EXISTENCE AND UNIQUENESS RESULTS FOR SINGULAR BOUNDARY VALUE PROBLEM OF WEIGHTED GENERALIZED FISHER TYPE DIFFERENTIAL EQUATION

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Abstract. In this paper, we give existence and uniqueness results of nontrivial positive solution for the boundary value problem of the weighted and generalized Fisher's type differential equation.

Key words and Phrases: Generalized Fisher's equation, positive solution, fixed point.

1. Introduction

Because of their applications in biology, the study of Fisher's differential equations have received a great deal of attention during the latter two decades; see [11], [12], [13], [15], [16] and [14], and references therein. It was derived in 1937 and takes the form

$$\frac{\partial u}{\partial t} = u\left(1 - u\right) + \frac{\partial^2 u}{\partial x^2}.$$

The generalized Fisher's equation .

$$\frac{\partial u}{\partial t} = u^\alpha \left(1 - u^\beta\right) + \frac{\partial}{\partial x}\left(u^m \frac{du}{dx}\right)$$

describes one-dimensional diffusion models for insect and animal dispersal and invasion, where t is time, x is a spatial coordinate, u is a population density, $\frac{\partial u}{\partial t}$ represents the growth of a population and the factor u^m characterizes the diffusion process.

In this paper, we consider the following singular boundary value problem of the weighted and generalized Fisher's type differential equation

$$\begin{cases} \frac{\partial}{\partial x}\left(h.u^m \frac{\partial u}{\partial x}\right) + a\left(x\right).\frac{\partial u}{\partial x} + b(x).u^\alpha \left(1 - u^\beta\right) + f(x, u, u') = 0, \ x > 0, \\ u(0) = 0 = \lim_{x \to +\infty} u(x), \end{cases} \tag{1}$$

where $\alpha, \ \beta \geq 1, \ m \geq 0, \ h \in C\left(\mathbb{R}^+, \mathbb{R}^+\right), \ f \in C\left(\mathbb{R}^+ \times (0, +\infty) \times \mathbb{R}, \mathbb{R}\right), \ a$ and b are the measurable and positive functions such that for all $x \geq 0, \ h(x) \neq 0$

$$\int_0^{+\infty} \max \left\{ \frac{1}{h\left(x\right)}, \frac{a\left(x\right)\gamma^{\frac{-m}{m+1}}\left(x\right)}{p(x)}, b\left(x\right) \right\} dx < \infty, \tag{2}$$

$$p\left(x\right) = \min \left\{ h(x)\int_x^\infty \frac{ds}{h(s)}, h(x)\int_0^x \frac{ds}{h(s)} \right\}$$

and

$$\gamma(x) = \min \left\{ \frac{\int_0^x \frac{ds}{h(s)}}{\int_0^\infty \frac{ds}{h(s)}}, \frac{\int_x^\infty \frac{ds}{h(s)}}{\int_0^\infty \frac{ds}{h(s)}} \right\}.$$

By using the Krein Rutman Theorem [17] and the homotopy method in the fixed point index theory [1, 6, 7, 10], existence and uniqueness results for the problem (1) are given.

In all this paper, we assume that

$$\begin{cases} \text{there exists a continuous function} \ \ q : \mathbb{R}^+ \to \mathbb{R}^+ \ \text{such that} \\ \int_0^{+\infty} q(t)dt < \infty \ \text{and for all} \ (t, x, y) \in \mathbb{R}^+ \backslash \{0\} \times \mathbb{R}^+ \backslash \{0\} \times \mathbb{R} \\ \qquad a(t).y + b(t).\left(x^\alpha - x^{\alpha+\beta}\right) + f\left(t, x, y\right) + q\left(t\right) \geq 0 \end{cases} \tag{3}$$

$$\begin{cases} \text{for all} \ r, R > 0 \ \text{with} \ r \leq R, \ \text{there exists a continuous function} \ g_{r,R} : \mathbb{R}^+ \to \mathbb{R}^+ \ \text{such that} \\ \qquad \int_0^{+\infty} g_{r,R}(t)dt < \infty \ \text{and} \\ \text{for all} \ (t, x, y) \in \mathbb{R}^+ \backslash \{0\} \times [\tilde{\gamma}\left(t\right)\left(r - w^*\right), R] \times [-R - w^*, R + w^*] \\ \qquad f\left(t, \rho\left(x\right), \rho'\left(x\right)\frac{y}{p(t)}\right) \leq g_{r,R}\left(t\right) \end{cases} \tag{4}$$

where

$$w^* = \sup_{x \geq 0} \left\{ \frac{w(x)}{\tilde{\gamma}(x)} \right\}, \ \ \tilde{\gamma}(x) = \frac{1}{2}\gamma(x),$$

with

$$w(x) = \int_0^{+\infty} G(x, t)q(t)dt, \ \rho\left(x\right) = x^{\frac{1}{m+1}}$$

and

$$G\left(x, t\right) = \frac{(m+1)}{\int_0^{+\infty} \frac{ds}{h\left(s\right)}} \cdot \begin{cases} \int_x^{+\infty} \frac{ds}{h\left(s\right)} \int_0^t \frac{ds}{h\left(s\right)} & x \geq t \\ \int_t^{+\infty} \frac{ds}{h\left(s\right)} \int_0^x \frac{ds}{h\left(s\right)} & x \leq t. \end{cases}$$

2. Preliminaries

For sake of completeness let us recall some basic facts needed in this paper. Let E be a real Banach space equipped with its norm noted $\|.\|$. A nonempty closed convex subset P of E is said to be a cone if $P \cap (-P) = 0$ and $(tP) \subset P$ for all $t \geq 0$. It is well known that a cone P induces a partial order in the Banach space E. We write for all $x; y \in E$; $x \leq y$ if $y - x \in P$.

The mapping $L : E \to E$ is said to be positive in P if $L(P) \subset P$, and compact if it is continuous and $L(B)$ is relatively compact in E for all bounded subset B of E. The real value

$$r(L) = \sup\{|\lambda| : \ \lambda \in Sp(L)\}$$

denotes the spectral radius of a linear and bounded operator L, where $Sp(L)$ is the spectrum of L, and we have

$$r(L) = \lim_{n \to \infty} \|L^n\|^{\frac{1}{n}}.$$

The main tool of this work is the following Guo-Krasnoselskii's version of fixed point theorems in a Banach space [10].

Theorem 2.1. *Let Ω_1, Ω_2 be open bounded subsets of E such that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. If $T : P \cap (\overline{\Omega_2}\backslash\Omega_1) \to P$ is a compact operator such that either:*

(1) $Tu \not\geq u$ for $u \in P \cap \partial\Omega_1$ and $Tu \not\leq u$ for $u \in P \cap \partial\Omega_2$, or
(2) $Tu \not\leq u$ for $u \in P \cap \partial\Omega_1$ and $Tu \not\geq u$ for $u \in P \cap \partial\Omega_2$,

Then T has a fixed point in $P \cap (\overline{\Omega_2}\backslash\Omega_2)$.

The following Krein Rutman Theorem has been established in [17]:

Theorem 2.2. *Let K be a cone in E and $L : E \to E$ be a linear, positive, and compact operator. Suppose that for some non-zero element $u \in K^*$, the following relation is satisfied:*

$$MLu \geq u, \text{ for some } M > 0.$$

Then L has a non-zero eigenvector $v \in K^*$:

$$Lv = \lambda_0 v,$$

where the positive eigenvalue λ_0 satisfies the inequality $\lambda_0 \geq M^{-1}$.

In what follows, we let E be a Banach space defined as

$$E = \left\{ u \in C^1(\mathbb{R}^+, \mathbb{R}) : \lim_{x \to \infty} u(x) = \lim_{x \to \infty} p(x).u'(x) = 0 \right\}$$

equipped with the norm

$$\|u\| = \|u\|_0 + \|p.u'\|_0, \ u \in E$$

where $\|u\|_0 = \sup_{x \geq 0} |u(x)|$ and

$$p(x) = \min \left\{ h(x) \int_x^\infty \frac{ds}{h(s)}, h(x) \int_0^x \frac{ds}{h(s)} \right\}.$$

Lemma 2.3. [7] *A non empty subset M of E is relatively compact if the following conditions hold :*

(1) M is bounded in E,
(2) M is locally equicontinuous on $[0, +\infty)$, and
(3) M is equiconvergent at ∞

3. RELATED LEMMAS

Let

$$E^+ = \left\{ u \in E : u \geq 0 \right\},$$
$$P = \left\{ u \in E^+ : u(0) = 0 \right\}$$

and

$$K = \left\{ u \in P : u \geq \tilde{\gamma} \left\| u \right\| \right\}$$

and

$$C = \left\{ u \in K : p \left| u' \right| \leq u \right\}$$

be the cones in E.

For $r > 0$, we consider the operator $T_r : K \backslash B(0, r) \to E$ defined by

$$T_r u(x) = \int_0^{+\infty} G(x, t) D(s, u, u') \, dt,$$

where for $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \backslash \{0\} \times \mathbb{R}$

$$D(t, x, y) = H(t, \rho(x - w), (y - w') \rho'(x - w))$$

with

$$H(t, x, y) = a(t).y + b(t).x^\alpha(1 - x^\beta) + f(t, x, y) + q(t)$$

and

$$\rho(x) = x^{\frac{1}{m+1}}, \text{ for all } x > 0$$

and for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^+$

$$G(x, t) = \frac{(m+1)}{\int_0^{+\infty} \frac{ds}{h(s)}} \cdot \begin{cases} \int_x^{+\infty} \frac{ds}{h(s)} \int_0^t \frac{ds}{h(s)} & x \geq t \\ \int_t^{+\infty} \frac{ds}{h(s)} \int_0^x \frac{ds}{h(s)} & x \leq t. \end{cases}$$

We remark that T_r can be written as

$$T_r u(x) = \int_x^{+\infty} \frac{m+1}{h(t)} \left(-\pi(u) + \int_0^t D(s, u, u') \, ds \right) dt$$

where

$$\pi(u) = \frac{\int_0^{+\infty} \left(\frac{1}{h(t)} \int_0^t D(s, u, u') \, ds \right) dt}{\int_0^{+\infty} \frac{dt}{h(t)}}. \tag{5}$$

