J. Indones. Math. Soc.
Vol. 23, No. 02 (2023), pp. 217–234.

C_5-FREE NONSPLIT GRAPHS WITH SPLIT MAXIMAL INDUCED SUBGRAPHS

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Abstract. A *split graph* is a graph in which the vertices can be partitioned into an independent set and a clique. A graph is split if and only if it has no induced subgraph isomorphic to C_5, C_4 or $2K_2$, which is a well-known characterization for split graph. A property of a graph G is *recognizable* if it can be recognized from the collection of all maximal proper induced subgraphs of G. We show that any nonsplit graph can have at most five split maximal induced subgraphs. Also we list out all C_5-free nonsplit graphs having split maximal induced subgraphs, which is the main and, in fact, tedious result of this paper.

Key words and Phrases: Split graph, Cycle, Clique, Independent set.

1. INTRODUCTION

All graphs considered in this paper are finite, simple and undirected. Terms not defined here are taken as in [7]. The set of all vertices adjacent to v in G is denoted by $N_G(v)$ and it is called the *neighbourhood* of v in G. A *clique* of a graph G is a vertex subset inducing a complete subgraph of G. A subset I of $V(G)$ is called an *independent set* if no pair of distinct vertices of I are adjacent in G. A *split graph* is a graph in which the vertices can be partitioned into an independent set and a clique. Split graphs were introduced by Foldes and Hammer [3]. Many characterizations and properties of split graphs were obtained over the past 25 years ([6]; Ch. 8 & 9).

An unlabeled maximal proper induced subgraph of a graph G is also called a *card* $G - v$ which is obtained from G by deleting a vertex v and all edges incident with v. The *deck* of a graph G, denoted by $\mathscr{D}(G)$, is the multiset of all its cards. The graph H is said to be a *reconstruction* of G if $\mathscr{D}(H) = \mathscr{D}(G)$. A graph G is said to be *reconstructible* if every reconstruction of G is isomorphic to G. A property of a graph

2020 Mathematics Subject Classification: 05C75, 05C60
Received: 21-02-2021, accepted: 30-07-2023.

G is *recognizable* if it can be recognized from the collection of all cards of G. One of the foremost unsolved problems in graph theory, the *Reconstruction Conjecture* (RC) [4], asserts that every graph on at least three vertices is reconstructible. The manuscripts [1] and [5] are surveys of work done on the RC and related problems. Recently, the manuscript [2] proved a reduction of the RC using diameter that the RC is true if and only if all non distance hereditary 2-connected graphs H such that $diam(H) = 2$ or $diam(H) = diam(\overline{H}) = 3$ are reconstructible. Most of the split graphs H have $diam(H) = 2$ or $diam(H) = diam(\overline{H}) = 3$ and the reconstructibility of split graphs are still open. In this paper, we show that any nonsplit graph can have at most five split cards. We also list out all C_5-free nonsplit graphs having k split cards for $k = 1, 2, 3, 4$.

2. NONSPLIT GRAPHS WITH SPLIT MAXIMAL INDUCED SUBGRAPHS

The following result was proved in [3] .

Theorem 2.1. *A graph is split if and only if it has no induced subgraph isomorphic to C_5, C_4 or $2K_2$.*

All one-vertex deleted subgraphs of each of the nonsplit graphs C_5, C_4 or $2K_2$ are split. We next prove that these are the only nonsplit graphs of this nature.

Theorem 2.2. *A graph G other than C_5, C_4 or $2K_2$ is a split graph if and only if all the cards of G are split.*

Proof. Necessity is obvious. For proving the sufficiency part, assume, to the contrary that, all the cards of G are split and that G is nonsplit. Then, by Theorem 2.1, the graph G must contain an induced subgraph K isomorphic to C_5, C_4 or $2K_2$. Consequently, at least one of the cards must contain a subgraph F isomorphic to K and hence the card is nonsplit, giving a contradiction. □

Kelly Lemma [1] will give the number of subgraphs (or induced subgraphs) of G isomorphic to a given graph F, where $|V(F)| < |V(G)|$, if its count the number of subgraphs (or induced subgraphs) isomorphic to F in its deck. So, we can decide whether any of the graphs C_5, C_4 and $2K_2$ is an induced subgraph of G or not. This leads to the next corollary.

Corollary 2.3. *Split graphs are recognizable.*

Lemma 2.4. *Any nonsplit graph G can have at most five split cards.*

Proof. By Theorem 2.1, the graph G has an induced subgraph H isomorphic to C_5, C_4 or $2K_2$. If $H \cong C_5$, then a card obtained from G by deleting a vertex from a copy of H in G may possibly containing no induced subgraphs isomorphic to H. Consequently, any nonsplit graph can have at most five vertices such that the corresponding five cards may not contain H as an induced subgraph. Since the other two have four vertices of the same type, the graph G has at most five split cards. □

Corollary 2.5. Split graphs G are recognizable by six cards.

Proof. Consider any six cards of G. If any one of these six cards is nonsplit, then G is nonsplit since every card of a split graph is split. Otherwise, G is split by Lemma 2.4. □

Now we proceed to find all nonsplit graphs with precisely k maximal proper induced split subgraphs, where $k = 1, 2, 3, 4$ or 5. Before it, few definitions and notation will be needed for the sake of clarity.

Let U and W be disjoint subsets of $V(G)$. By $U \sim W$ means that there is a vertex in U adjacent to at least one vertex in W; and by $U \sim\sim W$, we mean that every vertex in U is adjacent to every vertex in W; and by $U \nsim W$, we mean that there is a vertex in U not adjacent to at least one vertex in W; and by $U \nsim\nsim W$ means that no vertex in U is adjacent to a vertex in W. For $U = \{u\}$, we just write $u \sim W$ and $u \nsim W$ instead of $U \sim W$ and $U \nsim W$, respectively.

Every nonsplit graph G must contain an induced subgraph, say $\langle L(G)\rangle$, where $\langle L(G)\rangle \cong C_5,\ C_4$, or $2K_2$. Clearly no card, obtained from deleting a vertex from $V(G) - L(G)$, will be split and hence G can have at most five split cards as the order of $L(G)$ is at most five. Let $T(G) \subseteq V(G)$ such that $T(G) \supseteq L(G)$. If a card of G is split, then it must be obtained by deleting a vertex from $T(G)$ and so all the vertices of G that are not in $T(G)$ can be partitioned into a clique and an independent set. Let $\mathscr{G}$ be the collection of all nonsplit graphs G whose vertex set can be partitioned into $C(G)$, $I(G)$ and $T(G)$ such that $C(G)$ is a clique and $I(G)$ is an independent set, where $C(G)$ and $I(G)$ may be empty. If no confusion arise, we simply use $T,\ C,\ I$ instead of $T(G),\ C(G),\ I(G)$ respectively. By $\bar{\mathscr{F}}$, we mean the family of graphs whose complements are in the family of graphs $\mathscr{F}$.

In view of Theorem 2.1, we have the following three properties.

$R(C_5)$: In a nonsplit graph G containing an induced subgraph $K \cong C_5$, if $G - v$ is a split card, where $v \in K$, then neighbours of v that are in K must lie in the independent partition of $G - v$ and non-neighbours of v that are in K must lie in the clique partition of $G - v$.

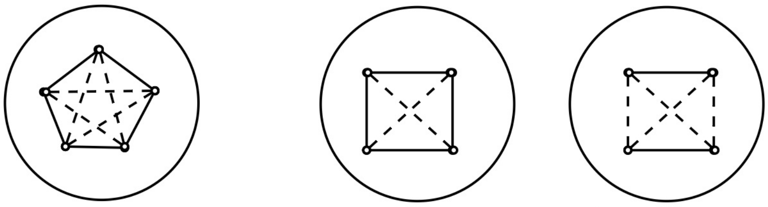
$R(C_4)$: In a nonsplit graph G containing an induced subgraph $K \cong C_4$, if $G - v$ is a split card, where $v \in K$, then non-neighbour of v that are in K must lie in the clique partition of $G - v$ and one of the neighbours of v that are in K must lie in an independent partition of $G - v$.

$R(2K_2)$: In a nonsplit graph G containing an induced subgraph $K \cong 2K_2$, if $G - v$ is a split card, where $v \in K$, then neighbour of v that are in K must lie in the independent partition of $G - v$ and one of the non-neighbours of v that are in K must lie in the clique partition of $G - v$.

A class of nonsplit graphs can be partitioned into the following two disjoint classes:

(I) All C_5- free nonsplit graphs.

