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LP-SASAKIAN MANIFOLDS EQUIPPED WITH ZAMKOVOY CONNECTION AND CONHARMONIC CURVATURE TENSOR

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Abstract. The paper concerns with some results on conharmonically flat, quasi-conharmonically flat and $\phi-$conharmonically flat LP-Sasakian manifolds with respect to Zamkovoy connection. Also, it contains study of generalized conharmonic $\phi-$recurrent LP-Sasakian manifolds with respect to Zamkovoy connection. Moreover, the paper deals with LP-Sasakian manifolds satisfying $\mathcal{K}^*(\xi, U).R^* = 0$, where $\mathcal{K}^*$ denotes conharmonic curvature tensor and R^* denotes Riemannian curvature tensor with respect to Zamkovoy connection, respectively.

Key words and Phrases: LP-Sasakian manifold, Zamkovoy connection, Conharmonic curvature tensor

1. Introduction

In 1989, K. Matsumoto [13] first introduced the notion of Lorentzian para-Sasakian manifolds (briefly, LP-Sasakian manifolds). Also, in 1992, I. Mihai and R. Rosca [14] introduced independently the notion of Lorentzian para-Sasakian manifolds in classical analysis. The generalized recurrent manifolds was introduced by Dubey [8] and it was studied by De and Guha et al. [6]. In this context, $\phi-$recurrent LP-Sasakian manifold was first studied by A. A. Shaikh, D. G. Prakasha and Helaluddin Ahmad [15]. On the other hand, $\phi-$conharmonically flat LP-Sasakian manifold was introduced by A. Taleshian [16]. Apart from these, the properties of LP-Sasakian manifolds were studied by several authors, namely U. C. De [7], C. Ozgur [17] and many others.

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In 2008, a new non-metric canonical connection on para contact manifold was introduced by S. Zamkovoy [18]. This connection named as Zamkovoy connection was further studied in Sasakian manifolds, LP-Sasakian manifolds and para-Kenmotsu manifolds by several researcher et al. ([3], [1], [2], [10], [11], [12], [5]). Zamkovoy connection ∇^* for an n-dimensional almost contact metric manifold [4] M equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a $1-$form η and a Riemannian metric g is given by

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y, \tag{1}$$

for all $X, Y \in \chi(M)$, where ∇ is the Levi-Civita connection and $\chi(M)$ is the set of all vector fields on M.

In 1957, Y. Ishii [9] first studied the notion of a conharmonic curvature tensor. A rank three tensor $\mathcal{K}$, that remains invariant under conharmonic transformation for an $n-$dimensional Riemannian manifold M is given by

$$\begin{aligned} \mathcal{K}(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y] \\ &\quad -\frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{2}$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the set of all vector fields of the manifold M and R denotes the Riemannian curvature tensor of type $(1,3)$, S denotes the Ricci tensor of type $(0,2)$, Q is the Ricci operator.

The conharmonic curvature tensor($\mathcal{K}^*$) with respect to Zamkovoy connection is given by

$$\begin{aligned} \mathcal{K}^*(X,Y)Z &= R^*(X,Y)Z - \frac{1}{n-2}[S^*(Y,Z)X - S^*(X,Z)Y] \\ &\quad -\frac{1}{n-2}[g(Y,Z)Q^*X - g(X,Z)Q^*Y], \end{aligned} \tag{3}$$

for all $X, Y, Z \in \chi(M)$, where R^*, S^* and Q^* are Riemannian curvature tensor, Ricci tensor and Ricci operator with respect to Zamkovoy connection, respectively.

Definition 1.1. *An n-dimensional LP-Sasakian manifold M is said to be generalized $\eta-$Einstein manifold if the Ricci tensor of type $(0,2)$ is of the form*

$$S(Y,Z) = k_1 g(Y,Z) + k_2\eta(Y)\eta(Z) + k_3\omega(Y,Z), \tag{4}$$

for all $Y, Z \in \chi(M)$, where k_1, k_2 and k_3 are scalars and ω is a $2-$form.