Lemma 3.1. *Let* $w : \mathbb{R}^+ \to \mathbb{R}^+$ *be the function defined by*

$$w(x) = \int_0^{+\infty} G(x,t)\, q(t) dt.$$

Then

$$\sup_{t \geq 0} \frac{w(t)}{\tilde{\gamma}(t)} = w^* < \infty.$$

Proof. Let $x_0 > 0$ such that

$$\int_{x_0}^{+\infty} \frac{ds}{h(s)} = \int_0^{x_0} \frac{ds}{h(s)}.$$

For all $t \geq 0$,

$$
\frac{w(x)}{\tilde{\gamma}(x)} = 2 \begin{cases} \frac{\int_0^{+\infty} \frac{ds}{h(s)}}{\int_0^x \frac{ds}{h(s)}} \int_0^{+\infty} G(x,t)\, q(t) dt & \text{if } x \leq x_0 \\ \frac{\int_0^{+\infty} \frac{ds}{h(s)}}{\int_x^{+\infty} \frac{ds}{h(s)}} \int_0^{+\infty} G(x,t)\, q(t) dt & \text{if } x \geq x_0 \end{cases}
$$

$$
\leq 2(m+1) \begin{cases} \int_x^{+\infty} \frac{ds}{h(s)} \int_0^x q(t) dt + \int_x^{+\infty} \left(\int_t^{+\infty} \frac{ds}{h(s)} \right) q(t) dt & \text{if } x \leq x_0 \\ \int_0^x \left(\int_0^t \frac{ds}{h(s)} \right) q(t) dt + \int_0^x \frac{ds}{h(s)} \int_x^{+\infty} q(t) dt & \text{if } x \geq x_0 \end{cases}
$$

$$
\leq 2(m+1) \int_0^{+\infty} \frac{ds}{h(s)} \int_0^{+\infty} q(t) dt < \infty.
$$

This completes the proof. □ □

Remark 3.2. *The positive function* $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and for all* $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$

$$0 \leq G(x,t) \leq G(x,x) \leq g_1(x) = 2\tilde{\gamma}(x) \cdot (m+1) \int_0^{+\infty} \frac{ds}{h(s)}.$$

Lemma 3.3. *The function G has the following properties:*

1: For all $x, \tau, t \in \mathbb{R}^+$

$$G(\tau,t) \geq \gamma(\tau) \cdot G(x,t),$$

where

$$\gamma(\tau) = \min \left\{ \frac{\int_0^\tau \frac{ds}{h(s)}}{\int_0^\infty \frac{ds}{h(s)}}, \frac{\int_\tau^\infty \frac{ds}{h(s)}}{\int_0^\infty \frac{ds}{h(s)}} \right\}.$$

2: For all $t, x, y \in \mathbb{R}^+$,

$$|G(x,t) - G(y,t)| \leq c_0 |x - y|$$

where $c_0 = (m+1) \sup \left\{ \frac{1}{h(t)}, \ t \geq 0 \right\}.$

3: For all $x, t \in \mathbb{R}^+$,

$$p(x) \left| \frac{\partial}{\partial x} G(x,t) \right| \leq G(x,t).$$

Proof. 1. For $x, \tau, t \in \mathbb{R}^+$

$$\frac{G(\tau,t)}{G(x,t)} = \begin{cases} \dfrac{\int_\tau^{+\infty} \frac{ds}{h(s)}}{\int_x^{+\infty} \frac{ds}{h(s)}} \geq \dfrac{\int_\tau^{+\infty} \frac{ds}{h(s)}}{\int_0^{+\infty} \frac{ds}{h(s)}} \geq \gamma(\tau) & \text{if } \tau \geq t, \\ & \quad x \geq t \\[3ex] \dfrac{\int_\tau^{+\infty} \frac{ds}{h(s)} \int_0^t \frac{ds}{h(s)}}{\int_t^{+\infty} \frac{ds}{h(s)} \int_0^x \frac{ds}{h(s)}} \geq \dfrac{\int_\tau^{+\infty} \frac{ds}{h(s)}}{\int_0^{+\infty} \frac{ds}{h(s)}} \geq \gamma(\tau) & \text{if } \tau \geq t, \\ & \quad x \leq t \\[3ex] \dfrac{\int_t^{+\infty} \frac{ds}{h(s)} \int_0^\tau \frac{ds}{h(s)}}{\int_x^{+\infty} \frac{ds}{h(s)} \int_0^t \frac{ds}{h(s)}} \geq \dfrac{\int_0^\tau \frac{ds}{h(s)}}{\int_0^\infty \frac{ds}{h(s)}} \geq \gamma(\tau) & \text{if } \tau \leq t, \\ & \quad x \geq t \\[3ex] \dfrac{\int_0^\tau \frac{ds}{h(s)}}{\int_0^x \frac{ds}{h(s)}} \geq \dfrac{\int_0^\tau \frac{ds}{h(s)}}{\int_0^\infty \frac{ds}{h(s)}} \geq \gamma(\tau) & \text{if } \tau \leq t, \\ & \quad x \leq t \end{cases}$$

Consequently, $\frac{G(\tau,t)}{G(x,t)} \geq \gamma(\tau)$.

2. For $x, y, t \in \mathbb{R}^+$, assume that $x > y$. We have

$$|G(x,t) - G(y,t)| = \frac{(m+1)}{\int_0^\infty \frac{ds}{h(s)}} \begin{cases} \int_0^t \frac{ds}{h(s)} \left(\int_y^x \frac{ds}{h(s)} \right) & \text{if } x > y \geq t \\[2ex] \left| \int_x^\infty \frac{ds}{h(s)} \int_0^t \frac{ds}{h(s)} - \int_0^y \frac{ds}{h(s)} \int_t^\infty \frac{ds}{h(s)} \right| & \text{if } x \geq t \geq y \\[2ex] \int_t^\infty \frac{ds}{h(s)} \left(\int_y^x \frac{ds}{h(s)} \right) & \text{if } t \geq x > y \end{cases} .$$

In the case $x \geq t \geq y$, if $\int_x^\infty \frac{ds}{h(s)} \int_0^t \frac{ds}{h(s)} \geq \int_0^y \frac{ds}{h(s)} \int_t^\infty \frac{ds}{h(s)}$ then

$$\begin{aligned} |G(x,t) - G(y,t)| &= \int_x^\infty \frac{ds}{h(s)} \int_0^t \frac{ds}{h(s)} - \int_0^y \frac{ds}{h(s)} \int_t^\infty \frac{ds}{h(s)} \\ &\leq \int_t^\infty \frac{ds}{h(s)} \left(\int_y^x \frac{ds}{h(s)} \right) \leq \int_0^\infty \frac{ds}{h(s)} \left(\int_y^x \frac{ds}{h(s)} \right) \end{aligned}$$

and if $\int_x^\infty \frac{ds}{h(s)} \int_0^t \frac{ds}{h(s)} \leq \int_0^y \frac{ds}{h(s)} \int_t^\infty \frac{ds}{h(s)}$ then

$$\begin{aligned} |G(x,t) - G(y,t)| &= \int_0^y \frac{ds}{h(s)} \int_t^\infty \frac{ds}{h(s)} - \int_x^\infty \frac{ds}{h(s)} \int_0^t \frac{ds}{h(s)} \\ &\leq \int_0^t \frac{ds}{h(s)} \left(\int_y^x \frac{ds}{h(s)} \right) \leq \int_0^\infty \frac{ds}{h(s)} \left(\int_y^x \frac{ds}{h(s)} \right) \end{aligned}$$

then for $x, y, t \in \mathbb{R}^+$

$$\begin{aligned} |G(x,t) - G(y,t)| &\leq (m+1) \int_y^x \frac{ds}{h(s)} \\ &\leq c_0 |x - y| . \end{aligned}$$

3. For all $x, t \in \mathbb{R}^+$,

$$\frac{p(x)\left|\frac{\partial}{\partial x}G(x,t)\right|}{G(x,t)} = \begin{cases} \frac{p(x)}{h(x)\int_x^\infty \frac{ds}{h(s)}} & \text{if } x \geq t, \\ \frac{p(x)}{h(x)\int_0^x \frac{ds}{h(s)}} & \text{if } x \leq t, \end{cases}$$

$$\leq 1$$

thus

$$p(x)\left|\frac{\partial}{\partial x}G(x,t)\right| \leq G(x,t)$$

completing the proof.   □   □

Let $r, R > 0$, where $r \leq R$. We denote by $\Theta_{r,R}$ and $\Psi_{r,R}$ the functions defined in $\mathbb{R}^+$ as

$$\begin{cases} \Theta_{r,R}(t) = R\frac{a(t)}{p(t)} + \left(R^\alpha + R^{\alpha+\beta}\right)b(t) + g_{r,R}(t) + q(t), \\ \Psi_{r,R}(t) = \frac{a(t)}{p(t)} \cdot \frac{R+w^*}{m+1}\left(\tilde{\gamma}(r-w^*)\right)^{\frac{-m}{m+1}} + b(t)\left(R^{\frac{\alpha}{m+1}} - \left(\tilde{\gamma}(r-w^*)\right)^{\frac{\alpha+\beta}{m+1}}\right) \\ \quad\quad + g_{r,R}(t) + q(t) \end{cases} \quad (6)$$

where $g_{r,R}$ is the function given in (4). We have from (2) that

$$\int_0^{+\infty} \Theta_{r,R}(t)dt < \infty \text{ and } \int_0^{+\infty} \Psi_{r,R}(t)dt < \infty.$$

Lemma 3.4. *Assume that Hypothesis (2), (3) and (4) hold true and let $R > r > w^*$. Then for all $u \in K \cap \partial B(0,r)$*

$$H(t,u,u') \leq \Theta_{r,r}(t), \text{ for all } t \geq 0$$

and for all $u \in K \cap \bar{B}(0,R) \setminus B(0,r)$

$$D(t,u,u') \leq \Psi_{r,R}(t), \text{ for all } t \geq 0.$$

Proof. Let $u \in K \setminus \{0\}$ and $r = \|u\|$. For $t > 0$, we have $u(t) \geq \gamma r > 0$, and from (4) it follows that for all $t > 0$, $H(t,u,u') \geq 0$ and

$$\begin{aligned} H(t,u,u') &\leq a(t).u'(t) + b(t)\left(u^\alpha + u^{\alpha+\beta}\right) + f(t,u,u') + q(t) \\ &\leq \|u\|\frac{a(t)}{p(t)} + \left(\|u\|^\alpha + \|u\|^{\alpha+\beta}\right)b(t) \\ &\quad + g_{r,r}(t) + q(t) \\ &= \Theta_{r,r}(t). \end{aligned}$$

Now, for $v \in K \cap \bar{B}(0,R) \setminus B(0,r)$, let $u = v - w$.
We have from Assertion (3) of Lemma (3.3) that $p|w'| \leq w \leq w^*$ and then

$$0 < \tilde{\gamma}.(r-w^*) \leq u \leq R$$

and

$$-(R+w^*) \leq p.u' \leq R+w^*.$$

Thus, for all $t > 0$

$$
\begin{aligned}
D\left(t, v, v'\right) & = H\left(t, (v-w)^{\frac{1}{m+1}}, \frac{(v'-w')}{m+1}(v-w)^{\frac{-m}{m+1}}\right) \\
& \leq \frac{a(t)}{p(t)} \cdot \frac{R+w^*}{m+1}\left(\tilde{\gamma}\left(r-w^*\right)\right)^{\frac{-m}{m+1}} + b\left(t\right)\left(R^{\frac{\alpha}{m+1}} - \left(\tilde{\gamma}\left(r-w^*\right)\right)^{\frac{\alpha+\beta}{m+1}}\right) \\
& \quad + g_{r,R}\left(t\right) + q(t) \\
& = \Psi_{r,R}(t).
\end{aligned}
$$