(II) All nonsplit graphs containing C_5 as an induced subgraph.



(i) A nonsplit graph containing induced C_5 (ii) A C_5-free nonsplit graph

Figure 1. Nonsplit graphs

Clearly, C_5- free nonsplit graphs has an induced subgraph, say T isomorphic to C_4 or $2K_2$. In the next section, we listed out all C_5- free nonsplit graphs having split maximal induced subgraphs.

2.1. C_5- free nonsplit graphs having split maximal induced subgraphs

2.1.1. C_5- free nonsplit graphs with one split card.

Our aim is to find C_5- free nonsplit graphs H with exactly one split card, say $H - v$. The graph H, being nonsplit, contains an induced subgraph T isomorphic to C_4 or $2K_2$. If an card $H - v'$, obtained by deleting a vertex v' not lying in any C_4 (or $2K_2$) in H, must contain an induced subgraph isomorphic to C_4 (or $2K_2$) and so $H - v'$ is nonsplit. Hence the vertex v must be a common vertex of all induced subgraphs isomorphic to C_4 or $2K_2$ in H. Therefore, we partite the collection of C_5-free nonsplit graphs with exactly one split card and its complements into the following nine types.

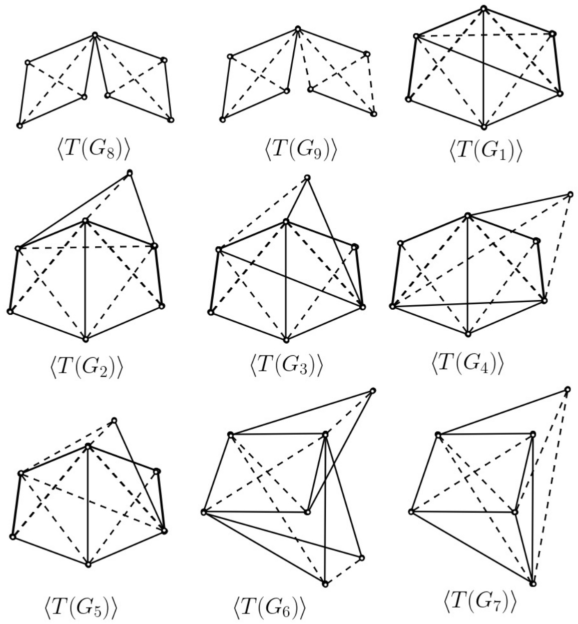


Figure 2. A structures of possible T

Let $G_1 \in \mathscr{G}$ with $T(G_1) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$,

$$a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_1a_5, a_5a_6, a_6a_2, a_4a_6 \in E(G_1)$$

and $a_1a_3, a_2a_4, a_4a_5, a_1a_6, a_2a_5 \notin E(G_1)$. Let $G_2 \in \mathscr{G}$ with $T(G_2) = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$,

$$a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_1a_5, a_5a_6, a_6a_2, a_4a_7, a_5a_7 \in E(G_2)$$

and $a_1a_3, a_2a_4, a_1a_6, a_2a_5, a_4a_5, a_1a_7 \notin E(G_2)$. Let $G_3 \in \mathscr{G}$ with $T(G_3) = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$,

$$a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_1a_5, a_5a_6, a_6a_2, a_1a_7, a_7a_6, a_6a_4 \in E(G_3)$$

and $a_1a_3, a_2a_4, a_1a_6, a_2a_5, a_4a_7 \notin E(G_3)$. Let $G_4 \in \mathscr{G}$ with $T(G_4) = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$,

$$a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_1a_5, a_5a_6, a_6a_2, a_1a_7, a_3a_6 \in E(G_4)$$

and $a_1a_3, a_2a_4, a_1a_6, a_2a_5, a_3a_7, a_6a_7 \notin E(G_4)$. Let $G_5 \in \mathscr{G}$ with $T(G_5) = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$,

$$a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_1a_5, a_5a_6, a_6a_2, a_6a_7 \in E(G_5)$$

and $a_1a_3, a_2a_4, a_1a_6, a_2a_5, a_1a_7, a_4a_7, a_4a_6 \notin E(G_5)$. Let $G_6 \in \mathscr{G}$ with $T(G_6) = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$,

$$a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_1a_5, a_5a_3, a_1a_6, a_6a_3, a_4a_7, a_7a_5 \in E(G_6)$$

and $a_1a_3, a_2a_4, a_4a_5, a_5a_6, a_1a_7 \notin E(G_6)$. Let $G_7 \in \mathscr{G}$ with $T(G_7) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$,

$$a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_2a_5, a_5a_4, a_3a_5, a_1a_6 \in E(G_7)$$

and $a_1a_3, a_2a_4, a_1a_5, a_5a_6, a_3a_6 \notin E(G_7)$.

A nonsplit graph G with such T has exactly one split card only if G belongs to any one of the following seventeen families ($\mathscr{F}10$ to $\mathscr{F}19'$) of graphs.

$\mathscr{F}10$: graphs in $\mathscr{G}$ containing two induced C_4 with exactly one common vertex.

$\mathscr{F}11$: graphs in $\mathscr{G}$ containing two induced $2K_2$ with exactly one common vertex.

$\mathscr{F}12$: graphs in $\mathscr{G}$ containing an induced C_4 and an induced $2K_2$ with exactly one common vertex.

$\mathscr{F}1(K+2)$: graphs containing G_K for $K = 1, 2, \ldots, 7$.

$\mathscr{F}1(K+2)'$: graphs containing $\overline{G_K}$ for $K = 1, 2, \ldots, 7$.

It is clear that $\overline{\mathscr{F}11} = \mathscr{F}10$, $\overline{\mathscr{F}13'} = \mathscr{F}13$, $\overline{\mathscr{F}14'} = \mathscr{F}14$, $\overline{\mathscr{F}15'} = \mathscr{F}15$, $\overline{\mathscr{F}16'} = \mathscr{F}16$, $\overline{\mathscr{F}17'} = \mathscr{F}17$, $\overline{\mathscr{F}18'} = \mathscr{F}18$ and $\overline{\mathscr{F}19'} = \mathscr{F}19$. Thus, we have only nine different families and hence nine subsections.

2.1.1.1. The family $\mathscr{F}10$

Let $G_8 \in \mathscr{G}$ with $T(G_8) = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$, $a_1a_2, a_2a_3, a_3a_4, a_4a_1$, $a_1a_5, a_5a_6, a_6a_7, a_7a_1 \in E(G_8)$ and $a_1a_3, a_2a_4, a_1a_6, a_5a_7 \notin E(G_8)$. We shall now construct G_8 such that $G_8 - a_1$ to be split.

<u>a_1-card</u>:

If a_3 was lying in an independent set of the card $G_8 - a_1$, then, since $a_2a_3, a_3a_4 \in E(G_8 - a_1)$ and $a_2a_4 \notin E(G_8 - a_1)$, both a_2 and a_4 would not lie in a clique of the card $G_8 - a_1$, giving a contradiction. Therefore a_3 lies in the clique partition of the card $G_8 - a_1$ and $a_3 \sim\sim C(G_8)$. Similarly, a_6 lies in the clique partition of the card $G_8 - a_1$ and $a_6 \sim\sim C(G_8)$. Therefore, one of the following nine conditions (X1-X9) must be a necessary condition for $G_8 - a_1$ to be a split card of G_8.

X1 : $\{a_3, a_6\} \sim\sim C(G_8)$ & $\{a_2, a_4, a_5, a_7\} \not\sim\not\sim I(G_8)$

X2 : $\{a_3, a_6, a_2\} \sim\sim C(G_8)$ & $\{a_4, a_5, a_7\} \not\sim\not\sim I(G_8)$

X3 : $\{a_3, a_6, a_4\} \sim\sim C(G_8)$ & $\{a_2, a_5, a_7\} \not\sim\not\sim I(G_8)$

X4 : $\{a_3, a_6, a_5\} \sim\sim C(G_8)$ & $\{a_2, a_4, a_7\} \not\sim\not\sim I(G_8)$

X5 : $\{a_3, a_6, a_7\} \sim\sim C(G_8)$ & $\{a_2, a_4, a_5\} \not\sim\not\sim I(G_8)$

X6 : $\{a_3, a_6, a_2, a_5\} \sim\sim C(G_8)$ & $\{a_4, a_7\} \not\sim\not\sim I(G_8)$

X7 : $\{a_3, a_6, a_2, a_7\} \sim\sim C(G_8)$ & $\{a_4, a_5\} \not\sim\not\sim I(G_8)$

X8 : $\{a_3, a_6, a_4, a_5\} \sim\sim C(G_8)$ & $\{a_2, a_7\} \not\sim\not\sim I(G_8)$

X9 : $\{a_3, a_6, a_4, a_7\} \sim\sim C(G_8)$ & $\{a_2, a_5\} \not\sim\not\sim I(G_8)$

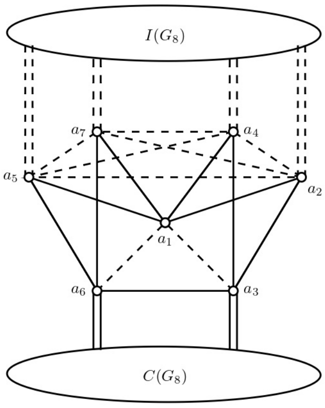


Figure 3. The graph G_8

In Figure 3, a single line denotes the existence of an edge, a double line denotes the existence of all possible edges, dashed single line denotes nonexistence of an edge, and dashed double line denotes the nonexistence of any edge.