Definition 1.2. *An n-dimensional LP-Sasakian manifold M is said to be conharmonically flat with respect to Zamkovoy connection if $\mathcal{K}^*(X,Y)Z = 0$, for all $X, Y, Z \in \chi(M)$.*

Definition 1.3. *An n-dimensional LP-Sasakian manifold M is said to be $\xi-$ conharmonically flat with respect to Zamkovoy connection if $\mathcal{K}(X,Y)\xi = 0$, for all $X, Y, Z \in \chi(M)$.*

Definition 1.4. *An $n-$dimensional LP-Sasakian manifold M is said to be generalized conharmonic $\phi-$recurrent with respect to Zamkovoy connection if*

$$\begin{aligned} \phi^2 \left(\nabla^*_W \mathcal{K}^*\right)(X,Y)Z &= A(W)K(X,Y)Z \\ &\quad + B(W)\left[g(Y,Z)X - g(X,Z)Y\right], \end{aligned}$$

for all $X,Y,Z,W \in \chi(M)$, where A and B are $1-$forms and B is non vanishing such that $A(W) = g(W,\rho_1), B(W) = g(W,\rho_2)$ and ρ_1, ρ_2 are vector fields associated with $1-$forms A and B, respectively.

This paper is structured as follows:

After introduction, a short description of LP-Sasakian manifold has been given in section (2). In section (3), we have obtained Riemannian curvature tensor R^*, Ricci tensor S^*, scalar curvature r^* with respect to Zamkovoy connection in LP-Sasakian manifold. Section (4) contains conharmonically flat and $\xi-$conharmonically flat LP-Sasakian manifolds with respect to Zamkovoy connection. In section (5), we have discussed quasi-conharmonically flat LP-Sasakian manifold with respect to Zamkovoy connection. Section (6) contains $\phi-$ conharmonically flat LP-Sasakian manifold with respect to ∇^*. Section (7) concerns with a generalized conharmonic $\phi-$recurrent LP-Sasakian manifold with respect to ∇^*. In section (8), we have discussed an LP-Sasakian manifold satisfying $\mathcal{K}^*(\xi,U).R^* = 0$.

2. Preliminaries

An n-dimensional differentiable manifold is called an LP-Sasakian manifold if it admits a $(1,1)$ tensor field ϕ, a vector field ξ, a $1-$form η and a Lorentzian metric g which satisfies:

$$\phi^2 Y = Y + \eta(Y)\xi, \eta(\xi) = -1, \eta(\phi X) = 0, \ \phi\xi = 0, \tag{5}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{6}$$

$$g(X, \phi Y) = g(\phi X, Y), \eta(Y) = g(Y,\xi), \tag{7}$$

$$\nabla_X \xi = \phi X, \quad g(X,\xi) = \eta(X), \tag{8}$$

$$(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{9}$$

for all $X, Y \in \chi(M)$,where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Let us introduced a symmetric $(0,2)$ tensor field ω such that

$$\omega(X,Y) = g(X,\phi Y). \tag{10}$$

Also, since the vector field η is closed in LP-Sasakian manifold M, we have

$$(\nabla_X \eta)Y = \omega(X,Y), \omega(X,\xi) = 0, \tag{11}$$

for all $X, Y \in \chi(M)$.

In LP-Sasakian manifold the following relations also hold:

$$\begin{aligned}\eta(R(X,Y)Z) &= g(Y,Z)\eta(X) - g(X,Z)\eta(Y), &(12)\\ R(X,Y)\xi &= \eta(Y)X - \eta(X)Y, &(13)\\ R(\xi,Y)Z &= g(Y,Z)\xi - \eta(Z)Y, &(14)\\ R(\xi,Y)\xi &= \eta(Y)\xi + Y, &(15)\\ S(X,\xi) &= (n-1)\eta(X), &(16)\\ S(\phi X,\phi Y) &= S(X,Y) + (n-1)\eta(X)\eta(Y), &(17)\end{aligned}$$

$$Q\xi = (n-1)\xi, Q\phi = \phi Q, S(X,Y) = g(QX,Y), S^2(X,Y) = S(QX,Y). \quad (18)$$