This completes the proof.   □   □

Let $A_r : E^+ \to E$ be an operator defined as

$$
A_r u\left(x\right) = \int_x^{+\infty} \frac{m+1}{h(t)}\left(-\pi\left(u\right) + \int_0^t g\left(s, u, u'\right) ds\right) dt
$$

where

$$
\pi_g\left(u\right) = \frac{\int_0^{+\infty}\left(\frac{1}{h(t)}\int_0^t g\left(s, u, u'\right) ds\right) dt}{\int_0^{+\infty} \frac{dt}{h(t)}} \tag{7}
$$

and

$$
g\left(s, u, u'\right) = H\left(t, u, u'\right) - q(t).
$$

Remark 3.5. *We have from Lemma (3.4) that $g\left(., u, u'\right) \in L^1\left(\mathbb{R}^+\right)$ for all $u \in E^+ \backslash \{0\}$.*

Lemma 3.6. *Assume that Hypothesis (2), (3) and (4) hold true and let $r \geq w^*$ and $v \in K \backslash B\left(0, r\right)$. If v is a fixed point of T_r then $u = (v - w)^{\frac{1}{m+1}}$ is a positive solution to the bvp (1).*

Proof. Let $r \geq w^*$ and assume that $v \in K \backslash B\left(0, r\right)$ is a fixed point of T_r. We have

$$
v\left(x\right) = \int_0^{+\infty} G\left(x, t\right) D\left(t, v, v'\right) dt
$$

where for $t > 0$

$$
D\left(t, v, v'\right) = H\left(t, (v-w)^{\frac{1}{m+1}}, \frac{(v'-w')}{m+1}(v-w)^{\frac{-m}{m+1}}\right).
$$

Let $u = (v - w)^{\frac{1}{m+1}}$. We have for $x \geq 0$

$$
\begin{aligned}
u^{m+1}\left(x\right) & = v\left(x\right) - w(x) = \int_0^{+\infty} G\left(x, t\right)\left(H\left(t, u, u'\right) - q(t)\right) dt \\
& = A_r u(x),
\end{aligned}
$$

where

$$
A_r u(x) = \int_x^{+\infty} \frac{m+1}{h(t)}\left(\int_0^t g\left(s, u, u'\right) ds - \pi_g\left(u\right)\right) dt
$$

$$\pi_g(u) = \frac{\int_0^{+\infty} \left(\frac{1}{h(t)} \int_0^t g(s, u, u')\, ds\right) dt}{\int_0^{+\infty} \frac{dt}{h(t)}}$$

and

$$g(t, u, u') = a(t).u' + b(t) u^\alpha (1 - u^\beta) + f(t, u, u').$$

We have from Lemma (3.4) that

$$g(., u, u') \in L^1\left(\mathbb{R}^+\right).$$

Then

$$\pi_g(u) < \infty$$

and so

$$\int_0^{+\infty} \left| \frac{m+1}{h(t)} \left(\int_0^t g(s, u, u')\, ds - \pi_g(u)\right)\right| < \infty.$$

Furthermore, the relation

$$u^{m+1} = A_r u \tag{8}$$

is equivalent to

$$\begin{cases} \frac{\partial}{\partial x}\left(\frac{h(x)}{m+1} \frac{\partial}{\partial x}\left(u^{m+1}\right)\right) = \frac{\partial}{\partial x}\left(\frac{h(x)}{m+1} \frac{\partial}{\partial x} A_r u(x)\right) = -g(x, u, u'),\ x > 0, \\ \lim_{x \to +\infty} u(x)^{m+1} = \lim_{x \to +\infty} A_r u(x) = 0.\ \text{and} \\ u^{m+1}(0) = \int_0^{+\infty} \frac{m+1}{h(t)}\left(\int_0^t g(s, u, u')\, ds - \pi_g(u)\right) \\ \quad = \int_0^{+\infty} \frac{m+1}{h(t)}\left(\int_0^t g(s, u, u')\, ds - \pi_g(u)\right) = 0. \end{cases}$$

This completes the proof.   □   □

Lemma 3.7. *Assume that Hypothesis (2) and (4) hold true. Let $r > w^*$. Then for all and $u \in K \backslash B(0, r)$, $T_r u \geq 0$ and for all $x \geq 0$*

$$p(x) \left|(T_r u)'(x)\right| \leq T_r u(x),$$

and

$$p(x) \left|(T_r u)'(x) - w'(x)\right| \leq \|T_r u\| + w^*,$$

where

$$p(x) = \min\left\{h(x) \int_x^\infty \frac{ds}{h(s)}, h(x) \int_0^x \frac{ds}{h(s)}\right\}.$$

Proof. Let $u \in K \backslash B(0, r)$ and let v be a function defined as

$$v = T_r u = \int_0^\infty G(x, t) H\left(t, (u - w)^{\frac{1}{m+1}}, \frac{(u' - w')}{m+1}(u - w)^{\frac{-m}{m+1}}\right) dt.$$

we have from (4) that $D(t, u, u') \geq 0$ for all $t \geq 0$. For $x \geq 0$,

$$|v'(x)| = \left|\int_0^\infty \frac{\partial}{\partial x} G(x, t). D(t, u, u')\, dt\right| \leq \int_0^\infty \left|\frac{\partial}{\partial x} G(x, t)\right| D(t, u, u')\, dt$$

with

$$\frac{p\left(x\right)\left|\frac{\partial}{\partial x}G\left(x,t\right)\right|}{G\left(x,t\right)} = \begin{cases} \frac{p(x)}{h(x)\int_x^\infty \frac{ds}{h(s)}} & \text{if } x \geq t, \\ \frac{p(x)}{h(x)\int_0^x \frac{ds}{h(s)}} & \text{if } x \leq t, \end{cases}$$
$$\leq 1.$$

Hence, for all $x \geq 0$,

$$\begin{aligned} p\left(x\right)\left|v'\left(x\right)\right| &\leq p\left(x\right)\int_0^\infty \left|\frac{\partial}{\partial x}G\left(x,t\right)\right|D\left(t,u,u'\right) \\ &\leq \int_0^\infty G\left(x,t\right)D\left(t,u,u'\right) = v(x), \end{aligned}$$

and

$$\begin{aligned} p\left(x\right)\left|v'\left(x\right)-w'\left(x\right)\right| &\leq p\left(x\right)\left|v'\left(x\right)\right|+p(x)\left|w'(x)\right| \\ &\leq \|v\|+w(x) \\ &\leq \|v\|+w^*. \end{aligned}$$

$\square$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 3.8. *Let $r > w^*$ and assume that Hypothesis (2) and (4) hold true. Then the operator $T_r : K\backslash B\left(0,r\right) \to C$ is compact.*

Proof. For $R \geq r$, let $M_{r,R} = T_r\left(\Omega_{r,R}\right)$, with $\Omega_{r,R} = K \cap \bar{B}\left(0,R\right)\backslash B\left(0,r\right)$.
1. In fact we show that the set $M_{r,R}$ is a subset of.E.
We have $f \in C\left(\mathbb{R}^+ \times \mathbb{R}^+\backslash\{0\} \times \mathbb{R}^+\right)$ and we see from continuity of the functions $\frac{1}{h}$ and G and so for

$$\frac{\partial}{\partial x}T_r u(x) = -\frac{m+1}{h(x)}\left(-\pi\left(u\right)+\int_0^x D\left(s,u,u'\right)ds\right)$$

that $M_{r,R} \subset C^1\left(\mathbb{R}^+,\mathbb{R}\right)$. Moreover, we have from from Hypothesis (3) that $D(.,u,u') \geq 0$, and from Assertion (3) of Lemma (3.3) that $p\left|w'\right| \leq w$, and from Lemma (3.4) follows that for all $u \in \Omega_{r,R}$

$$D(.,u,u') \leq \Psi_{r,R} \in L^1\left(\mathbb{R}^+\right)$$

where $\Psi_{r,R}$ is the integrable function given in (6).
For $x \in \mathbb{R}^+$

$$\begin{aligned} T_r\left(u\right)\left(x\right) &= \int_0^{+\infty} G(x,t)D(t,u,u')dt \\ &\leq G(x,x)\int_0^{+\infty} D(t,u,u')dt \\ &= \frac{m+1}{\int_0^\infty \frac{ds}{h(s)}}\int_x^\infty \frac{ds}{h(s)}\int_0^x \frac{ds}{h(s)}\int_0^{+\infty} D(t,u,u')dt \end{aligned}$$

and from Lemma (3.7) we have

$$p(x)\left|\frac{\partial}{\partial x}T_r(u)(x)\right| \leq T_r(u)(x)\,, \text{ for all } x \geq 0$$

then

$$\lim_{x\to\infty} p(x)\left|\frac{\partial}{\partial x}T_r(u)(x)\right| = \lim_{x\to\infty} T_r(u)(x) = 0.$$

2. We show that $M_{r,R}$ is relatively compact.

In first, we show that the set $M_{r,R}$ is bounded. Let $u \in \Omega_{r,R}$ and let $\Psi_{r,R}$ be the integrable function given in (6). We have from Assertion (3) of Lemma (3.3) that $p|w'| \leq w$ then

$$(u - w, p(u' - w')) \in [\tilde{\gamma}(x)(r - w^*), r] \times [-(R + w^*), R + w^*].$$

By using Lemma (3.4) we have that for all $x \geq 0$

$$
\begin{aligned}
T_r(u)(x) &\leq G(x, x)\int_0^{+\infty} D(t, u, u')dt \\
&\leq (m + 1)\int_0^{+\infty}\frac{ds}{h(s)}\int_0^{+\infty}\Psi_{r,R}(t)dt = N_{r,R}
\end{aligned}
$$

and from Lemma (3.7) we deduce that

$$\|T_r(u)\| \leq 2\|T_r(u)\|_0 \leq 2N_{r,R}$$

proving the boundeness of $M_{r,R}$.