A nonsplit graph G_8 has only one split card $G_8 - a_4$ if and only if it satisfies one of the following adjacency conditions (1C.1) to (1C.3).

1C.1: $\{a_3, a_6\} \sim\sim C(G_8), \{a_2, a_4, a_5, a_7\} \nsim\nsim I(G_8), a_3 \sim a_6$ and $\{a_2, a_4\} \nsim\nsim \{a_5, a_7\}$ (Figure 3).
(Here we use the label 1C to mean a **condition** under **one** split card case.)

1C.2: $\{a_2, a_3, a_6\} \sim\sim C(G_8)$, $\{a_4, a_5, a_7\} \nsim\nsim I(G_8)$, $\{a_2, a_3\} \sim\sim a_6$, and $\{a_4\} \nsim\nsim \{a_5, a_7\}$.

1C.3: $\{a_2, a_3, a_5, a_6\} \sim\sim C(G_8)$, $\{a_4, a_7\} \nsim\nsim I(G_8)$, $\{a_2, a_3\} \sim\sim \{a_5, a_6\}$ and $a_4 \nsim a_7$.

2.1.1.2. The family $\mathscr{F}12$

Let $G_9 \in \mathscr{G}$ with $T(G_9) = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$, $a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_1a_5, a_6a_7 \in E(G_9)$ and $a_1a_3, a_2a_4, a_1a_6, a_1a_7, a_5a_6, a_5a_7 \notin E(G_9)$. Now by proceeding as in Section 2.1.1.1, we get that a nonsplit graph G_9 has only one split card $G_9 - a_1$ if and only if it satisfies one of the following adjacency conditions (1C.4) to (1C.6).

1C.4: $\{a_3, a_6\} \sim\sim C(G_9)$, $\{a_2, a_4, a_5, a_7\} \nsim\nsim I(G_9)$, $a_3 \sim a_6$ and $\{a_2, a_4\} \nsim\nsim \{a_5, a_7\}$.

1C.5: $\{a_2, a_3, a_6\} \sim\sim C(G_9)$, $\{a_4, a_5, a_7\} \nsim\nsim I(G_9)$, $\{a_2, a_3\} \sim\sim a_6$ and $a_4 \nsim\nsim \{a_5, a_7\}$.

1C.6: $\{a_3, a_6, a_7\} \sim\sim C(G_9)$, $\{a_2, a_4, a_5\} \nsim\nsim I(G_9)$, $a_3 \sim\sim \{a_6, a_7\}$ and $a_5 \nsim\nsim \{a_2, a_4\}$.

2.1.1.3. The family $\mathscr{F}13$

By proceeding as in Section 2.1.1.1, we get that a nonsplit graph G_1 has only one split card $G_1 - a_1$ if and only if it satisfies the following condition 1C.7.

1C.7: $\{a_3, a_6\} \sim\sim C(G_1)$, $\{a_2, a_4, a_5\} \nsim\nsim I(G_1)$ and $a_3 \sim a_6$.

2.1.1.4. The family $\mathscr{F}14$

We get, as in Section 2.1.1.1, that a nonsplit graph G_2 has only one split card $G_2 - a_1$ if and only if it satisfies the following condition 1C.8.

1C.8: $\{a_3, a_6, a_7\} \sim\sim C(G_2)$, $\{a_2, a_4, a_5\} \nsim\nsim I(G_2)$ and $a_3 \sim \{a_6, a_7\}$ and $a_6 \sim a_7$

2.1.1.5. The family $\mathscr{F}15$

We get, as in Section 2.1.1.1, that every nonsplit graph G_3 with one split card lies in one of the family of graphs $\mathscr{F}10 - \mathscr{F}14$.

2.1.1.6. The family $\mathscr{F}16$

By proceeding as in Section 2.1.1.1, we get that a nonsplit graph G_4 has only one split card $G_4 - a_1$ if and only if it satisfies one of the following adjacency conditions (1C.9) to (1C.11).

1C.9: $\{a_3, a_6, a_2\} \sim\sim C(G_4)$, $\{a_4, a_5, a_7\} \nsim\nsim I(G_4)$, $a_4 \nsim a_5$ and $\{a_4, a_5\} \nsim\nsim \{a_7\}$.

1C.10: $\{a_3, a_4, a_6\} \sim\sim C(G_4)$, $\{a_2, a_5, a_7\} \nsim\nsim I(G_4)$, $a_4 \sim a_6$ and $a_7 \nsim\nsim \{a_2, a_5\}$.

1C.11: $\{a_3, a_4, a_5, a_6\} \sim\sim C(G_4)$, $\{a_2, a_7\} \nsim\nsim I(G_4)$, $\{a_3, a_4\} \sim\sim \{a_5, a_6\}$ and $a_2 \nsim a_7$.

2.1.1.7. The family $\mathscr{F}17$

We get, as in Section 2.1.1.1, that a nonsplit graph G_5 has only one split card $G_5 - a_1$ if and only if it satisfies one of the following adjacency conditions (1C.12) to (1C.14).

1C.12: $\{a_3, a_6, a_2\} \sim\sim C(G_5)$, $\{a_4, a_5, a_7\} \nsim\nsim I(G_5)$, $a_3 \sim a_6$ and $\{a_4, a_7\} \nsim\nsim \{a_5\}$.

1C.13: $\{a_3, a_6, a_7\} \sim\sim C(G_5)$, $\{a_2, a_4, a_5\} \nsim\nsim I(G_5)$, $a_3 \sim\sim \{a_6, a_7\}$ and $a_4 \nsim a_5$.

1C.14: $\{a_2, a_3, a_6, a_7\} \sim\sim C(G_5)$, $\{a_4, a_5\} \nsim\nsim I(G_5)$, $\{a_2, a_3\} \sim\sim \{a_6, a_7\}$ and $a_4 \nsim a_5$.

2.1.1.8. The family $\mathscr{F}18$

By proceeding as in Section 2.1.1.1, we get that every nonsplit graph G_6 with one split card lies in the family of graphs $\mathscr{F}10$.

2.1.1.9. The family $\mathscr{F}19$

We get, as in Section 2.1.1.1, that a nonsplit graph G_7 has only one split card $G_7 - a_1$ if and only if it satisfies the condition 1C.15.

1C.15: $\{a_2, a_3, a_5\} \sim\sim C(G_7)$, $\{a_4, a_6\} \nsim\nsim I(G_7)$ and $a_4 \nsim a_6$

In Case 2.1.1 , we have 30 classes of graphs of which 15 classes of graphs obtained by applying conditions 1C.1 to 1C.15 and the rest of the graphs are their complements. Clearly, each C_5-free nonsplit graph in these 30 classes only has exactly one split card. Thus, we proved the next theorem.

Theorem 2.6. A C_5-free nonsplit graph G has exactly one split card if and only if either G or $\bar{G}$ lies in the class of graphs satisfying the conditions 1C.1 to 1C.15.

2.1.2. C_5-free nonsplit graphs with two split cards:

A C_5-free nonsplit graph G with such T has exactly two split cards only if G belongs to any one of the following six families of graphs.

$\mathscr{F}20$: Graphs in $\mathscr{G}$ containing two induced C_4 with exactly two nonadjacent common vertices.

$\mathscr{F}21$: Graphs in $\mathscr{G}$ containing two induced C_4 with exactly two adjacent common vertices.

$\mathscr{F}22$: Graphs $G \in \mathscr{G}$ containing two induced $2K_2$ with exactly two nonadjacent common vertices.