Lemma 2.1. *The relation between Zamkovoy connection and Levi-Civita connection in an LP-Sasakian manifold is given by*

$$\nabla^*_X Y = \nabla_X Y + g(X,\phi Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y, \quad (19)$$

where the torsion tensor of Zamkovoy connection is

$$T^*(X,Y) = 2[\eta(X)\phi Y - \eta(Y)\phi X]. \quad (20)$$

Proof. In view of (1) and (11), we have

$$(\nabla^*_X g)(Y,Z) = -2g(Y,\phi Z)\eta(X). \quad (21)$$

Suppose that the Zamkovoy connection ∇^* defined on an $n-$ dimensional LP-Sasakian manifold M is connected with the Levi-Civita connection ∇ by the relation

$$\nabla^*_X Y = \nabla_X Y + P(X,Y), \quad (22)$$

where $P(X,Y)$ is a tensor field of type $(1,1)$. Then by definition of torsion tensor, we have

$$T^*(X,Y) = P(X,Y) - P(Y,X). \quad (23)$$

Zamkovoy connection is a non-metric connection and hence from (22), we get

$$\begin{aligned}g(P(X,Y),Z) + g(P(X,Z),Y) &= 2g(Y,\phi Z)\eta(X), &(24)\\ g(P(Y,X),Z) + g(P(Y,Z),X) &= 2g(X,\phi Z)\eta(Y), &(25)\\ g(P(Z,X),Y) + g(P(Z,Y),X) &= 2g(X,\phi Y)\eta(Z). &(26)\end{aligned}$$

In view of (24), (25), (26) and (23), we have

$$\begin{aligned}& g(T^*(X,Y),Z) + g(T^*(Z,X),Y) + g(T^*(Z,Y),X)\\ = \ & g(P(X,Y),Z) - g(P(Y,X),Z) + g(P(Z,X),Y)\\ & -g(P(X,Z),Y) + g(P(Z,Y),X) - g(P(Y,Z),X)\\ = \ & 2g(P(X,Y),Z) - 2g(Y,\phi Z)\eta(X)\\ & -2g(X,\phi Z)\eta(Y) + 2g(X,\phi Y)\eta(Z). \quad (27)\end{aligned}$$

Setting

$$\begin{aligned}g(T^*(Z,X),Y) &= g\left(\overline{T}(X,Y),Z\right), &(28)\\ g(T^*(Z,Y),X) &= g\left(\overline{T}(Y,X),Z\right), &(29)\end{aligned}$$

in (27), we have

$$\begin{aligned} & g\left(T^{*}\left(X,Y\right),Z\right)+g\left(\overline{T}\left(X,Y\right),Z\right)+g\left(\overline{T}\left(Y,X\right),Z\right) \\ = \; & 2g\left(P\left(X,Y\right),Z\right)-2g\left(Y,\phi Z\right)\eta\left(X\right) \\ & -2g\left(X,\phi Z\right)\eta\left(Y\right)+2g\left(X,\phi Y\right)\eta\left(Z\right), \end{aligned} \tag{30}$$

which implies that

$$\begin{aligned} P\left(X,Y\right) &= \frac{1}{2}\left[T^{*}\left(X,Y\right)+\overline{T}\left(X,Y\right)+\overline{T}\left(Y,X\right)\right] \\ & +\eta\left(X\right)\phi Y+\eta\left(Y\right)\phi X-g\left(X,\phi Y\right)\xi. \end{aligned} \tag{31}$$

In reference to (20), (28) and (29), we have

$$\overline{T}\left(X,Y\right) = 2g\left(X,\phi Y\right)\xi-2\eta\left(X\right)\phi Y, \tag{32}$$

$$\overline{T}\left(Y,X\right) = 2g\left(X,\phi Y\right)\xi-2\eta\left(Y\right)\phi X. \tag{33}$$

Using (20), (32) and (33) in (31), we obtain

$$P\left(X,Y\right)=g\left(X,\phi Y\right)\xi-\eta\left(Y\right)\phi X+\eta\left(X\right)\phi Y. \tag{34}$$

In reference to (22) and (34), we can easily bring out the equation (19). □

From the equation (19), it is obvious that

$$\nabla^{*}_{X}\xi=2\phi X. \tag{35}$$

Proposition 2.2. *The Zamkovoy connection on an $n-$dimensional LP-Sasakian manifold is a non-metric linear connection with torsion tensor given by equation (20).*