Let $I = [\eta, \zeta]$ be a compact interval in $\mathbb{R}^+$ and let $t_1, t_2 \in [\eta, \zeta] \subset \mathbb{R}^+$, such that $t_1 < t_2$. For all $u \in \Omega$ we have

$$|T_r u(t_2) - T_r u(t_1)| \leq \int_0^{+\infty}|G(t_2, s) - G(t_1, s)|\,\Psi_{r,R}(s)ds$$

and by Assertion 2 of Lemma (3.3)

$$|T_r u(t_2) - T_r u(t_1)| \leq \left(c_0\int_0^{+\infty}\Psi_{r,R}(s)ds\right)|t_2 - t_1|,$$

where

$$c_0 = (m + 1)\sup\left\{\frac{1}{h(t)},\ t \geq 0\right\}.$$

Moreover

$$
\begin{aligned}
\left|p(t_2)\frac{\partial}{\partial x}T_r u(t_2) - p(t_1)\frac{\partial}{\partial x}T_r u(t_1)\right| &\leq \int_0^{+\infty}\left|p(t_2)\frac{\partial}{\partial x}G(t_2, s) - p(t_1)\frac{\partial}{\partial x}G(t_1, s)\right|\Psi_{r,R}(s)ds \\
&\leq |p(t_2) - p(t_1)|\int_0^{+\infty}\left|\frac{\partial}{\partial x}G(t_1, s)\right|\Psi_{r,R}(s)ds \\
&\quad + p(t_2)\int_0^{+\infty}\left|\frac{\partial}{\partial x}G(t_2, s) - \frac{\partial}{\partial x}G(t_1, s)\right|\Psi_{r,R}(s)ds.
\end{aligned}
$$

As

$$\left|\frac{\partial}{\partial x}G(t_2,s)-\frac{\partial}{\partial x}G(t_1,s)\right| = \frac{m+1}{\int_0^{+\infty}\frac{d\tau}{h(\tau)}}\begin{cases}\left|\frac{1}{h(t_2)}-\frac{1}{h(t_1)}\right|\int_0^s\frac{d\tau}{h(\tau)} & t_2>t_1\geq s\\[2mm]\frac{1}{h(t_2)}\int_0^s\frac{ds}{h(s)}+\frac{1}{h(t_1)}\int_s^{+\infty}\frac{d\tau}{h(\tau)} & t_1\leq s\leq t_2\\[2mm]\left|\frac{1}{h(t_2)}-\frac{1}{h(t_1)}\right|\int_s^{+\infty}\frac{d\tau}{h(\tau)} & t_1<t_2\leq s\end{cases}$$

$$\leq (m+1)\begin{cases}\left|\frac{1}{h(t_2)}-\frac{1}{h(t_1)}\right| & \begin{array}{c}t_2>t_1\geq s\text{ or}\\ t_1<t_2\leq s\end{array}\\[2mm]\frac{1}{h(t_2)}+\frac{1}{h(t_1)} & t_1\leq s\leq t_2\end{cases}$$

and

$$\left|\frac{\partial}{\partial x}G(t_1,s)\right| = \frac{m+1}{\int_0^{+\infty}\frac{d\tau}{h(\tau)}}\begin{cases}\frac{1}{h(t_1)}\int_0^s\frac{d\tau}{h(\tau)} & t_1\geq s\\[2mm]\frac{1}{h(t_1)}\int_s^{+\infty}\frac{d\tau}{h(\tau)} & t_1\leq s\end{cases}$$

$$\leq (m+1)\sup_{x\geq 0}\left(\frac{1}{h(x)}\right)$$

then

$$\left|p(t_2)\frac{\partial}{\partial x}T_r u\,(t_2)-p(t_1)\frac{\partial}{\partial x}T_r u\,(t_1)\right|\leq c_1\,|p(t_2)-p(t_1)|+c_2\left|\frac{1}{h(t_2)}-\frac{1}{h(t_1)}\right|+c_3\int_{t_1}^{t_2}\Psi_{r,R}(s)ds$$

where

$$c_1=(m+1)\sup_{x\geq 0}\left(\frac{1}{h(x)}\right)\int_0^{+\infty}\Psi_{r,R}(s)ds,\ \ c_2=2\,(m+1)\sup_{x\in[\eta,\zeta]}(p(x))\int_0^{+\infty}\Psi_{r,R}(s)ds$$

and

$$c_3=2\,(m+1)\sup_{x\in[\eta,\zeta]}(p(x))\sup_{x\geq 0}\left(\frac{1}{h(x)}\right).$$

Because that $p,\frac{1}{h}$ and $t\to\int_0^t\Psi_{r,R}(s)ds$ are uniformly continuous on compact intervals, the above estimates prove that $M_{r,R}$ is equicontinuous on compact intervals.

Finally, let $u\in\Omega_{r,R}$. By using Lemma (3.7) we have for $x\geq 0$

$$\left|p(x)\frac{\partial}{\partial x}T_r(u)\,(x)\right|\leq|T_r(u)\,(x)|\leq G(x,x)\int_0^{+\infty}\Psi_{r,R}(s)ds.$$

With the fact that

$$\lim_{x\to+\infty}G(x,x)=0$$

the equiconvergence of $M_{r,R}$ holds. By Lemma (2.3), we deduce that $M_{r,R}$ is relatively compact.

3. We show that T_r is continuous in $\Omega_{r,R}$.

Let $(u_n)_n$ be a sequence in $\Omega_{r,R}$ such that

$$\lim_{n\to\infty}u_n=u\in\Omega_{r,R}.$$

For all $t>0$, we have

$$u\,(t)-w\,(t)\geq\tilde{\gamma}\,(t)\,.\,(r-w^*)>0$$

and

$$\begin{aligned}
|T_r(u_n)(t) - T_r(u)(t)| &\leq \int_0^{+\infty} G(t,s)\,|D(s,u_n,u'_n) - D(s,u,u')|\,ds \\
&\leq c \int_0^{+\infty} |D(s,u_n,u'_n) - D(s,u,u')|\,ds
\end{aligned}$$

where

$$c = (m+1) \int_0^{+\infty} \frac{ds}{h(s)}$$

and by using Assertion (3) of Lemma (3.3) we obtain

$$\begin{aligned}
p(t)\left|\frac{\partial}{\partial t}T_r(u_n)(t) - \frac{\partial}{\partial t}T_r(u)(t)\right| &\leq p(t) \int_0^{+\infty} \left|\frac{\partial}{\partial t}G(t,s)\right|\,|D(s,u_n,u'_n) - D(s,u,u')|\,ds \\
&\leq \int_0^{+\infty} G(t,s)\,|D(s,u_n,u'_n) - D(s,u,u')|\,ds \\
&\leq c \int_0^{+\infty} |D(s,u_n,u'_n) - D(s,u,u')|\,ds
\end{aligned}$$

leading to

$$\|T_r(u_n) - T_r(u)\| \leq 2c \int_0^{+\infty} |D(s,u_n,u'_n) - D(s,u,u')|\,ds.$$

Let $t > 0$. Because of $f \in C\left(\mathbb{R}^+ \times \mathbb{R}^+ \backslash \{0\} \times \mathbb{R}\right), \rho \in C^1\left(\mathbb{R}^+ \backslash \{0\}\right)$ and

$$\tilde{\gamma}(t)r > \tilde{\gamma}(t)w^* \geq w(t) > 0,$$

we have that the function $(x,y) \to D(t,x,y)$ defined by

$$\begin{aligned}
D(t,x,y) &= a(t)(y - w'(t))\rho'(x - w(t)) + b(t)\left(\rho^\alpha(x - w(t)) - \rho^{\alpha+\beta}(x - w(t))\right) \\
&\quad + f(t, \rho(x - w(t)), (y - w'(t))\rho'(x - w(t)))
\end{aligned}$$

is continuous in $[\tilde{\gamma}r, +\infty) \times \mathbb{R}$, and then

$$\lim_{n \to \infty} |D(t, u_n(t), u'_n(t)) - D(t, u(t), u'(t))| = 0 \ a.e. \ in \ \mathbb{R}^+.$$

Moreover, we have

$$|D(t, u_n(t), u'_n(t)) - D(t, u(t), u'(t))| \leq 2\Psi_{r,R}(t)$$

where $\Psi_{r,R} \in L^1\left(\mathbb{R}^+\right)$ is the function given in (6). Then the Lebesgue dominated convergence theorem guarantees that

$$\lim_{n \to \infty} \|T_r(u_n) - T_r(u)\| = 0$$

which shows the continuity of T_r.

4. Finally, we prove that $T_r\left(K \backslash B\left(0, r\right)\right) \subset C$.

Set $v = T_r u$, $u \in K \backslash B(0, r)$, and let $\tau \in \mathbb{R}^+$. Assertion 1 of Lemma (3.3) gives

$$
\begin{aligned}
v(\tau) & = \int_0^{+\infty} G(\tau, s) H(s, u, u') ds \\
& \geq \int_0^{+\infty} \gamma(\tau) . G(x, s) H(s, u(s)) ds \\
& = \gamma(\tau) v(x)
\end{aligned}
$$

this is for all $x \in \mathbb{R}^+$, then

$$
v(\tau) \geq \gamma(\tau) \|v\|_0
$$

and with Lemma (3.7) holds

$$
v(\tau) \geq \gamma(\tau) \|p.v'\|_0
$$

leading to

$$
v(\tau) \geq \tilde{\gamma}(\tau) \|v\|
$$

then $T_r(K \backslash B(0, r)) \subset K$, and with Lemma (3.7) it follows that

$$
T_r(K \backslash B(0, r)) \subset C
$$

completing the proof. □ □

Lemma 3.9. *For $\varphi \in L^1(\mathbb{R}^+, \mathbb{R}^+)$, let $L_\varphi : E \to E$ be a linear operator defined as*

$$
L_\varphi u(t) = \int_0^{+\infty} G(t, s) \varphi(s) u(s) ds. \tag{9}
$$

Then L_φ is compact and $L_\varphi(E^+ \backslash \{0\}) \subset C \backslash \{0\}$. Moreover

$$
r(L_\varphi) > 0.
$$

Proof. First, it's clear that L_φ is compact and $L_\varphi(E) \subset E$, and from Assertion (1) of Lemma (3.3), we have that for all $u \in E^+ \backslash \{0\}$,

$$
L_\varphi u \geq \gamma \|L_\varphi u\|_0
$$

and from Assertion (3) of Lemma (3.3) that

$$
L_\varphi u \geq \tilde{\gamma} \|L_\varphi u\|
$$

then $L_\varphi(E^+ \backslash \{0\}) \subset K \backslash \{0\}$, and Lemma (3.7) leads $L_\varphi(E^+ \backslash \{0\}) \subset C \backslash \{0\}$. Now we show that $r(L_\varphi) > 0$. Let $u = \tilde{\gamma}$. Since $L_\varphi(E^+) \subset K$, we have

$$
L_\varphi u \geq u. \|L_\varphi u\| > 0
$$

then from Lemma (2.2) we deduce that L_φ has a positive eigenvalue $\lambda \geq M^{-1} = \|L_\varphi \tilde{\gamma}\|$. Thus

$$
r(L_\varphi) \geq \lambda \geq \|L_\varphi(\tilde{\gamma})\| > 0.
$$

□ □

Let $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be a function defined as

$$\eta(t, x, y) = a(t)y + b(t)x^\alpha \left(1 - x^\beta\right).$$

In what follows, we consider the following Hypothesis ;