$\mathscr{F}23$: Graphs $G \in \mathscr{G}$ containing two induced $2K_2$ with exactly two adjacent common vertices.

$\mathscr{F}24$: Graphs $G \in \mathscr{G}$ containing an induced C_4 and an induced $2K_2$ with exactly two nonadjacent common vertices.

$\mathscr{F}25$: Graphs $G \in \mathscr{G}$ containing an induced C_4 and an induced $2K_2$ with exactly two adjacent common vertices.

It is clear that $\overline{\mathscr{F}21} = \mathscr{F}22$, $\overline{\mathscr{F}20} = \mathscr{F}23$, $\overline{\mathscr{F}24} = \mathscr{F}25$ and so we have three subsections.

2.1.2.1. The family $\mathscr{F}22$

Let $G_{10} \in \mathscr{G}$ with $T(G_{10}) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $a_1a_2, a_3a_4, a_2a_5, a_4a_6 \in E(G_{10})$ and $a_1a_3, a_1a_4, a_2a_3, a_2a_4, a_2a_6, a_4a_5, a_5a_6 \notin E(G_{10})$. We shall now construct G_{10} such that $G_{10} - a_2$ and $G_{10} - a_4$ are to be split.

<u>a_2-card:</u>

By $R(2K_2)$, a_1 and a_5 lie in an independent set of the card $G_{10} - a_2$ and $\{a_1, a_5\} \not\sim\not\sim I(G_{10})$. Hence one of the following five conditions (X1-X5) must be a necessary condition for $G_{10} - a_2$ to be a split card of G_{10}.

X1 : $\{a_3, a_4, a_6\} \sim\sim C(G_{10})$ & $\{a_1, a_5\} \not\sim\not\sim I(G_{10})$
X2 : $\{a_3, a_4\} \sim\sim C(G_{10})$ & $\{a_1, a_5, a_6\} \not\sim\not\sim I(G_{10})$
X3 : $\{a_4, a_6\} \sim\sim C(G_{10})$ & $\{a_1, a_5, a_3\} \not\sim\not\sim I(G_{10})$
X4 : $\{a_3, a_6\} \sim\sim C(G_{10})$ & $\{a_1, a_5, a_4\} \not\sim\not\sim I(G_{10})$
X5 : $\{a_4\} \sim\sim C(G_{10})$ & $\{a_1, a_5, a_3, a_6\} \not\sim\not\sim I(G_{10})$

<u>a_4-card:</u>

By $R(2K_2)$, a_3 and a_6 lie in an independent set of the card $G_{10} - a_4$ and $\{a_3, a_6\} \not\sim\not\sim I(G_{10})$. Hence one of the following five conditions (Y1-Y5) must be a necessary condition for $G_{10} - a_4$ to be a split card of G_{10}.

Y1 : $\{a_1, a_2, a_5\} \sim\sim C(G_{10})$ & $\{a_3, a_6\} \not\sim\not\sim I(G_{10})$
Y2 : $\{a_1, a_2\} \sim\sim C(G_{10})$ & $\{a_3, a_6, a_5\} \not\sim\not\sim I(G_{10})$
Y3 : $\{a_2, a_5\} \sim\sim C(G_{10})$ & $\{a_3, a_6, a_1\} \not\sim\not\sim I(G_{10})$
Y4 : $\{a_1, a_5\} \sim\sim C(G_{10})$ & $\{a_3, a_6, a_2\} \not\sim\not\sim I(G_{10})$
Y5 : $\{a_2\} \sim\sim C(G_{10})$ & $\{a_3, a_6, a_1, a_5\} \not\sim\not\sim I(G_{10})$

In the graph G_{10} with split cards $G_{10} - a_2$ and $G_{10} - a_4$, a vertex not in $T(G_{10})$ may lie in a clique of $G_{10} - a_2$ and may lie in an independent set of $G_{10} - a_2$ and vice versa. So we have the following types for the structure of G_{10}.

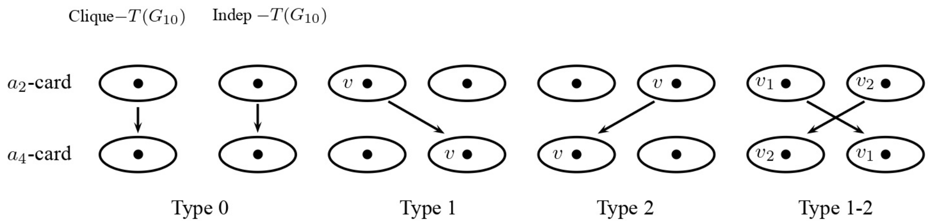


Figure 4. Pattern of clique and independent set in a_2-card and a_4-card

In Type 0, left upper oval denotes the set of all vertices in the clique in the a_2-card but not in $T(G_{10})$; the right upper oval denotes the set of all vertices in the independent set in the a_2- card but not in $T(G_{10})$; the lower ovals denote respective sets in the a_4- card. Analogously we define the ovals for the other types. The arrow denotes the change of role of the some vertex from one card to the other card.
For Type 1 and Type 2, let $C(G_i) \cup I(G_i) = C_1(G_i) \cup I_1(G_i) \cup \{v\}$ such that $C_1(G_i)$ is a clique and $I_1(G_i)$ is an independent set, $v \sim\sim C_1(G_i)$ and $v \nsim\nsim I_1(G_i)$.
For Type 1-2, let $C(G_j) \cup I(G_j) = C_2(G_j) \cup I_2(G_j) \cup \{v_1, v_2\}$ such that $C_2(G_j)$ is a clique and $I_2(G_j)$ is an independent set, $\{v_1, v_2\} \sim\sim C_2(G_j)$ and $\{v_1, v_2\} \nsim\nsim I_2(G_j)$.

Table 1 : All mutually nonequivalent conditions (X_i, Y_i) for each type.

Types	Nonequivalent conditions
Type 0	[X2,Y2], [X2,Y3], [X2,Y5], [X5,Y5]
Type 1	[X5,Y5]
Type 2	[X2,Y5], [X5,Y5]
Type 1-2	[X5,Y5]

Type 0:

Using Table 1, a nonsplit graph G_{10} has only two split cards $G_{10} - a_2$ and $G_{10} - a_4$ if it satisfies one of the following adjacency conditions (2C.1) – (2C.2).

2C.1: $\{a_2, a_3, a_4, a_5\} \sim\sim C(G_{10}), \{a_1, a_3, a_5, a_6\} \nsim\nsim I(G_{10}), a_1 \nsim\nsim \{a_5, a_6\}$ and $a_3 \nsim\nsim a_6$

2C.2: $\{a_2, a_4\} \sim\sim C(G_{10})$, $\{a_1, a_3, a_5, a_6\} \nsim\nsim I(G_{10})and\{a_1, a_3\} \nsim\nsim \{a_5, a_6\}$.

Type 1 and Type 2:

Using Table 1, a nonsplit graph G_{10} has only two split cards $G_{10} - a_2$ and $G_{10} - a_4$ if it satisfies one of the following adjacency conditions (2C.3) - (2C.4).

2C.3: $\{a_2, a_3, a_4\} \sim\sim C_1(G_{10})$, $\{a_1, a_3, a_5, a_6\} \nsim\nsim I_1(G_{10})$, $\{a_1, a_3\} \nsim\nsim \{a_5, a_6\}$ and $v \sim a_2$ and $v \nsim\nsim \{a_1, a_5, a_6\}$ (Figure 5(i)).

2C.4: $\{a_2, a_4\} \sim\sim C_1(G_{10})$, $\{a_1, a_3, a_5, a_6\} \nsim\nsim I_1(G_{10})$, $\{a_1, a_3\} \nsim\nsim \{a_5, a_6\}$ and $v \sim a_4$ and $v \nsim\nsim \{a_1, a_3, a_5, a_6\}$.