3. Some properties of LP-Sasakian manifold with respect to Zamkovoy connection

Let R^{*} be the Riemannian curvature tensor with respect to Zamkovoy connection and it be defined as

$$R^{*}\left(X,Y\right)Z=\nabla^{*}_{X}\nabla^{*}_{Y}Z-\nabla^{*}_{Y}\nabla^{*}_{X}Z-\nabla^{*}_{[X,Y]}Z. \tag{36}$$

Using (5), (8), (9) and (19) in (36), we get the Riemannian curvature R^{*} with respect to Zamkovoy connection as

$$\begin{aligned} R^{*}\left(X,Y\right)Z &= R\left(X,Y\right)Z+3g\left(X,Z\right)\eta\left(Y\right)\xi \\ & -3g\left(Y,Z\right)\eta\left(X\right)\xi+3g\left(Y,\phi Z\right)\phi X-3g\left(X,\phi Z\right)\phi Y \\ & -\eta\left(X\right)\eta\left(Z\right)Y+\eta\left(Y\right)\eta\left(Z\right)X. \end{aligned} \tag{37}$$

Consequently, one can easily bring out the followings:

$$S^{*}\left(Y,Z\right) = S\left(Y,Z\right)+\left(n-1\right)\eta\left(Y\right)\eta\left(Z\right)+3\psi g\left(Y,\phi Z\right), \tag{38}$$

$$S^{*}\left(\xi,Z\right) = S^{*}\left(Z,\xi\right)=0, \tag{39}$$

$$Q^{*}Y = QY+\left(n-1\right)\eta\left(Y\right)\xi+3\psi\phi Y, \tag{40}$$

$$Q^{*}\xi = 0, \tag{41}$$

$$r^{*} = r-n+1+3\psi^{2}, \tag{42}$$

$$R^*(X,Y)\xi = 0, \tag{43}$$
$$R^*(\xi,Y)Z = 4g(\phi Y,\phi Z)\xi, \tag{44}$$
$$R^*(X,\xi)Z = -4g(\phi X,\phi Z)\xi, \tag{45}$$

for all $X, Y, Z \in \chi(M)$, where $\psi = trace(\phi)$.

Proposition 3.1. *Let M be an n-dimensional LP-Sasakian manifold admitting Zamkovoy connection ∇^*, then*

(i) The curvature tensor R^* of ∇^* is given by (37),
(ii) The Ricci tensor S^* of ∇^* is given by (38),
(iii) The scalar curvature r^* of ∇^* is given by (42),
(iv) The Ricci tensor S^* of ∇^* is symmetric,
(v) R^* satisfies: $R^*(X,Y)Z + R^*(Y,Z)X + R^*(Z,X)Y = 0$.

4. Conharmonically flat and ξ−conharmonically flat LP-Sasakian manifolds with respect to Zamkovoy connection

Theorem 4.1. *If an $n-$dimensional LP-Sasakian manifold $M\,(n > 2)$ is conharmonically flat with respect to Zamkovoy connection, then the scalar curvature is given by $r = n - 1 - 3\psi^2$.*

Proof. In view of (2) and (3), we have

$$\begin{aligned}
&\mathcal{K}^*(X,Y)Z \\
= \;& \mathcal{K}(X,Y)Z + 3g(X,Z)\eta(Y)\xi - 3g(Y,Z)\eta(X)\xi \\
&+3g(Y,\phi Z)\phi X - 3g(X,\phi Z)\phi Y - \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X \\
&-\frac{n-1}{n-2}\left[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi\right] \\
&-\frac{3\psi}{n-2}\left[g(Y,Z)\phi X - g(X,Z)\phi Y\right] \\
&-\frac{n-1}{n-2}\left[\eta(Y)X - \eta(X)Y\right]\eta(Z) \\
&-\frac{3\psi}{n-2}\left[g(Y,\phi Z)X - 3\psi g(X,\phi Z)Y\right].
\end{aligned} \tag{46}$$

