(L+2) + 3t \tag{46}$$

equality holds if and only if G is complete or complete bipartite graph. From Eq.(46) and Eq.(28) we get the result Eq.(37), If $g \geq 4$

$$M_1(G) \quad \leq \quad m^2 \tag{47}$$

equality holds if and only if G is complete or complete bipartite graph. From Eq.(47) and Eq.(28) we get the result Eq.(38). □

Remark. If $m = n-1$, then the (41) is equal to the (40). If $m \geq n$, then (41) is better than (40). Usually (39) is finer than the bound (41).

Theorem 5.4. *Let G be a connected graph with order n and size m. Let Δ and δ are maximum and minimum degree of G. Then*

$$DSSE(G) \leq 2\sqrt{\frac{2m}{n}\left(F(G)+(n-1)(\Delta^3-\delta^3)\right) - \left(\frac{\Delta^3-\delta^3}{n}\right) M_1(G)} \tag{48}$$

Further

$$DSSE(G) \leq 2\sqrt{P + \frac{4m^2}{n}(\Delta^2 - \delta^2)(n-2) + 2m(\Delta^3 - \delta^3)(n - 1 - \frac{2m}{n}) - \frac{16m^4}{n^2}} \tag{49}$$

where $P = \frac{8m^3(2m+(n-1)(\Delta-\delta))}{n(n+\Delta-\delta)}$

Proof. In [6], by using the Theorem 3, we get

$$\sum_{j=1}^{n} d_i^4 \leq \frac{2m}{n}\left(\sum_{j=1}^{n} d_i^3 + (n-1)(\Delta^3 - \delta^3)\right) - (\frac{\Delta^3 - \delta^3)}{n}\sum_{j=1}^{n} d_i^2 \tag{50}$$

From Eq.(50) we get the result Eq.(48). And by using Theorem 3 in [6], we get

$$\sum_{j=1}^{n} d_i^3 \leq \frac{2m}{n}\left(\sum_{j=1}^{n} d_i^2 + (n-1)(\Delta^2 - \delta^2)\right) - (\frac{\Delta^2 - \delta^2)}{n}\sum_{j=1}^{n} d_i \tag{51}$$

In [6], by using the remark 5, we get

$$\sum_{j=1}^{n} d_i^2 \leq \frac{2m(2m + (n-1)(\Delta - \delta))}{n + \Delta - \delta}$$

and since

$$\sum_{j=1}^{n} d_i = 2m.$$

Therefore from Eq.(51)

$$\sum_{j=1}^{n} d_i^3 \leq \frac{2m}{n}\left(\frac{2m(2m + (n-1)(\Delta - \delta))}{n + \Delta - \delta}\right) + \frac{2m}{n}(n-2)(\Delta^2 - \delta^2) \tag{52}$$

In [6], by using the remark 4, we get

$$\sum_{j=1}^{n} d_i^2 \geq \frac{4m^2}{n} \tag{53}$$

Using Eq.(50), Eq.(52), and Eq.(53) we get the result Eq.(49). □

6. CONCLUSION

In this paper authors have introduced a new matrix called degree square subtraction matrix. Authors have obtained the characteristic polynomial, spectra, and energy of $DSS(G)$. Furthermore authors have obtained bounds for spectral radius and bounds for $DSS-$energy of graph. As a future work, authors plan to explore $DSS-$ spectra and energy for graph operation, study the behaviour of $DSS-$energy of graphs.

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