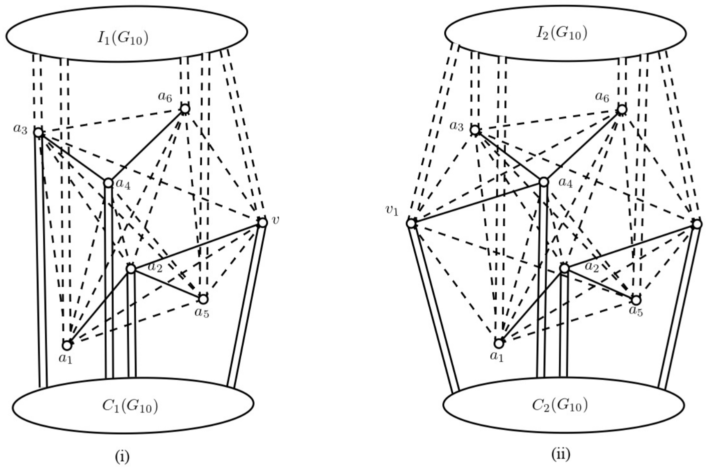


Figure 5. The graph G_{10}

Type 1-2:

Using Table 1, a nonsplit graph G_{10} has only two split cards $G_{10} - a_2$ and $G_{10} - a_4$ if it satisfies the condition 2C.5.

2C.5: $\{a_2, a_4\} \sim\sim C_2(G_{10})$, $\{a_1, a_3, a_5, a_6\} \nsim\nsim I_2(G_{10})$, $\{a_1, a_3\} \nsim\nsim \{a_5, a_6\}$, $v_1 \sim a_4$, $v_2 \sim a_2$ and $\{v_1, v_2\} \nsim\nsim \{a_1, a_3, a_5, a_6\}$ (Figure 5(ii)).

2.1.2.2. The family $\mathscr{F}23$

Let $G_{11} \in \mathscr{G}$ with $T(G_{11}) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $a_1a_2, a_3a_4, a_5a_6 \in E(G_{11})$ and $a_1a_3, a_1a_4, a_1a_5, a_1a_6, a_2a_3, a_2a_4, a_2a_5, a_2a_6, \notin E(G_{11})$. Now by proceeding as in Section 2.1.2.1, we get the following types.

Type 0:

A nonsplit graph G_{11} has only two split cards $G_{11} - a_1$ and $G_{11} - a_2$ if it satisfies one of the following adjacency conditions (2C.6) - (2C.8).

2C.6: $\{a_3, a_4, a_5, a_6\} \sim\sim C(G_{11})$, $\{a_1, a_2\} \nsim\nsim I(G_{11})$ and $\{a_3, a_4\} \sim\sim \{a_5, a_6\}$.

2C.7: $\{a_3, a_4, a_5\} \sim\sim C(G_{11})$, $\{a_1, a_2, a_6\} \nsim\nsim I(G_{11})$ and $\{a_3, a_4\} \sim\sim a_5$.

2C.8: $\{a_3, a_5\} \sim\sim C(G_{11})$, $\{a_1, a_2, a_4, a_6\} \nsim\nsim I(G_{11})$, $a_3 \sim a_5$ and $a_4 \nsim a_6$.

Type 1 and Type 2:

A nonsplit graph G_{11} has only two split cards $G_{11} - a_1$ and $G_{11} - a_2$ if it satisfies one of the following adjacency conditions (2C.9) - (2C.13).

2C.9: $\{a_3, a_4, a_5, a_6\} \sim\sim C_1(G_{11})$, $\{a_1, a_2\} \nsim\nsim I_1(G_{11})$, $\{a_3, a_4\} \sim\sim \{a_5, a_6\}$, $v \sim\sim \{a_3, a_4, a_5, a_6\}$ and $v \nsim a_1$.

2C.10: $\{a_3, a_4, a_5, a_6\} \sim\sim C_1(G_{11})$, $\{a_1, a_2, a_6\} \nsim\nsim I_1(G_{11})$, $\{a_3, a_4\} \sim\sim \{a_5, a_6\}$, $v \sim\sim \{a_3, a_4, a_5\}$ and $v \nsim a_2$.

2C.11: $\{a_3, a_4, a_5\} \sim\sim C_1(G_{11})$, $\{a_1, a_2, a_6\} \nsim\nsim I_1(G_{11})$, $\{a_3, a_4\} \sim\sim a_5$, $v \sim\sim \{a_3, a_4, a_5\}$ and $v \nsim\nsim \{a_1, a_6\}$.

2C.12: $\{a_3, a_4, a_5\} \sim\sim C_1(G_{11})$, $\{a_1, a_2, a_4, a_6\} \nsim\nsim I_1(G_{11})$, $\{a_3, a_4\} \sim\sim a_5$, $a_4 \nsim a_6$, $v \sim\sim \{a_3, a_5\}$ and $v \nsim\nsim \{a_2, a_6\}$.

2C.13: $\{a_3, a_5\} \sim\sim C_1(G_{11})$, $\{a_1, a_2, a_4, a_6\} \nsim\nsim I_1(G_{11})$, $a_3 \sim a_5$, $a_4 \nsim a_6$, $v \sim\sim \{a_3, a_5\}$ and $v \nsim\nsim \{a_1, a_4, a_6\}$.

Type 1-2:

A nonsplit graph G_{11} has only two split cards $G_{11} - a_1$ and $G_{11} - a_2$ if it satisfies one of the following adjacency conditions (2C.14) - (2C.16).

2C.14: $\{a_3, a_4, a_5, a_6\} \sim\sim C_2(G_{11})$, $\{a_1, a_2\} \nsim\nsim I_2(G_{11})$, $\{a_3, a_4\} \sim\sim \{a_5, a_6\}$, $\{v_1, v_2\} \sim\sim \{a_3, a_4, a_5, a_6\}$, $v_1 \nsim a_1$ and $v_2 \nsim a_2$.

2C.15: $\{a_3, a_4, a_5\} \sim\sim C_2(G_{11})$, $\{a_1, a_2, a_6\} \nsim\nsim I_2(G_{11})$, $\{a_3, a_4\} \sim\sim a_5$, $\{v_1, v_2\} \sim\sim \{a_3, a_4, a_5\}$, $v_1 \nsim\nsim \{a_1, a_6\}$ and $v_2 \nsim\nsim \{a_2, a_6\}$.

2C.16: $\{a_3, a_5\} \sim\sim C_2(G_{11})$, $\{a_1, a_2, a_4, a_6\} \nsim\nsim I_2(G_{11})$, $a_3 \sim a_5$, $a_4 \nsim a_6$, $\{v_1, v_2\} \sim\sim \{a_3, a_5\}$, $v_1 \nsim\nsim \{a_1, a_4, a_6\}$ and $v_2 \nsim\nsim \{a_2, a_4, a_6\}$.

2.1.2.3. The family $\mathscr{F}25$

Let $G_{12} \in \mathscr{G}$ with $T(G_{12}) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_5a_6 \in E(G_{12})$ and $a_1a_3, a_2a_4, a_3a_5, a_3a_6, a_4a_5, a_4a_6 \notin E(G_{12})$. Now by proceeding as in Section 2.1.2.1, we get the following types.

Type 0:

A nonsplit graph G_{12} has only two split cards $G_{12} - a_3$ and $G_{12} - a_4$ if it satisfies one of the following adjacency conditions (2C.17) - (2C.18).

2C.17: $\{a_1, a_2, a_5, a_6\} \sim\sim C(G_{12})$, $\{a_3, a_4\} \nsim\nsim I(G_{12})$ and $\{a_1, a_2\} \sim\sim \{a_5, a_6\}$.

2C.18: $\{a_1, a_2, a_5\} \sim\sim C(G_{12})$, $\{a_3, a_4, a_6\} \nsim\nsim I(G_{12})$ and $\{a_1, a_2\} \sim a_5$.

Type 1 and Type 2:

A nonsplit graph G_{12} has only two split cards $G_{12} - a_3$ and $G_{12} - a_4$ if it satisfies one of the following adjacency conditions (2C.19) - (2C.23).

2C.19: $\{a_1, a_2, a_5, a_6\} \sim\sim C_1(G_{12})$, $\{a_3, a_4\} \nsim\nsim I_1(G_{12})$, $\{a_1, a_2\} \sim\sim \{a_5, a_6\}$, $v \sim\sim \{a_1, a_2, a_5, a_6\}$ and $v \nsim\nsim a_3$.

2C.20: $\{a_1, a_2, a_5, a_6\} \sim\sim C_1(G_{12})$, $\{a_3, a_4, a_6\} \nsim\nsim I_1(G_{12})$, $\{a_1, a_2\} \sim\sim \{a_5, a_6\}$, $v \sim\sim \{a_1, a_2, a_5\}$ and $v \nsim a_4$.

2C.21: $\{a_1, a_2, a_5, a_6\} \sim\sim C_1(G_{12})$, $\{a_1, a_3, a_4\} \nsim\nsim I_1(G_{12})$, $\{a_1, a_2\} \sim\sim \{a_5, a_6\}$, $v \sim\sim \{a_2, a_5, a_6\}$ and $v \nsim a_4$.