Let us consider an LP-Sasakian manifold M which is conharmonically flat with respect to Zamkovoy connection, then from (3), we have

$$\begin{aligned}
R^*(X,Y)Z = \;& \frac{1}{n-2}\left[S^*(Y,Z)X - S^*(X,Z)Y\right] \\
&+\frac{1}{n-2}\left[g(Y,Z)Q^*X + g(X,Z)Q^*Y\right].
\end{aligned} \tag{47}$$

Taking inner product of (47) with a vector field V, we get

$$\begin{aligned} & g\left(R^*\left(X,Y\right)Z,V\right) \\ = & \frac{1}{n-2}\left[S^*\left(Y,Z\right)g\left(X,V\right)-S^*\left(X,Z\right)g\left(Y,V\right)\right] \\ & +\frac{1}{n-2}\left[g\left(Y,Z\right)S^*\left(X,V\right)-g\left(X,Z\right)S^*\left(Y,V\right)\right]. \end{aligned} \tag{48}$$

Taking an orthonormal frame field of M and contracting (48) over X and V, we obtain

$$r = n-1-3\psi^2.$$

This gives the theorem. □

Corollary 4.2. *If an LP-Sasakian manifold is conharmonically flat with respect to Zamkovoy connection, then its scalar curvature is constant, provided that trace* $(\phi) = 0$.

Theorem 4.3. *An $n-$dimensional LP-Sasakian manifold $(n>2)$ is $\xi-$ conharmonically flat with respect to Zamkovoy connection if and only if it is so with respect to Levi-Civita connection, provided that the vector fields are horizontal vector fields.*

Proof. Setting $Z=\xi$ in (46), we have

$$\begin{aligned} & \mathcal{K}^*\left(X,Y\right)\xi \\ = & \mathcal{K}\left(X,Y\right)\xi+\frac{1}{n-2}\left[\eta\left(Y\right)X-\eta\left(X\right)Y\right] \\ & -\frac{3\psi}{n-2}\left[\eta\left(Y\right)\phi X-\eta\left(X\right)\phi Y\right] \\ = & \mathcal{K}\left(X,Y\right)\xi, \quad \textit{if } X,Y \textit{ are horizontal vector fields on } M. \end{aligned} \tag{49}$$

This gives the theorem. □

Theorem 4.4. *If an $n-$dimensional LP-Sasakian manifold $(n>2)$ is $\xi-$ conharmonically flat with respect to Zamkovoy connection, then its scalar curvature with respect to Zamkovoy connection vanishes.*

Proof. Setting $Z=\xi$ in (3), we have

$$\mathcal{K}^*\left(X,Y\right)\xi = \frac{1}{n-2}\left[\eta\left(Y\right)Q^*X-\eta\left(X\right)Q^*Y\right]. \tag{50}$$

If M is $\xi-$conharmonically flat with respect to Zamkovoy connection, then it follows from (50) that

$$0=\eta\left(Y\right)Q^*X-\eta\left(X\right)Q^*Y. \tag{51}$$

Taking inner product of (51) with a vector field V, we obtain

$$0=\eta\left(Y\right)S^*\left(X,V\right)-\eta\left(X\right)S^*\left(Y,V\right). \tag{52}$$

Setting $Z=\xi$ in (52)

$$S^*\left(X,V\right)=0. \tag{53}$$

Taking an orthonormal frame field of M and contracting (53) over X and V, we get

$$r^{*}=0.$$

This gives the theorem. □

5. Quasi-conharmonically flat LP-Sasakian manifold with respect to Zamkovoy connection.

Theorem 5.1. *If an $n-$dimensional LP-Sasakian manifold M $(n>2)$ is quasi-conharmonically flat with respect to Zamkovoy connection, then its scalar curvature with respect to Zamkovoy connection vanishes.*

Proof. Let us consider an LP-Sasakian manifold M which is quasi-conharmonically flat with respect to Zamkovoy connection, i.e.,

$$g\left(\mathcal{K}^{*}\left(\phi X,Y\right)Z,\phi V\right)=0, \tag{54}$$

for all $X,Y,Z,V\in\chi\left(M\right)$.