$$\begin{cases} \text{there exist } R_0 > w^*, \ 0 < \theta_1 < \theta_2 \text{ and } \sigma_1, \sigma_2 \in L^1\left(\mathbb{R}^+\right) \text{ such that} \\ \sup_{t \geq 0} \left\{ \int_{\theta_1}^{\theta_2} G(t, s) \left(\sigma_1\left(s\right) \left(\tilde{\gamma}\left(s\right)\right)^N - \sigma_2\left(s\right) - b(s)\right) ds \right\} > 1 \\ \text{and for all } (t, x, y) \in [\theta_1, \theta_2] \times [R_0, \infty) \times \mathbb{R} \\ f(t, \rho\left(x\right), \frac{y}{p(t)}\rho'\left(x\right)) + h(t) \geq \sigma_1\left(t\right) x^N - \sigma_2\left(t\right) |y|^N \end{cases} \quad (10)$$

where

$$N = \max \left\{ 1, \frac{\alpha + \beta}{m + 1} \right\}.$$

Lemma 3.10. *Assume that Hypothesis (10) holds true and let $r > w^*$. Then there exists $R > \max\{r, R_0\}$ such that for all $u \in C \backslash B(0, R)$,*

$$\|T_r u\|_0 > \|u\|_0.$$

Proof. In the contrary, we suppose that for all $n \geq \left[\max\left(\frac{R_0}{\min_{t \in [\theta_1, \theta_2]} \tilde{\gamma}(t)} + w^*, r \right) \right] + 1$, there exists $u_n \in C \backslash B(0, n)$, $t_n \geq 0$ such that

$$u_n\left(t_n\right) \ = \ \|u_n\|_0 \geq \|T_r u_n\|_0 \qquad (11)$$

$$\geq \ T_r u_n\left(t\right) = \int_0^{+\infty} G(t, s) D(s, u_n, u_n') ds \qquad (12)$$

$$\geq \ \int_{\theta_1}^{\theta_2} G(t, s) D(s, u_n, u_n') ds, \ \forall t \geq 0. \qquad (13)$$

As $u_n - w \geq \min_{t \in [\theta_1, \theta_2]} \tilde{\gamma}\left(t\right)\left(n - w^*\right) \geq R_0$ for all $t \in [\theta_1, \theta_2]$, then we have from Hypothesis (10) that for $s \in [\theta_1, \theta_2]$

$$D(s, u_n, u_n') \geq \eta(s, \rho\left(u_n - w\right), (u_n' - w')\rho'(u_n - w)) + \sigma_1\left(s\right)\left(u_n - w\right)^N - \sigma_2\left(s\right)\left(p(s) |u_n' - w'|\right)^N$$

with

$$\eta(s, \rho\left(u_n - w\right), (u_n' - w')\rho'(u_n - w)) \ = \ D(s, u_n, u_n') - f\left(s, \rho\left(u_n - w\right), (u_n' - w')\rho'(u_n - w)\right) - h(s)$$

$$= \ \frac{a(s)}{p(s)} p\left(s\right)\left(u_n' - w'\right)\rho'(u_n - w) + b(s)\rho^\alpha\left(u_n - w\right)\left(1 - \rho^\beta\left(u_n - w\right)\right)$$

From Assertion (3) of Lemma (3.3) we have $p|w'| \leq w$, then

$$-(w + u_n) \leq p(s)(u_n' - w') \leq u_n + w$$

and so

$$\eta(s, \rho\left(u_n - w\right), (u_n' - w')\rho'(u_n - w)) \geq -\frac{a(s)}{p(s)}(w + u_n)\rho'(u_n - w) + b(s)\rho^\alpha\left(u_n - w\right)\left(1 - \rho^\beta\left(u_n - w\right)\right).$$

By dividing in (11) by $\|u_n\|^N$ we obtain for $t \geq 0$

$$
\begin{aligned}
v_n(t_n) &= \frac{u_n(t_n)}{\|u_n\|} \geq \frac{u_n(t_n)}{\|u_n\|^N} \\
&\geq \int_{\theta_1}^{\theta_2} G(t,s) \frac{\eta(s, \rho(u_n - w), (u'_n - w')\rho'(u_n - w)) + \sigma_1(s)(u_n - w)^N - \sigma_2(s)(u_n + w)^N}{\|u_n\|^N} ds \\
&\geq \int_{\theta_1}^{\theta_2} G(t,s) \frac{-\frac{a(s)}{p(s)}(w + u_n)\rho'(u_n - w)}{\|u_n\|^N} ds \\
&\quad + \int_{\theta_1}^{\theta_2} G(t,s) \frac{b(s)\rho^\alpha(u_n - w)\left(1 - \rho^\beta(u_n - w)\right)}{\|u_n\|^N} ds \\
&\quad + \int_{\theta_1}^{\theta_2} G(t,s)\sigma_1(s)\left(v_n - \frac{w}{\|u_n\|}\right)^N - \sigma_2(s)\left(v_n + \frac{w}{\|u_n\|}\right)^N ds.
\end{aligned}
$$

As $1 \geq v_n \geq \tilde{\gamma}$, we have

$$
\begin{aligned}
\int_{\theta_1}^{\theta_2} G(t,s)\sigma_1(s)\left(v_n - \frac{w}{\|u_n\|}\right)^N - \sigma_2(s)\left(v_n + \frac{w}{\|u_n\|}\right)^N ds &\geq \int_{\theta_1}^{\theta_2} G(t,s)\sigma_1(s)\left(\tilde{\gamma} - \frac{w}{\|u_n\|}\right)^N ds \\
&\quad - \int_{\theta_1}^{\theta_2} G(t,s)\sigma_2(s)\left(1 + \frac{w}{\|u_n\|}\right)^N ds
\end{aligned}
$$

and then

$$
\lim_{n\to\infty} \int_{\theta_1}^{\theta_2} G(t,s)\sigma_1(s)\left(v_n - \frac{w}{\|u_n\|}\right)^N - \sigma_2(s)\left(v_n + \frac{w}{\|u_n\|}\right)^N ds \geq \int_{\theta_1}^{\theta_2} G(t,s)\left(\sigma_1(s)(\tilde{\gamma})^N - \sigma_2(s)\right) ds.
$$

Since $N = \max\left\{1, \frac{\alpha+\beta}{m+1}\right\}$, then

$$
\lim_{n\to\infty} \frac{\rho(u_n - w)}{\|u_n\|^N} = \lim_{n\to\infty} \frac{(u_n - w)^{\frac{1}{m+1}}}{\|u_n\|^N} = \lim_{n\to\infty} \frac{(u_n)^{\frac{1}{m+1}}}{\|u_n\|^N} = 0,
$$

$$
\lim_{n\to\infty} \frac{(w + u_n)\rho'(u_n - w)}{\|u_n\|^N} = \lim_{n\to\infty} \frac{\left(\frac{u_n}{\|u_n\|^N}\right)\left(\frac{u_n}{\|u_n\|}\right)^{\frac{-m}{m+1}}}{(m+1)\|u_n\|^{\frac{m}{m+1}}} = 0,
$$

$$
\lim_{n\to\infty} \frac{\rho^\alpha(u_n - w)}{\|u_n\|^N} = \frac{(u_n - w)^{\frac{\alpha}{m+1}}}{\|u_n\|^N} = 0
$$

and

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\rho^{\alpha+\beta}(u_n - w)}{\|u_n\|^N} &= \frac{(u_n - w)^{\frac{\alpha+\beta}{m+1}}}{\|u_n\|^N} = \begin{cases} 0 & if \ \frac{\alpha+\beta}{m+1} < 1 \\ (v_n)^{\frac{\alpha+\beta}{m+1}} & if \ \frac{\alpha+\beta}{m+1} \geq 1 \end{cases} \\
&\leq 1.
\end{aligned}
$$

Then we have

$$
1 \geq \lim_{n\to\infty} v_n(t_n) \geq \int_{\theta_1}^{\theta_2} G(t,s)\left(\sigma_1(s)(\tilde{\gamma})^N - \sigma_2(s) - b(s)\right) ds
$$

this is for all $t \geq 0$, and with (10) we obtain the following contradiction

$$1 \geq \sup_{t \geq 0} \left\{ \int_{\theta_1}^{\theta_2} G(t,s) \left(\sigma_1(s)(\tilde{\gamma})^N - \sigma_2(s) - b(s) \right) ds \right\} > 1.$$

Then there exists $R > R_0$ such that for all $u \in \partial B(0,R) \cap C$,

$$\|T_r u\|_0 > \|u\|_0.$$

This completes the proof. $\square$ $\square$

4. Main results

4.1. **Existence results.**

Theorem 4.1. *Assume that Hypothesis (2), (3), (4) and (10) hold true.*
If there exist $r > w^*$, λ_1, $\lambda_2 \geq 0$ and ϕ_1, $\phi_2 \in L^1(\mathbb{R}^+)$ such that for all $t > 0$ and all $(x,y) \in [\tilde{\gamma}(t)(r - w^*), r] \times [-r - w^*, r + w^*]$

$$f(t, \rho(x), \frac{y}{p(t)} \rho'(x)) + q(t) \leq \lambda_1 \phi_1(t) x + \lambda_2 \phi_2(t) y - \eta(t, \rho(x), \frac{y}{p(t)} \rho'(x)), \quad (14)$$

and there exists $i \in \{1, 2\}$ such that

$$0 < \lambda_i < \frac{1}{r(L_\psi)}$$

where

$$\psi = \frac{\lambda_1 \phi_1 + 2\lambda_2 \phi_2}{\lambda_i}$$

then bvp (1) admits at least one positive solution.