2C.22: $\{a_1, a_2, a_5\} \sim\sim C_1(G_{12})$, $\{a_3, a_4, a_6\} \nsim\nsim I_1(G_{12})$, $\{a_1, a_2\} \sim\sim a_5$, $v \sim\sim \{a_1, a_2, a_5\}$ and $v \nsim\nsim \{a_3, a_6\}$.

2C.23: $\{a_1, a_2, a_5\} \sim\sim C_1(G_{12})$, $\{a_1, a_3, a_4, a_6\} \nsim\nsim I_1(G_{12})$, $\{a_1, a_2\} \sim\sim a_5$, $a_1 \nsim a_6$, $v \sim\sim \{a_2, a_5\}$ and $v \nsim\nsim \{a_4, a_6\}$.

Type 1-2:

A nonsplit graph G_{12} has only two split cards $G_{12} - a_3$ and $G_{12} - a_4$ if it satisfies one of the following adjacency conditions (2C.24) - (2C.25).

2C.24: $\{a_1, a_2, a_5, a_6\} \sim\sim C_2(G_{12})$, $\{a_3, a_4\} \nsim\nsim I_2(G_{12})$, $\{a_1, a_2\} \sim\sim \{a_5, a_6\}$, $\{v_1, v_2\} \sim\sim \{a_1, a_2, a_5, a_6\}$, $v_1 \nsim a_3$ and $v_2 \nsim a_4$.

2C.25: $\{a_1, a_2, a_5\} \sim\sim C_2(G_{12})$, $\{a_3, a_4, a_6\} \nsim\nsim I_2(G_{12})$, $\{a_1, a_2\} \sim\sim a_5$, $\{v_1, v_2\} \sim\sim \{a_1, a_2, a_5\}$, $v_1 \nsim\nsim \{a_3, a_6\}$ and $v_2 \nsim\nsim \{a_4, a_6\}$.

In view of the above discussion in Case 2.1.2, we have 50 classes of graphs of which 25 classes of graphs obtained by applying conditions 2C.1 to 2C.25 and the rest are their complements. Clearly C_5-free nonsplit graphs in these 50 classes only have exactly two split cards. These arguments prove the next theorem.

Theorem 2.7. A C_5-free nonsplit graph G has exactly two split cards if and only if either G or $\bar{G}$ lies in the class of graphs satisfying the conditions 2C.1 to 2C.25.

2.1.3. C_5-free nonsplit graphs with three split cards:

A C_5-free nonsplit graph G with such T has exactly three split cards only if G belongs to any one of the following two family of graphs.
$\mathscr{F}30$: graphs in $\mathscr{G}$ containing two induced C_4 with exactly three common vertices.
$\mathscr{F}31$: graphs in $\mathscr{G}$ containing two induced $2K_2$ with exactly three common vertices.
It is clear that $\overline{\mathscr{F}30} = \mathscr{F}31$.

The family $\mathscr{F}31$:

Let $G_{13} \in \mathscr{G}$ with $T(G_{13}) = \{a_1, a_2, a_3, a_4, a_5\}$, $a_1a_2, a_3a_4, a_4a_5 \in E(G_{13})$ and $a_1a_3, a_1a_4, a_2a_3, a_2a_4, a_1a_5, a_2a_5 \notin E(G_{13})$. Now we construct G_{13} such that cards $G_{13} - a_1$, $G_{13} - a_2$ and $G_{13} - a_4$ are to be split.

a_4-card:

By $R(2K_2)$, a_3 and a_5 lies in an independent set of the card $G_{13} - a_4$ and $\{a_3, a_5\} \nsim\nsim I(G_{13})$. Hence one of the following three conditions (X1-X3) must be a necessary condition for $G_{13} - a_4$ to be a split card of G_{13}.
X1 : $\{a_1, a_2\} \sim\sim C(G_{13})$ & $\{a_3, a_5\} \nsim\nsim I(G_{13})$
X2 : $\{a_1\} \sim\sim C(G_{13})$ & $\{a_3, a_5, a_2\} \nsim\nsim I(G_{13})$
X3 : $a_2 \sim\sim C(G_{13})$ & $\{a_3, a_5, a_1\} \nsim\nsim I(G_{13})$

a_1-card:

By $R(2K_2)$, a_2 lies in an independent set of the card $G_{13} - a_1$ and $a_2 \nsim\nsim I(G_{13})$. Hence one of the following five conditions (Y1-Y5) must be a necessary condition for $G_{13} - a_1$ to be a split card of G_{13}.
Y1 : $\{a_3, a_4, a_5\} \sim\sim C(G_{13})$ & $a_2 \nsim\nsim I(G_{13})$
Y2 : $\{a_3, a_4\} \sim\sim C(G_{13})$ & $\{a_2, a_5\} \nsim\nsim I(G_{13})$
Y3 : $\{a_4, a_5\} \sim\sim C(G_{13})$ & $\{a_2, a_3\} \nsim\nsim I(G_{13})$
Y4 : $\{a_3, a_5\} \sim\sim C(G_{13})$ & $\{a_2, a_4\} \nsim\nsim I(G_{13})$
Y5 : $\{a_4\} \sim\sim C(G_{13})$ & $\{a_2, a_3, a_5\} \nsim\nsim I(G_{13})$

a_2-card:

By $R(2K_2)$, a_1 lies in an independent set of the card $G_{13} - a_2$ and $a_1 \nsim\nsim I(G_{13})$. Hence one of the following five conditions (Z1-Z5) must be a necessary condition for $G_{13} - a_2$ to be a split card of G_{13}.
Z1 : $\{a_3, a_4, a_5\} \sim\sim C(G_{13})$ & $a_1 \nsim\nsim I(G_{13})$
Z2 : $\{a_3, a_4\} \sim\sim C(G_{13})$ & $\{a_1, a_5\} \nsim\nsim I(G_{13})$
Z3 : $\{a_4, a_5\} \sim\sim C(G_{13})$ & $\{a_1, a_3\} \nsim\nsim I(G_{13})$
Z4 : $\{a_3, a_5\} \sim\sim C(G_{13})$ & $\{a_1, a_4\} \nsim\nsim I(G_{13})$
Z5 : $\{a_4\} \sim\sim C(G_{13})$ & $\{a_1, a_3, a_5\} \nsim\nsim I(G_{13})$

In the graph G_{13} with split cards $G_{13} - a_1$, $G_{13} - a_2$ and $G_{13} - a_4$, a vertex not in $T(G_{13})$ may lie in a clique of $G_{13} - a_i$ and may lie in an independent set of the split card $G_{13} - a_j$ and vice versa for $i, j \in \{1, 2, 4\}$ and $i \neq j$. So we have the following types for the structure of G_{13}.

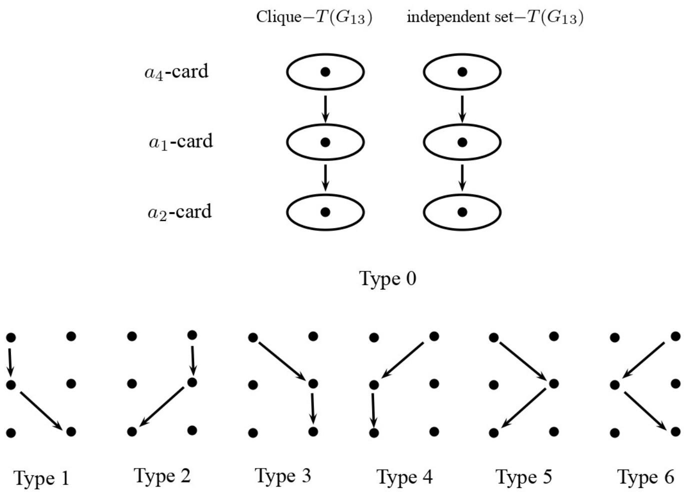


Figure 6. Pattern of clique and independent set in a_1-card, a_2-card and a_4-card

Table 2 : All mutually nonequivalent conditions (Xi, Yj, Zk) for each type.