Then, in view of (3), we have

$$\begin{aligned} & g\left(R^{*}\left(\phi X,Y\right)Z,\phi V\right) \\ = & \frac{1}{n-2}\left[S^{*}\left(Y,Z\right)g\left(\phi X,\phi V\right)-S^{*}\left(\phi X,Z\right)g\left(Y,\phi V\right)\right] \\ & +\frac{1}{n-2}\left[g\left(Y,Z\right)S^{*}\left(\phi X,\phi V\right)-g\left(\phi X,Z\right)S^{*}\left(Y,\phi V\right)\right]. \end{aligned} \tag{55}$$

Let $\{e_i\}\,(1\le i\le n)$ be an orthonormal basis of the tangent space at any point of the manifold M. Setting $Y=Z=e_i$ in the equation (55) and taking summation over $i\,(1\le i\le n)$, we get

$$\begin{aligned} & \sum_{i=1}^{n}g\left(R^{*}\left(\phi X,e_i\right)e_i,\phi V\right) \\ = & \frac{1}{n-2}\left[\sum_{i=1}^{n}S^{*}\left(e_i,e_i\right)g\left(\phi X,\phi V\right)-\sum_{i=1}^{n}S^{*}\left(\phi X,e_i\right)g\left(e_i,\phi V\right)\right] \\ & +\frac{1}{n-2}\left[\sum_{i=1}^{n}g\left(e_i,e_i\right)S^{*}\left(\phi X,\phi V\right)-\sum_{i=1}^{n}g\left(\phi X,e_i\right)S^{*}\left(e_i,\phi V\right)\right]. \end{aligned} \tag{56}$$

It can be easily seen that

$$\sum_{i=1}^{n}g\left(e_i,e_i\right) = n, \tag{57}$$

$$\sum_{i=1}^{n}S^{*}\left(\phi X,e_i\right)g\left(e_i,\phi V\right) = S^{*}\left(\phi X,\phi V\right), \tag{58}$$

$$\sum_{i=1}^{n}S^{*}\left(e_i,e_i\right) = r^{*}. \tag{59}$$

Using (57), (58) and (59) in (56), we get

$$r^* = 0.$$

This gives the theorem. □

6. $\phi-$conharmonically flat LP-Sasakian manifold with respect to Zamkovoy connection

Theorem 6.1. *If an $n-$dimensional LP-Sasakian manifold M $(n > 2)$ is $\phi-$conharmonically flat with respect to Zamkovoy connection, then M is a generalized $\eta-$Einstein manifold.*

Proof. Let us consider an LP-Sasakian manifold M which is $\phi-$conharmonically flat with respect to Zamkovoy connection, i.e.,

$$g\left(\mathcal{K}^*\left(\phi X, \phi Y\right)\phi Z, \phi V\right) = 0, \tag{60}$$

for all $X, Y, Z, V \in \chi\left(M\right)$

Then in view of (3), we have

$$\begin{aligned} & g\left(R^*\left(\phi X, \phi Y\right)\phi Z, \phi V\right) \\ = \; & \frac{1}{n-2}\left[S^*\left(\phi Y, \phi Z\right)g\left(\phi X, \phi V\right) - S^*\left(\phi X, \phi Z\right)g\left(\phi Y, \phi V\right)\right] \\ & + \frac{1}{n-2}\left[g\left(\phi Y, \phi Z\right)S^*\left(\phi X, \phi V\right) - g\left(\phi X, \phi Z\right)S^*\left(\phi Y, \phi V\right)\right]. \end{aligned} \tag{61}$$

Let $\{e_i, \xi\}\,(1 \leq i \leq n-1)$ be a local orthonormal basis of the tangent space at any point of the manifold M. Using the fact that $\{\phi e_i, \xi\}\,(1 \leq i \leq n-1)$ is also a local orthonormal basis of the tangent space and setting $Y = Z = e_i$ and taking summation over $i(1 \leq i \leq n-1)$ it follows from (61) that

$$\begin{aligned} & \sum_{i=1}^{n-1} R^*\left(\phi X, \phi e_i, \phi e_i, \phi V\right) \\ = \; & \frac{1}{n-2}\left[\sum_{i=1}^{n-1} S^*\left(\phi e_i, \phi e_i\right)g\left(\phi X, \phi V\right) - \sum_{i=1}^{n-1} S^*\left(\phi X, \phi e_i\right)g\left(\phi e_i, \phi V\right)\right] \\ & + \frac{1}{n-2}\left[\sum_{i=1}^{n-1} g\left(\phi e_i, \phi e_i\right)S^*\left(\phi X, \phi V\right) - \sum_{i=1}^{n-1} g\left(\phi X, \phi e_i\right)S^*\left(\phi e_i, \phi V\right)\right] \end{aligned} \tag{62}$$