Proof. Fisrt, we show that for all $u \in \partial B(0,r) \cap C$

$$T_r u \not\geq u.$$

Suppose in the contrary that there exist $u \in \partial B(0,r) \cap C$ and $t > 0$ such that

$$T_r u \geq u$$

then for $t \geq 0$

$$u(t) \leq T_r u(t) = \int_0^{+\infty} G(t,s) D(s, u, u') ds.$$

We have $pw' \geq -w$ and

$$(u - w, p(u' - w')) \in [\tilde{\gamma}(t)(r - w^*), r] \times [-r - w^*, r + w^*]$$

and from the condition (14) follows that for all $s > 0$,

$$f(s, \rho(u - w), (u' - w') \rho'(u - w)) + q(t) \leq \lambda_1 \phi_1(s)(u - w) + \lambda_2 \phi_2(s) p(s)(u' - w')$$
$$- \eta(s, \rho(u - w), (u' - w') \rho'(u - w))$$

leading to

$$D\left(s, u, u'\right) \leq \lambda_1 \phi_1\left(s\right)\left(u - w\right) + \lambda_2 \phi_2\left(s\right) p(s)\left(u' - w'\right),$$

then

$$
\begin{aligned}
u(t) \quad &\leq \quad \int_0^{+\infty} G(t, s)\left[\lambda_1 \phi_1\left(s\right)\left(u - w\right) + \lambda_2 \phi_2\left(s\right) p(s)\left(u' - w'\right)\right] ds \\
&\leq \quad \int_0^{+\infty} G(t, s)\left[\left(\lambda_1 \phi_1\left(s\right) + \lambda_2 \phi_2\left(s\right)\right) u - \left(\lambda_1 \phi_1\left(s\right) - \lambda_2 \phi_2\left(s\right)\right) w\right] ds \\
&\leq \quad \int_0^{+\infty} G(t, s)\left(\lambda_1 \phi_1\left(s\right) + 2\lambda_2 \phi_2\left(s\right)\right) u\left(s\right) ds.
\end{aligned}
$$

Let $i \in \{1, 2\}$ such that $0 < \lambda_i < \frac{1}{r(L_\psi)}$, with

$$\psi = \frac{\lambda_1 \phi_1 + 2\lambda_2 \phi_2}{\lambda_i}.$$

We have for $t \geq 0$

$$u(t) \leq \lambda_i \int_0^{+\infty} G(t, s)\psi\left(s\right) u\left(s\right) ds = L_\psi u$$

leading from Lemma (2.2) for $M = \lambda$, to the following contradiction

$$\lambda_i^{-1} \leq \lambda_0 \leq r(L_\psi) < \lambda_i^{-1}.$$

Then for all $u \in \partial B\left(0, r\right) \cap C$

$$T_r u \ngeq u.$$

Moreover, as Hypothesis (10) is verified, then Lemma (3.10) guarantees that there exists $R > r$ such that for all $u \in \partial B\left(0, R\right) \cap C$

$$\|T_r u\|_0 > \|u\|_0,$$

which means that

$$T_r u \nleq u.$$

Thus, it follows from Theorem (2.1) that T_r admits a fixed point $v \in C$ such that $r \leq \|v\| \leq R$. Hence by Lemma (3.6), $u = (v - w)^{\frac{1}{m + 1}}$ is a positive solution to the bvp (1). $\qquad \square \qquad \qquad \square$

Let Λ be the following positive value

$$\Lambda = \left(\frac{1}{2 \sup_{x \geq 0} \int_0^{+\infty} G(x, s)\left[\left(\delta_1\left(s\right) + 2^N \delta_2\left(s\right)\right) + 2\frac{a(s)\hat{\gamma}^{-\frac{m}{m+1}}}{p(s)} + b(s)\right] ds}\right)^{\frac{1}{N-1}}.$$

Theorem 4.2. *Assume that Hypothesis (2), (3), (4) and (10) hold true.*
If there exist $r_0 \geq w^* + 1$ and $\delta_1, \delta_2 \in L^1(\mathbb{R}^+, \mathbb{R}^+) \setminus \{0\}$ such that

$$\sup_{x \geq 0} \int_0^{+\infty} G(x, s) \left(\left(\delta_1(s) + 2^{N_O} \delta_2(s) \right) + 2 \frac{a(s) \tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)} + b(s) \right) ds < \frac{1}{2(1 + w^*)^{N_0 - 1}} \tag{15}$$

and for all $t > 0$, and all $(x, y) \in [\tilde{\gamma}(t), r_0] \times [-r_0 - w^*, r_0 + w^*]$

$$f(t, \rho(x), \frac{y}{p(t)} \rho'(x)) + q(t) \leq \delta_1(t) x^{N_0} + \delta_2(t) |y|^{N_0}, \tag{16}$$

where

$$N_0 = \max \left\{ 1, \frac{\alpha}{m+1} \right\}.$$

then bvp (1) admits at least one positive solution.

Proof. Let $r \in [1 + w^*, r_0]$ be a real number such that

$$r < \Lambda, \tag{17}$$

where

$$\Lambda = \left(\frac{1}{2 \sup_{x \geq 0} \int_0^{+\infty} G(x, s) \left[\left(\delta_1(s) + 2^{N_0} \delta_2(s) \right) + 2 \frac{a(s) \tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)} + b(s) \right] ds} \right)^{\frac{1}{N_0 - 1}}.$$

We show that for all $u \in \partial B(0, r) \cap C$

$$T_r u \not\geq u.$$

In the contrary, we assume that there exists $u \in \partial B(0, r) \cap C$ such that $T_r u \geq u$ and let $t \geq 0$. We have

$$u(t) \leq T_r u(t) = \int_0^{+\infty} G(t, s) D(s, u, u') ds. \tag{18}$$

We have

$$(u - w, p(u' - w')) \in [\tilde{\gamma}(t)(r - w^*), r] \times [-r - w^*, r + w^*] \subset [\tilde{\gamma}(t), r_0] \times [-r_0 - w^*, r_0 + w^*]$$

and from the condition (16) follows that for all $s > 0$,

$$\begin{aligned}
f(s, \rho(u - w), (u' - w') \rho'(u - w)) + q(t) &\leq \delta_1(s)(u - w)^{N_0} + \delta_2(s)(p(s)|u' - w'|)^{N_0} \\
&\leq \delta_1(s)(u - w)^{N_0} + \delta_2(s)(u + w)^{N_0} \\
&\leq \left(\delta_1(s) + 2^{N_0} \delta_2(s) \right) u^{N_0}.
\end{aligned}$$

and

$$\eta(s, \rho(u-w), (u'-w')\rho'(u-w)) = a(s)(u'-w')\rho'(u-w) + b(s)\rho^\alpha(u-w)(1-\rho^\beta(u-w))$$

$$\leq \frac{a(s)}{p(s)}(u+w)\rho'(u-w) + b(s)\rho^\alpha(u-w)(1-\rho^\beta(u-w))$$

$$\leq \frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)}(2u)\rho'(r-w^*) + b(s)\rho^\alpha(r).$$

As $r \geq 1 + w^*$, we have $\rho'(r-w^*) \leq 1$. Then

$$\eta(s, \rho(u-w), (u'-w')\rho'(u-w)) \leq \frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)}(2r) + b(s)\rho^\alpha(r),$$

leading to

$$D(s, u, u') \leq (\delta_1(s) + 2^{N_0}\delta_2(s)) r^{N_0} + \frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)}(2r) + b(s)\rho^\alpha(r),$$

and by dividing (18) by r^{N_0} we obtain

$$\frac{u(t)}{r^{N_0}} \leq \int_0^{+\infty} G(t,s)\left[(\delta_1(s) + 2^{N_0}\delta_2(s)) + 2\frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)}(r)^{1-N_0} + b(s)(r)^{\frac{\alpha}{m+1}-N_0}\right]ds$$

and with the fact that $N_0 = \max\left\{1, \frac{\alpha}{m+1}\right\}$ we have

$$\frac{u(t)}{r^{N_0}} \leq \sup_{x \geq 0}\left(\int_0^{+\infty} G(x,s)\left[(\delta_1(s) + 2^{N_0}\delta_2(s)) + 2\frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)} + b(s)\right]ds\right)$$

this is for all $t \geq 0$, then

$$\frac{\|u\|_0}{r^{N_0}} \leq \sup_{x \geq 0}\left(\int_0^{+\infty} G(x,s)\left[(\delta_1(s) + 2^{N_0}\delta_2(s)) + 2\frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)} + b(s)\right]ds\right)$$

$$= \frac{1}{2\Lambda^{N_0-1}}.$$

Since $\|u\|_0 \geq \|pu'\|_0$, we have

$$\frac{r}{2r^{N_0}} \leq \frac{1}{2\Lambda^{N_0-1}}$$

giving

$$r \geq \Lambda$$

which contradicts (17). Then for all $u \in \partial B(0,r) \cap C$

$$T_r u \not\geq u.$$

Moreover, Lemma (3.10) guarantees that there exists $R > r$ such that for all $u \in \partial B(0,R) \cap C$

$$\|T_r u\|_0 > \|u\|_0,$$

which means that

$$T_r u \not\leq u,$$

and Theorem (2.1) guarantees that T_r admits a fixed point $v \in C$ such that $r \le \|v\| \le R$. Hence by Lemma (3.6), $u = (v - w)^{\overline{\frac{1}{m+1}}}$ is a positive solution to the bvp (1). $\qquad \square \qquad\qquad\qquad\qquad\qquad\qquad \square$

4.2. **Uniqueness results.** For $0 < \theta < \theta'$, $\psi \in L^1(\mathbb{R}^+)$ and $r > 0$, let

$$\lambda_\psi(\theta, \theta') = \left(\sup_{t \ge 0} \int_\theta^{\theta'} G(t,s)\psi(s) \right)^{-1}$$

and $\sum_r = \left\{ (u - w)^{\overline{\frac{1}{m+1}}}, \ u \in K \backslash B(0,r) \right\}$.

Theorem 4.3. *Assume that Hypothesis (2), (3) and (4) hold true. If there exist $r, \theta, \theta' > 0$, $c \ge 0$, and $\phi \in L^1(\mathbb{R}^+)$ such that*

$$c < \frac{1}{r(L_\phi)},$$

for all $(x, p(t)y) \in [\tilde\gamma(t)r, +\infty) \times [-r, r]$

$$D(t,x,y) \ge \lambda_\psi(\theta, \theta')\psi(t)x, \ \text{for } t \in [\theta, \theta'] \tag{19}$$

and for all $x_1, x_2 \in [\tilde\gamma(t)r, +\infty)^2$, $y_1, y_2 \in \mathbb{R}$ and all $t > 0$

$$|D(t,x_1,y_1) - D(t,x_2,y_2)| \le c\phi(t)|x_1 - x_2|. \tag{20}$$

Then bvp (1) has a unique positive solution in $\sum_r$.

Proof. The case $c = 0$ is obvious, so we suppose that $c > 0$:

Existence. Let $v \in K \cap \partial B(0,r)$ and consider the sequence $(u_n)_n$ defined by

$$\begin{cases} u_{n+1} = T_r u_n \\ u_0 = v. \end{cases}$$

We show that $(u_n)_n \subset C \backslash B(0,r)$.

Let $u \in P \backslash B(0,r)$. We have from Lemma (3.8) that $T_r(C \backslash B(0,r)) \subset C \backslash \{0\}$. For $t \ge 0$, we have $u \ge \tilde\gamma(t)r$, $p(t)|u'| \le r$ and from the condition (19)

$$\begin{aligned} T_r u &\ge \lambda(\theta) \int_\theta^{\theta'} G(t,s)\psi(s)u(s)\,ds \\ &\ge r.\lambda(\theta) \int_\theta^{\theta'} G(t,s)\psi(s)\tilde\gamma(s)\,ds \end{aligned}$$

this is for all $t \ge 0$, then

$$\|T_r u\| \ge r.\lambda(\theta) \sup_{t \ge 0} \left(\int_\theta^{\theta'} G(t,s)\psi(s)\tilde\gamma(s)\,ds \right) = r.$$

Then $T_r(C \backslash B(0,r)) \subset C \backslash B(0,r)$, which means that $(u_n)_n \subset C \backslash B(0,r)$.