Types	Nonequivalent conditions
Type 0	[X1,Y5,Z5], [X2,Y2,Z2], [X2,Y2,Z3], [X2,Y2,Z5], [X2,Y5,Z2], [X2,Y5,Z5]
Type 2	[X1,Y5,Z5], [X2,Y2,Z5], [X2,Y5,Z5]
Type 4	[X1,Y5,Z5], [X2,Y5,Z5]
Type 5	[X1,Y5,Z5], [X2,Y2,Z5], [X2,Y5,Z5]
Type 6	[X1,Y5,Z5], [X2,Y5,Z2], [X2,Y5,Z5]
Type 2-6	[X1,Y5,Z5], [X2,Y5,Z5]
Type 5-6	[X2,Y5,Z5]

Type 0:

Using Table 2, a nonsplit graph G_{13} has only three split cards $G_{13} - a_4$, $G_{13} - a_1$ and $G_{13} - a_2$ if it satisfies the condition 3C.1.

3C.1: $\{a_1, a_4\} \sim\sim C(G_{13})$, $\{a_1, a_2, a_3, a_5\} \not\sim\not\sim I(G_{13})$ and $a_3 \not\sim\not\sim a_5$.

Type 2, Type 4, Type 5 and Type 6:

Let $C(G_{13}) \cup I(G_{13}) = C_1(G_{13}) \cup I_1(G_{13}) \cup \{v\}$ such that $C_1(G_{13})$ is a clique and $I_1(G_{13})$ is an independent set, $v \sim\sim C_1(G_{13})$ and $v \not\sim\not\sim I_1(G_{13})$. Using Table 2, a nonsplit graph G_{13} has only three split cards $G_{13} - a_4$, $G_{13} - a_1$ and $G_{13} - a_2$ if it satisfies one of the following adjacency conditions (3C.2) - (3C.4).

3C.2: $\{a_1, a_2, a_4\} \sim\sim C_1(G_{13})$, $\{a_1, a_2, a_3, a_5\} \not\sim\not\sim I_1(G_{13})$, $a_3 \not\sim a_5$, $v \sim a_4$ and $v \not\sim\not\sim \{a_3, a_5\}$.

3C.3: $\{a_1, a_3, a_4\} \sim\sim C_1(G_{13})$, $\{a_1, a_2, a_3, a_5\} \not\sim\not\sim I_1(G_{13})$, $a_3 \not\sim a_5$, $v \sim\sim \{a_1, a_4\}$ and $v \not\sim\not\sim \{a_2, a_5\}$.

3C.4: $\{a_1, a_4\} \sim\sim C_1(G_{13})$, $\{a_1, a_2, a_3, a_5\} \nsim\nsim I_1(G_{13})$, $a_3 \nsim a_5$, $v \sim a_4$ and $v \nsim\nsim \{a_2, a_3, a_5\}$.

Type 2-6 and Type 5-6:

Let $C(G_{13}) \cup I(G_{13}) = C_2(G_{13}) \cup I_2(G_{13}) \cup \{v_1, v_2\}$ such that $C_2(G_{13})$ is a clique and $I_2(G_{13})$ is an independent set, $\{v_1, v_2\} \sim\sim C_2(G_{13})$ and $\{v_1, v_2\} \nsim\nsim I_2(G_{13})$.

Using Table 2, a nonsplit graph G_{13} has only three split cards $G_{13} - a_4$, $G_{13} - a_1$ and $G_{13} - a_2$ if it satisfies one of the following adjacency conditions (3C.5 - 3C.7).

3C.5: $\{a_1, a_2, a_4\} \sim\sim C_2(G_{13})$, $\{a_1, a_2, a_3, a_5\} \nsim\nsim I_2(G_{13})$, $a_3 \nsim a_5$, $\{v_1, v_2\} \sim\sim a_4$, $v_1 \nsim\nsim \{v_2, a_2, a_3, a_5\}$ and $v_2 \nsim\nsim \{a_1, a_3, a_5\}$.

3C.6: $\{a_1, a_4\} \sim\sim C_2(G_{13})$, $\{a_1, a_2, a_3, a_5\} \nsim\nsim I_2(G_{13})$, $a_3 \nsim a_5$, $\{v_1, v_2\} \sim\sim a_4$, $v_1 \nsim\nsim \{v_2, a_2, a_3, a_5\}$ and $v_2 \nsim\nsim \{a_1, a_2, a_3, a_5\}$.

3C.7: $\{a_1, a_4\} \sim\sim C_2(G_{13})$, $\{a_1, a_2, a_3, a_5\} \nsim\nsim I_2(G_{13})$, $a_3 \nsim a_5$, $v_1 \sim\sim \{a_1, a_4\}$, $v_2 \sim a_4$, $v_1 \nsim\nsim \{a_2, a_3, a_5\}$ and $v_2 \nsim\nsim \{a_1, a_2, a_3, a_5\}$.

In view of the above discussion in Case 2.1.3 , we have 14 classes of graphs of which 7 classes of graphs obtained by applying conditions 3C.1 to 3C.7 and the rest are their complements. Clearly C_5-free nonsplit graphs in these 14 classes only have exactly three split cards. Thus, we proved the next theorem.

Theorem 2.8. A C_5-free nonsplit graph G has exactly three split cards if and only if either G or $\bar{G}$ lies in the class of graphs satisfying the conditions 3C.1 to 3C.7.

2.1.4. C_5-free nonsplit graphs with four split cards:

A C_5-free nonsplit graph G with such T has exactly three split cards only if G belongs to any one of the following two family of graphs.

$\mathscr{F}40$: graphs in $\mathscr{G}$ containing an induced cycle on four vertices.

$\mathscr{F}41$: graphs in $\mathscr{G}$ containing an induced complement of cycle on four vertices.

It is clear that $\overline{\mathscr{F}40} = \mathscr{F}41$.

The family $\mathscr{F}41$:

Let $G_{14} \in \mathscr{G}$ with $T(G_{14}) = \{a_1, a_2, a_3, a_4\}$, $a_1a_2, a_3a_4 \in E(G_{14})$ and $a_1a_3, a_1a_4, a_2a_3, a_2a_4 \notin E(G_{14})$. We now construct G_{14} such that $G_{14}-a_1$, $G_{14}-a_2$, $G_{14} - a_3$ and $G_{14} - a_4$ are to be split.

<u>a_1-card:</u>

By $R(2K_2)$, a_2 lies in an independent set of the card $G_{14} - a_1$ and $a_2 \nsim\nsim I(G_{14})$. Hence one of the following three conditions (X1-X3) must be a necessary condition for $G_{14} - a_1$ to be a split card of G_{14}.

X1 : $\{a_3, a_4\} \sim\sim C(G_{14})$ & $\{a_2\} \nsim\nsim I(G_{14})$

X2 : $\{a_3\} \sim\sim C(G_{14})$ & $\{a_2, a_4\} \nsim\nsim I(G_{14})$

X3 : $\{a_4\} \sim\sim C(G_{14})$ & $\{a_2, a_5\} \nsim\nsim I(G_{14})$

<u>a_2-card:</u>

Similarly, by $R(2K_2)$, a_1 lies in an independent set of the card $G_{14} - a_2$ and $a_1 \nsim\nsim I(G_{14})$. Hence one of the following three conditions (Y1-Y3) must be a necessary condition for $G_{14} - a_2$ to be a split card of G_{14}.

Y1 : $\{a_3, a_4\} \sim\sim C(G_{14})$ & $\{a_1\} \nsim\nsim I(G_{14})$

Y2 : $\{a_3\} \sim\sim C(G_{14})$ & $\{a_1, a_4\} \nsim\nsim I(G_{14})$

Y3 : $\{a_4\} \sim\sim C(G_{14})$ & $\{a_1, a_5\} \not\sim\not\sim I(G_{14})$

a_3-card:

Similarly, by $R(2K_2)$, a_4 lies in an independent set of the card $G_{14} - a_3$ and $a_4 \not\sim\not\sim I(G_{14})$. Hence one of the following three conditions (Z1-Z3) must be a necessary condition for $G_{14} - a_3$ to be a split card of G_{14}.

Z1 : $\{a_1, a_2\} \sim\sim C(G_{14})$ & $\{a_4\} \not\sim\not\sim I(G_{14})$

Z2 : $\{a_1\} \sim\sim C(G_{14})$ & $\{a_4, a_2\} \not\sim\not\sim I(G_{14})$

Z3 : $\{a_2\} \sim\sim C(G_{14})$ & $\{a_4, a_1\} \not\sim\not\sim I(G_{14})$

a_4-card:

Similarly, by $R(2K_2)$, a_3 lies in an independent set of the card $G_{14} - a_4$ and $a_3 \not\sim\not\sim I(G_{14})$. Hence one of the following three conditions (W1-W3) must be a necessary condition for $G_{14} - a_4$ to be a split card of G_{14}.