It can be easily seen that

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n-1, \quad (63)$$

$$\sum_{i=1}^{n-1} S^*(\phi X, \phi e_i)\, g(\phi e_i, \phi V) = S^*(\phi X, \phi V), \quad (64)$$

$$\sum_{i=1}^{n-1} S^*(\phi e_i, \phi e_i) = r^*. \quad (65)$$

Using (63), (64) and (65) in (62), we have

$$\begin{aligned} S(X,V) &= \left(r-n+1+3\psi^2\right) g(X,V) \\ &+ \left(r-2n+2+3\psi^2\right)\eta(X)\,\eta(V) - 3\psi\omega(X,V), \end{aligned} \quad (66)$$

where $\omega(X,V) = g(X,\phi V)$ and $\psi = trace(\phi)$.
Therefore M is a generalized $\eta-$Einstein manifold. □

7. Generalized conharmonic ϕ-recurrent LP-Sasakian manifold with respect to Zamkovoy connection

Theorem 7.1. *If an $n-$dimensional LP-Sasakian manifold M $(n>2)$ is generalized conharmonic $\phi-$recurrent with respect to Zamkovoy connection, then $1-$forms A and B are related as $B(W) = \left[\frac{r-n+1+3\psi^2}{(n-2)(n-1)}\right] A(W)$, where W is an arbitrary vector field on M and $\psi = trace(\phi)$.*

Proof. Let M be a generalized conharmonic $\phi-$recurrent LP-Sasakian manifold with respect to Zamkovoy connection, then

$$\begin{aligned} &\phi^2\left(\nabla^*_W \mathcal{K}^*\right)(X,Y)Z \\ = \;& A(W)\,\mathcal{K}^*(X,Y)Z + B(W)\left[g(Y,Z)X - g(X,Z)Y\right], \end{aligned} \quad (67)$$

where the $1-$forms are given by $A(W) = g(W,\rho_1)$, $B(W) = g(W,\rho_2)$, $B(W) \neq 0$ and ρ_1, ρ_2 are vector fields associated with $1-$forms A and B, respectively.

Using (5) in (67), we have

$$\begin{aligned} &\left(\nabla^*_W \mathcal{K}^*\right)(X,Y)Z \\ = \;& -\eta\left(\left(\nabla^*_W \mathcal{K}^*\right)(X,Y)Z\right)\xi A(W)\,\mathcal{K}^*(X,Y)Z \\ &+ B(W)\left[g(Y,Z)X - g(X,Z)Y\right]. \end{aligned} \quad (68)$$

The inner product of the equation (68) with vector field V gives

$$\begin{aligned} &g\left(\left(\nabla^*_W \mathcal{K}^*\right)(X,Y)Z, V\right) \\ = \;& -\eta\left(\left(\nabla^*_W \mathcal{K}^*\right)(X,Y)Z\right)\eta(V) + A(W)\,g\left(\mathcal{K}^*(X,Y)Z, V\right) \\ &+ B(W)\left[g(Y,Z)\,g(X,V) - g(X,Z)\,g(Y,V)\right]. \end{aligned} \quad (69)$$

In view of (3), it is easily seen that

$$\begin{aligned} & g\left(\left(\nabla_W^* \mathcal{K}^*\right)(X,Y)\,Z,V\right) \\ = \; & g\left(\left(\nabla_W^* R^*\right)(X,Y)\,Z,V\right) \\ & -\frac{1}{n-2}\left[\left(\nabla_W^* S^*\right)(Y,Z)\,g(X,V)-\left(\nabla_W^* S^*\right)(X,Z)\,g(Y,V)\right] \\ & -\frac{1}{n-2}\left[g(Y,Z)\left(\nabla_W^* S^*\right)(X,V)-g(X,Z)\left(\nabla_W^* S^*\right)(Y,V)\right], \end{aligned} \tag{70}$$