Now, by (20) we have for all $n \geq 1$

$$
\begin{aligned}
|u_{n+1} - u_n| &= |T_r u_n - T_r u_{n-1}| \leq \int_0^{+\infty} G(t,s) \left| D\left(s, u_n, u_n'\right) - D\left(s, u_{n-1}, u_{n-1}'\right) \right| ds \\
&\leq cL_\phi |u_n - u_{n-1}|.
\end{aligned}
$$

Then, for all $n \geq 0$

$$
|u_{n+1} - u_n| \leq c^n L_\phi^n |u_1 - u_0|
$$

and from Lemma (3.3) we have

$$
\begin{aligned}
p\left|u_{n+1}' - u_n'\right| &= p\left|(T_r u_n)' - (T_r u_{n-1})'\right| \leq p(t) \int_0^{+\infty} \left| \frac{\partial}{\partial t} G(t,s) \right| \left| D\left(s, u_n, u_n'\right) - D\left(s, u_{n-1}, u_{n-1}'\right) \right| ds \\
&\leq \int_0^{+\infty} G(t,s) \left| D\left(s, u_n, u_n'\right) - D\left(s, u_{n-1}, u_{n-1}'\right) \right| ds \\
&\leq cL_\phi |u_n - u_{n-1}|.
\end{aligned}
$$

leads

$$
p\left|u_{n+1}' - u_n'\right| \leq c^n L_\phi^n |u_1 - u_0|
$$

then

$$
\|u_{n+1} - u_n\| \leq 2c^n \left\| L_\phi^n |u_1 - u_0| \right\|_0.
$$

Therefore, for $m > n \geq 1$,

$$
\begin{aligned}
\|u_m - u_n\| &\leq \|u_m - u_{m-1}\| + \|u_{m-1} - u_{m-2}\| + \dots + \|u_{n+1} - u_n\| \\
&\leq 2c^{m-1} \left\| L_\phi^{m-1} |u_1 - u_0| \right\|_0 + 2c^{m-2} \left\| L_\phi^{m-2} |u_1 - u_0| \right\|_0 + \dots + 2c^n \left\| L_\phi^n |u_1 - u_0| \right\|_0 \\
&= 2 \left(S_{m-1} - S_{n-1} \right),
\end{aligned}
$$

where

$$
S_n = \sum_{n=0}^{n=+\infty} c^n \left\| L_\phi^n w \right\|_0, \quad \text{with } w = |u_1 - u_0|.
$$

Since $c < (r(L_\phi))^{-1}$, we have that

$$
\lim_{n \to \infty} \sqrt[n]{c^n \left\| L_\phi^n w \right\|_0} \leq c \lim_{n \to \infty} \sqrt[n]{\left\| L_\phi^n \right\|} = c.r(L_\phi) < 1,
$$

then $(S_n)_n$ converges and

$$
\lim_{n \to \infty} \|u_m - u_n\| \leq 2 \lim_{n \to \infty} \left(S_{m-1} - S_{n-1} \right) = 0.
$$

Therefore, the sequence $(u_n)_n$ is also a cauchy sequence and the completeness of E leads to $\lim_{n \to \infty} u_n = u \in C$, with

$$
\|u\| \geq r > 0.
$$

At the end, passing to the limit in $u_{n+1} = T_r u_n$, and by continuity of T_r in $C \backslash B(0, r)$, we obtain $u = T_r u$, and $u \in C \backslash B(0, r)$ is a fixed point of T_r, and so, $v = (u - w)^{\frac{1}{m+1}}$ is a positive solution to the bvp (1) in $\sum_r$.

Uniqueness. If u_1, $u_2 \in K \backslash B(0, r)$ are two solutions of (1) with $u_1 \neq u_2$, then u_1, u_2 are fixed points of T_r where

$$r = \min \{\|u_1\|, \|u_2\|\}.$$

For all $t \in \mathbb{R}$,

$$u_i(t) \geq \tilde{\gamma}(t) . r, \text{ for } i \in \{1, 2\}$$

and so

$$|u_1 - u_2| = |T_r u_1 - T_r u_2| \leq \int_0^{+\infty} G(t, s) |D(s, u_1, u_1') - D(s, u_2, u_2')| \, ds \quad (21)$$

$$\leq c \int_0^{+\infty} G(t, s) \phi(s) |u_1 - u_2| \, ds.$$

Let L_ϕ be the operator defined in Lemma (3.9) by

$$L_\phi u(t) = \int_0^{+\infty} G(t, s) \phi(s) u(s) \, ds$$

and let $w = L_\phi |u_1 - u_2|$. The inequality (21) leads

$$w \leq c L_\phi w.$$

As $|u_1 - u_2| \in E^+ \backslash \{0\}$, then by Lemma (3.9) we have $w \in K \backslash \{0\}$. Then by Theorem (2.2) we deduce that L_ϕ admits a positive eigenvalue λ_0 such that

$$\lambda_0 \geq c^{-1}$$

leading to the following contradiction

$$r(L_\phi) \geq \lambda_0 \geq c^{-1} > r(L_\phi).$$

The proof is complete. $\square$ $\square$

In what follows, we assume that

$$\left\{ \begin{array}{c} \alpha + \beta \leq m + 1, \ a \in C^1(\mathbb{R}^+, \mathbb{R}^+), \ a(t) \leq \min\left(\int_0^t \frac{ds}{h(s)}, \int_t^{+\infty} \frac{ds}{h(s)}\right) \\ e(t) = \max\left(b(t), \frac{1}{h(t)}, |a'(t)|\right) \in L^1_{\gamma^{-1}}(\mathbb{R}^+, \mathbb{R}^+) \text{ and} \\ \text{there exists a function } F \text{ sucth that for all } (t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \backslash \{0\} \times \mathbb{R} \\ f(t, x, y) = F(t, x) \text{ and} \\ c(t)x + b(t)x^\alpha(1 - x^\beta) + F(t, x) + q(t) \geq 0 \end{array} \right.$$

$$(22)$$

where

$$c(t) = \frac{-a(t)}{h(t) \min\left(\int_0^t \frac{ds}{h(s)}, \int_t^{+\infty} \frac{ds}{h(s)}\right)} - a'(t)$$

and

$$L^1_{\gamma^{-1}}(\mathbb{R}^+, \mathbb{R}) = \left\{ u : \mathbb{R}^+ \to \mathbb{R} \text{ such that } \int_0^{+\infty} |u(s)| \gamma^{-1}(s) \, ds < \infty \right\}$$

and consider the Banach space $E_0 = \{u \in C(\mathbb{R}^+), \ \lim_{t \to \infty} u(t) = 0\}$ equipped with its sup-norm $\|.\|_0$ and the cone

$$K_0 = \{u \in E_0, \ u(0) = 0 \text{ and } u \geq \gamma \|u\|_0\}.$$

Let $\tilde{T}_r : K_0 \to E_0$ be an operator defined by

$$\tilde{T}_r u(t) = \int_0^{+\infty} J(t,s)\rho(u-w)\,ds + \int_0^{+\infty} G(t,s)D_0(s,u)ds$$

where for $t, s \in \mathbb{R}^+$, $x > 0$,

$$J(t,s) = -\frac{\partial}{\partial s}(G(t,s).a(s))$$

and

$$D_0(s,x) = b(s)\rho^\alpha (x-w)\left(1 - \rho^\beta (x-w)\right) + F(s, \rho(x-w)) + q(s).$$

Lemma 4.4. *Assume that the condition (22) holds true.*
The function $J : \mathbb{R}^+ \times \mathbb{R}^+$ is continuous and verifies for all $x, t \in \mathbb{R}^+$,

$$|J(x,t)| \leq \mu(t)G(x,t),$$

and

$$\frac{J(t,s)}{G(t,s)} \geq c(t)$$

where

$$\mu(t) = \frac{1}{h(t)} + |a'(t)|.$$

Moreover, for all $u \in C^1(\mathbb{R}^+) \cap K_0$, $\tilde{T}_r u = T_r u$.

Proof. Beacause that a, a', $\frac{1}{h} \in L^1(\mathbb{R}^+)$, J is continuous. Moreover, for $x, t \in \mathbb{R}^+ \backslash \{0\}$

$$J(x,t) = -\frac{m+1}{\int_0^{+\infty} \frac{ds}{h(s)}} \begin{cases} \int_x^{+\infty} \frac{ds}{h(s)}\left[\frac{a(t)}{h(t)} + a'(t)\int_0^t \frac{ds}{h(s)}\right] & \text{if } x \geq t \\ \int_0^x \frac{ds}{h(s)}\left[-\frac{a(t)}{h(t)} + a'(t)\int_t^{+\infty} \frac{ds}{h(s)}\right] & \text{if } x \leq t \end{cases}$$

leading to

$$|J(x,t)| \leq \frac{m+1}{\int_0^{+\infty} \frac{ds}{h(s)}} \begin{cases} \int_x^{+\infty} \frac{ds}{h(s)}\left[\frac{a(t)}{h(t)} + |a'(t)|\int_0^t \frac{ds}{h(s)}\right] & \text{if } x \geq t \\ \int_0^x \frac{ds}{h(s)}\left[\frac{a(t)}{h(t)} + |a'(t)|\int_t^{+\infty} \frac{ds}{h(s)}\right] & \text{if } x \leq t \end{cases}.$$

We have from the condition (22) that $a(t) \leq \min\left(\int_0^t \frac{ds}{h(s)}, \int_t^{+\infty} \frac{ds}{h(s)}\right)$, then

$$\begin{aligned} |J(x,t)| &\leq \frac{m+1}{\int_0^{+\infty} \frac{ds}{h(s)}} \begin{cases} \int_x^{+\infty} \frac{ds}{h(s)} \int_0^t \frac{ds}{h(s)}\left[\frac{1}{h(t)} + |a'(t)|\right] & \text{if } x \geq t \\ \int_0^x \frac{ds}{h(s)} \int_t^{+\infty} \frac{ds}{h(s)}\left[\frac{1}{h(t)} + |a'(t)|\right] & \text{if } x \leq t \end{cases} \\ &= G(x,t)\left[\frac{1}{h(t)} + |a'(t)|\right] \end{aligned}$$

and

$$\frac{J(x,t)}{G(x,t)} = \begin{cases} \left[-\frac{a(t)}{h(t)\int_0^t \frac{ds}{h(s)}} - a'(t) \right] & \text{if } x \geq t \\ \left[\frac{a(t)}{h(t)\int_t^{+\infty} \frac{ds}{h(s)}} - a'(t) \right] & \text{if } x \leq t \end{cases}$$

$$\geq -\frac{a(t)}{h(t)\min\left(\int_0^t \frac{ds}{h(s)}, \int_t^{+\infty} \frac{ds}{h(s)}\right)} - a'(t).$$