W1 : $\{a_1, a_2\} \sim\sim C(G_{14})$ & $\{a_3\} \not\sim\not\sim I(G_{14})$

W2 : $\{a_1\} \sim\sim C(G_{14})$ & $\{a_3, a_2\} \not\sim\not\sim I(G_{14})$

W3 : $\{a_2\} \sim\sim C(G_{14})$ & $\{a_3, a_1\} \not\sim\not\sim I(G_{14})$

In the graph G_{14} with split cards $G_{14} - a_1$, $G_{14} - a_2$, $G_{14} - a_3$ and $G_{14} - a_4$, a vertex not in $T(G_{14})$ may lie in a clique of $G_{14} - a_i$ and may lie in an independent set of the split card $G_{14} - a_j$ and vice versa for $i, j \in \{1, 2, 3, 4\}$ and $i \neq j$. So we have the following types for the structure of G_{14}.

Table 3 : All mutually nonequivalent conditions (Xi, Yj, Zk, Wl) for each type.

Types	**Nonequivalent conditions**
Type 1	[X3,Y3,Z1,W1], [X3,Y3,Z2,W1], [X3,Y3,Z2,W2], [X3,Y3,Z3,W1], [X3,Y3,Z3,W3]
Type 6	[X1,Y1,Z2,W2], [X1,Y2,Z2,W2], [X1,Y3,Z2,W2], [X2,Y2,Z2,W2], [X3,Y3,Z2,W2]
Type 7	[X2,Y2,Z1,W1], [X2,Y2,Z1,W2], [X2,Y2,Z1,W3], [X2,Y2,Z2,W2], [X2,Y2,Z3,W3]
Type 9	[X1,Y1,Z3,W3], [X2,Y1,Z3,W3], [X2,Y2,Z3,W3], [X3,Y1,Z3,W3], [X3,Y3,Z3,W3]
Type 11	[X2,Y1,Z1,W3], [X2,Y1,Z3,W3], [X2,Y2,Z1,W3], [X2,Y2,Z3,W3]
Type 12	[X1,Y3,Z2,W1], [X1,Y3,Z2,W2], [X3,Y3,Z2,W1], [X3,Y3,Z2,W2]
Type 13	[X3,Y1,Z3,W1], [X3,Y1,Z3,W3], [X3,Y3,Z3,W1], [X3,Y3,Z3,W3]
Type 14	[X1,Y2,Z1,W2], [X1,Y2,Z2,W2], [X2,Y2,Z1,W2], [X2,Y2,Z2,W2]
Type 1-6	[X3,Y3,Z2,W2]
Type 1-9	[X3,Y3,Z3,W3]
Type 6-7	[X2,Y2,Z2,W2]
Type 7-9	[X2,Y2,Z3,W3]

Type 0:

From the necessary conditions of four cards to be split, a nonsplit graph G_{14} has only four split cards $G_{14} - a_1$, $G_{14} - a_2$, $G_{14} - a_3$ and $G_{14} - a_4$ if it satisfies the condition 4C.1.

4C.1: $\{a_1, a_3\} \sim\sim C(G_{14})$ and $\{a_1, a_2, a_3, a_4\} \not\sim\not\sim I(G_{14})$.

Type 1, Type 6, Type 7, Type 9, Type 11, Type 12, Type 13 and Type 14:

Let $C(G_{14}) \cup I(G_{14}) = C_1(G_{14}) \cup I_1(G_{14}) \cup \{v\}$ such that $C_1(G_{14})$ is a clique and $I_1(G_{14})$ is an independent set, $v \sim\sim C_1(G_{14})$ and $v \not\sim\not\sim I_1(G_{14})$. Using Table 3, a nonsplit graph G_{14} has only four split cards $G_{14} - a_1$, $G_{14} - a_2$, $G_{14} - a_3$ and $G_{14} - a_4$ if it satisfies one of the following adjacency conditions (4C.2) - (4C.3).

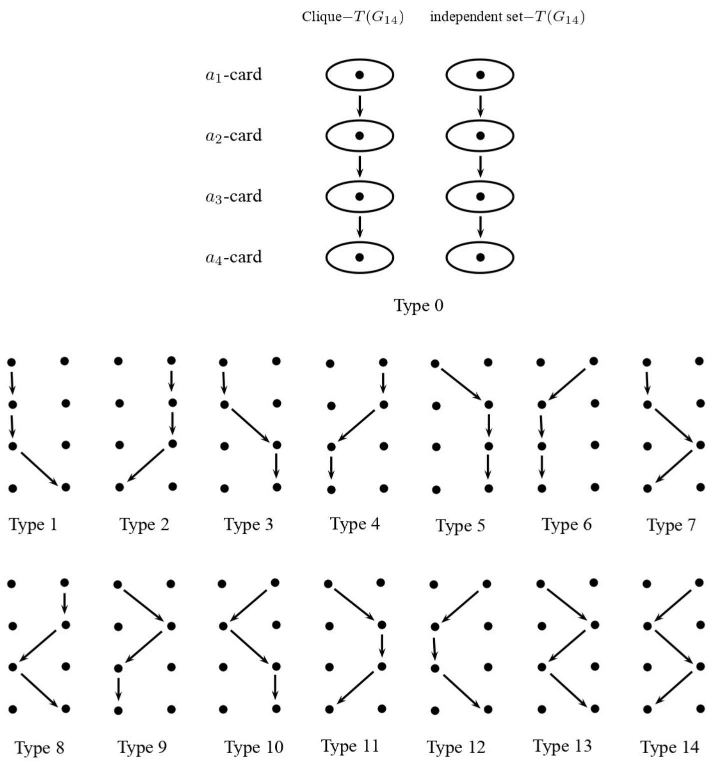


Figure 7. Pattern of clique and independent set in a_1-card, a_2-card, a_3-card and a_4-card

4C.2: $\{a_1, a_3, a_4\} \sim\sim C_1(G_{14})$, $\{a_1, a_2, a_3, a_4\} \nsim\nsim I_1(G_{14})$, $v \sim\sim \{a_1, a_3\}$ and $v \nsim\nsim \{a_3, a_2\}$.

4C.3: $\{a_1, a_3\} \sim\sim C_1(G_{14})$, $\{a_1, a_2, a_3, a_4\} \nsim\nsim I_1(G_{14})$, $v \sim\sim \{a_1, a_3\}$ and $v \nsim\nsim \{a_2, a_4\}$.

Type 1-6, Type 1-9, Type 6-7 and Type 7-9:

Let $C(G_{14}) \cup I(G_{14}) = C_2(G_{14}) \cup I_2(G_{14}) \cup \{v_1, v_2\}$ such that $C_2(G_{14})$ is a clique and $I_2(G_{14})$ is an independent set, $\{v_1, v_2\} \sim\sim C_2(G_{14})$ and $\{v_1, v_2\} \nsim\nsim I_2(G_{14})$. Using Table 3, a nonsplit graph G_{14} has only four split cards $G_{14} - a_1$, $G_{14} - a_2$, $G_{14} - a_3$ and $G_{14} - a_4$ if it satisfies the condition 4C.4.

4C.4: $\{a_1, a_3\} \sim\sim C_2(G_{14})$, $\{a_1, a_2, a_3, a_4\} \nsim\nsim I_2(G_{14})$, $v_1 \sim v_2$, $\{v_1, v_2\} \sim\sim \{a_1, a_3\}$ and $\{v_1, v_2\} \nsim\nsim \{a_2, a_4\}$.

In view of the above discussion in Case 2.1.4, we have 8 classes of graphs of which 4 classes of graphs obtained by applying conditions 4C.1 to 4C.4 and the rest are their complements. Clearly C_5-free nonsplit graphs in these 8 classes only have exactly four split cards. These arguments prove the next theorem.

Theorem 2.9. A C_5-free nonsplit graph G has exactly four split cards if and only if either G or $\bar{G}$ lies in the class of graphs satisfying the conditions 4C.1 to 4C.4.

3. CONCLUDING REMARKS

In the above sections, we have proved that there are thirty, fifty, fourteen and eight classes of C_5-free nonsplit graphs having exactly one, two, three and four split cards respectively. In a similar technique, one can find the list of C_4 or $2K_2$-free nonsplit graphs having split cards. If it is found, then we will have the list of all nonsplit graphs having split cards, which will defintely help us to find the reconstruction number of all split graphs.

Acknowledgement. Monikandan's research is supported by the National Board for Higher Mathematics(NBHM), Government of India. Grant No. 02011/14/2022/ NBHM(R.P)/R&D II.

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