Now, let $u \in C^1(\mathbb{R}^+) \cap K_0$. By integrating by parts we obtain

$$\int_0^{+\infty} G(t,s)a(s)\left(\rho(u-w)\right)' ds = [G(t,s)a(s)\rho(u-w)]_0^\infty - \int_0^{+\infty} \frac{\partial}{\partial s}\left(G(t,s)a(s)\right)\rho(u-w)\, ds$$

$$= \int_0^{+\infty} J(t,s)\rho(u-w)\, ds,$$

then

$$\tilde{T}_r u(t) = \int_0^{+\infty} J(t,s)\rho(u-w)\, ds + \int_0^{+\infty} G(t,s)D_0(s,u)ds$$

$$= \int_0^{+\infty} G(t,s)a(s)\left(\rho(u-w)\right)' ds + \int_0^{+\infty} G(t,s)D_0(s,u)ds$$

$$= \int_0^{+\infty} G(t,s)D(s,u,u')ds$$

hence $\tilde{T}_r u = T_r u$. $\qquad \square \qquad\qquad\qquad \square$

Theorem 4.5. *Assume that Hypothesis (4), (10) and (22) hold true.*
If there exist $r \geq w^* + 1$, $\lambda > 0$ and $\sigma \in L^1_{\gamma^{-1}}(\mathbb{R}^+, \mathbb{R}^+)\setminus\{0\}$ such that for all
$(t,x) \in \mathbb{R}^+\setminus\{0\} \times [\gamma(t)r, r]$

$$|F(t,\rho(x)) - F(t,\rho(y))| \leq \lambda\sigma(t)|x-y|, \tag{23}$$

and

$$\lambda < \frac{1}{r(L_\delta)} \tag{24}$$

where

$$\delta = \gamma^{-1}(s)\left(\frac{\mu(s) + 2b(s)}{\lambda} + \sigma(s)\right)$$

then bvp (1) admits a unique positive solution in $\sum_r$.

Proof. The case $c = 0$ is obvious, so we suppose that $c > 0$:
    Existence. From Hypothesis (10) and Lemma (3.10) we have that there
exists $R > r$ such that for all $u \in C$, if $\|u\| \geq R$ then

$$\|T_r u\|_0 > \|u\|_0.$$

Let $v \in C$ such that $\|v\|_0 \geq R$ and consider the sequence $(u_n)_n$ defined by

$$
\begin{cases}
u_{n+1} = \tilde{T}_r u_n \\
u_0 = v.
\end{cases}
$$

For all $n \geq 1$, $u_n \in C^1(\mathbb{R}^+)$, then $u_{n+1} = \tilde{T}_r u_n = T_r u_n \in C$ and since $\|v\| \geq \|v\|_0 \geq R$, we have

$$
\|u_{n+1}\|_0 = \|T_r u_n\|_0 > \|u_n\|_0 \geq R > r, \ \forall n \geq 1
$$

.

then $(u_n)_n \subset K_0 \backslash B(0, r)$.

Now, by (23) we have for all $n \geq 1$

$$
\begin{aligned}
|u_{n+1} - u_n| &= \left| \tilde{T}_r u_n - \tilde{T}_r u_{n-1} \right| \leq \int_0^{+\infty} G(t, s) \mu(s) \left| \rho(u_n - w) - \rho(u_{n-1} - w) \right| ds \quad (25) \\
&+ \int_0^{+\infty} G(t, s) \left| D_0(s, u_n) - D_0(s, u_{n-1}) \right| ds. \quad (26)
\end{aligned}
$$

Let $v_n = u_n - w$, $n \geq 1$. We have

$$
\begin{aligned}
|\rho(v_n) - \rho(v_{n-1})| &\leq \frac{1}{m+1} (\gamma r - \tilde{\gamma} w^*)^{\frac{-m}{m+1}} |v_n - v_{n-1}| \\
&\leq \frac{\gamma^{-1}}{m+1} \left(r - \frac{1}{2} w^* \right)^{\frac{-m}{m+1}} |v_n - v_{n-1}| \\
&\leq \gamma^{-1} |v_n - v_{n-1}| \\
|\rho^\alpha(v_n) - \rho^\alpha(v_{n-1})| &\leq \frac{\alpha}{m+1} \left(r - \frac{1}{2} w^* \right)^{\left(\frac{\alpha}{m+1} - 1 \right)} |v_n - v_{n-1}| \\
&\leq \gamma^{-1} |v_n - v_{n-1}| \\
|\rho^\alpha(v_n) - \rho^\alpha(v_{n-1})| &\leq \frac{\alpha + \beta}{m+1} \left(r - \frac{1}{2} w^* \right)^{\left(\frac{\alpha+\beta}{m+1} - 1 \right)} |v_n - v_{n-1}| \\
&\leq \gamma^{-1} |v_n - v_{n-1}|
\end{aligned}
$$

leading to

$$
\begin{aligned}
|v_{n+1} - v_n| &\leq \int_0^{+\infty} G(t, s) (\mu(s) + 2b(s) + \lambda \sigma(s)) |v_n - v_{n-1}| ds \quad (27) \\
&= \lambda L_\delta (|v_n - v_{n-1}|)
\end{aligned}
$$

Then, for all $n \geq 0$

$$
|u_{n+1} - u_n| \leq \lambda^n L_\delta |u_1 - u_0|.
$$

Therefore, for $m > n \geq 1$,

$$
\begin{aligned}
|u_m - u_n| &\leq |u_m - u_{m-1}| + |u_{m-1} - u_{m-2}| + ... + |u_{n+1} - u_n| \\
&\leq \lambda^{m-1} L_\delta{}^{m-1} |u_1 - u_0| + \lambda^{m-2} L_\delta{}^{m-2} |u_1 - u_0| + ... + \lambda^n L_\delta^n |u_1 - u_0|
\end{aligned}
$$

then

$$
\begin{aligned}
\|u_m - u_n\|_0 &\leq \lambda^{m-1} \left\| L_\delta^{m-1} w \right\|_0 + \lambda^{m-2} \left\| L_\delta^{m-2} w \right\|_0 + ... + \lambda^n \left\| L_\delta^n w \right\|_0 \\
&= S_{m-1} - S_{n-1},
\end{aligned}
$$

where

$$
S_n = \sum_{n=0}^{n=+\infty} c^n \left\| L_\delta^n w \right\|_0, \text{ with } w = |u_1 - u_0|.
$$

Since $\lambda < (r(L_\delta))^{-1}$, we have that

$$
\lim_{n \to \infty} \sqrt[n]{c^n \left\| L_\delta^n w \right\|_0} \leq c \lim_{n \to \infty} \sqrt[n]{\left\| L_\delta^n \right\|} = c.r(L_\delta) < 1,
$$

then $(S_n)_n$ converges and

$$
\lim_{n \to \infty} \|u_m - u_n\|_0 = \lim_{n \to \infty} S_{m-1} - S_{n-1} = 0.
$$

Therefore, the sequence $(u_n)_n$ is also a cauchy sequence and the completeness of E leads to $\lim_{n \to \infty} u_n = u \in C$, with

$$
\|u\|_0 \geq r > 0.
$$

At the end, passing to the limit in $u_{n+1} = \tilde{T}_r u_n$, and by continuity of T_r in $C \backslash B(0, r)$, we obtain

$$
u = \tilde{T}_r u = T_r u
$$

and u is a fixed point of T_r in $C \backslash B(0, r)$, which means that $v = (u - w)^{\frac{1}{m+1}}$ is a positive solution of bvp (1) in $\sum_r$.

Uniqueness. If $u_1, u_2 \in K_0 \backslash B(0, r)$ are two solutions of (1) with $u_1 \neq u_2$, then u_1, u_2 are fixed points of T_r.

Then $u_1, u_2 \in C^1(\mathbb{R}^+)$ and by Lemma (4.4) we have that u_1, u_2 are fixed points of $\tilde{T}_r$.

For all $t \in \mathbb{R}$,

$$
u_i(t) \geq \gamma(t) . r, \text{ for } i \in \{1, 2\}
$$

and so

$$
|u_1 - u_2| = \left| \tilde{T}_r u_1 - \tilde{T}_r u_2 \right| \leq \int_0^{+\infty} G(t,s)\mu(s) \left| \rho(u_1 - w) - \rho(u_2 - w) \right| ds \tag{28}
$$

$$
+ \int_0^{+\infty} G(t,s) \left| D_0(s, u_1) - D_0(s, u_2) \right| ds. \tag{29}
$$

Let $v_i = u_i - w$, $i \in \{1, 2\}$. We have

$$
\begin{aligned}
|\rho(v_1) - \rho(v_2)| &\leq \frac{1}{m+1}(\gamma r - \tilde{\gamma}w^*)^{\frac{-m}{m+1}}|v_1 - v_2| \\
&\leq \frac{\gamma^{-1}}{m+1}\left(r - \frac{1}{2}w^*\right)^{\frac{-m}{m+1}}|v_1 - v_2| \\
&\leq \gamma^{-1}|v_1 - v_2| \\
|\rho^\alpha(v_1) - \rho^\alpha(v_2)| &\leq \frac{\alpha}{m+1}(\gamma r - \tilde{\gamma}w^*)^{\left(\frac{\alpha}{m+1}-1\right)}|v_1 - v_2| \\
&\leq \gamma^{-1}|v_1 - v_2| \\
\left|\rho^{\alpha+\beta}(v_1) - \rho^{\alpha+\beta}(v_2)\right| &\leq \frac{\alpha+\beta}{m+1}(\gamma r - \tilde{\gamma}w^*)^{\left(\frac{\alpha+\beta}{m+1}-1\right)}|v_1 - v_2| \\
&\leq \gamma^{-1}|v_1 - v_2|
\end{aligned}
$$

leading to

$$
\begin{aligned}
|v_1 - v_2| &\leq \int_0^{+\infty} G(t,s)\gamma^{-1}(s)(\mu(s) + 2b(s) + \lambda\sigma(s))|v_1 - v_2|\,ds \quad (30) \\
&= \lambda L_\delta(|v_1 - v_2|)
\end{aligned}
$$

where L_δ is the operator defined in Lemma (3.9) by

$$
L_\delta u(t) = \int_0^{+\infty} G(t,s)\delta(s)u(s)ds
$$

where

$$
\delta(s) = \gamma^{-1}(s)\left(\frac{\mu(s) + 2b(s)}{\lambda} + \sigma(s)\right)
$$

and let $w = L|u_1 - u_2|$. The inequality (30) leads

$$
w \leq \lambda L_\delta w.
$$

As $|u_1 - u_2| \in E^+ \backslash \{0\}$, then by Lemma (3.9) we have $w \in K_0 \backslash \{0\}$. Then by Theorem (2.2) we deduce that L_δ admits a positive eigenvalue λ_0 such that

$$
\lambda_0 \geq \lambda^{-1}
$$

leading to the following contradiction

$$
r(L) \geq \lambda_0 \geq \lambda^{-1} > r(L).
$$

This completes the proof.   □   □

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