$$\begin{aligned} & \eta\left(\left(\nabla_W^* \mathcal{K}^*\right)(X,Y)\,Z\right) \\ = \; & g\left(\left(\nabla_W^* R^*\right)(X,Y)\,Z,\xi\right) \\ & -\frac{1}{n-2}\left[\left(\nabla_W^* S^*\right)(Y,Z)\,\eta(X)-\left(\nabla_W^* S^*\right)(X,Z)\,\eta(Y)\right], \end{aligned} \tag{71}$$

$$\begin{aligned} & g\left(\mathcal{K}^*(X,Y)\,Z,V\right) \\ = \; & g\left(R^*(X,Y)\,Z,V\right) \\ & -\frac{1}{n-2}\left[S^*(Y,Z)\,g(X,V)-S^*(X,Z)\,g(Y,V)\right] \\ & -\frac{1}{n-2}\left[g(Y,Z)\,S^*(X,V)-g(X,Z)\,S^*(Y,V)\right]. \end{aligned} \tag{72}$$

Using (70), (71) and (72) in (69), we get

$$\begin{aligned} & g\left(\left(\nabla_W^* R^*\right)(X,Y)\,Z,V\right) \\ = \; & \frac{1}{n-2}\left[\left(\nabla_W^* S^*\right)(Y,Z)\,g(X,V)-\left(\nabla_W^* S^*\right)(X,Z)\,g(Y,V)\right] \\ & +\frac{1}{n-2}\left[g(Y,Z)\left(\nabla_W^* S^*\right)(X,V)-g(X,Z)\left(\nabla_W^* S^*\right)(Y,V)\right] \\ & +\frac{1}{n-2}\left[\left(\nabla_W^* S^*\right)(Y,Z)\,\eta(X)-\left(\nabla_W^* S^*\right)(X,Z)\,\eta(Y)\right]\eta(V) \\ & +g\left(R^*(X,Y)\,Z,V\right)A(W)-g\left(\left(\nabla_W^* R^*\right)(X,Y)\,Z,\xi\right)\eta(V) \\ & -\frac{1}{n-2}\left[S^*(Y,Z)\,g(X,V)-S^*(X,Z)\,g(Y,V)\right]A(W) \\ & -\frac{1}{n-2}\left[g(Y,Z)\,S^*(X,V)-g(X,Z)\,S^*(Y,V)\right]A(W) \\ & +\left[g(Y,Z)\,g(X,V)-g(X,Z)\,g(Y,V)\right]B(W). \end{aligned} \tag{73}$$

Taking an orthonormal frame field of M and contracting (73) over Y and Z, we get

$$\begin{aligned}
& \left(\nabla_W^* S^*\right)(X,V) \\
= \; & \frac{1}{n-2}\left[\nabla_W^* r^* g(X,V) - \left(\nabla_W^* S^*\right)(X,V)\right] \\
& + \frac{1}{n-2}\left[n\left(\nabla_W^* S^*\right)(X,V) - \left(\nabla_W^* S^*\right)(X,V)\right] - g\left(\nabla_W^* S^*\right)(X,\xi)\,\eta(V) \\
& + \frac{1}{n-2}\left[\nabla_W^* r^* \eta(X)\,\eta(V) - \left(\nabla_W^* S^*\right)(X,\xi)\,\eta(V)\right] \\
& + \left(\nabla_W^* S^*\right)(X,V)\,A(W) - \frac{1}{n-2}\left[r^* g(X,V) - S^*(X,V)\right]A(W) \\
& - \frac{n-1}{n-2} S^*(X,V)\,A(W) + (n-1)\,g(X,V)\,B(W). \qquad (74)
\end{aligned}$$

Setting $V = \xi$ in (74)

$$B(W) = \left[\frac{r - n + 1 + 3\psi^2}{(n-2)(n-1)}\right] A(W). \qquad (75)$$

This gives the theorem. □

8. LP-Sasakian manifold satisfying $\mathcal{K}^*(\xi,U).R^* = 0$

Theorem 8.1. *If in an n-dimensional $(n > 2)$ LP-Sasakian manifold M, the condition $\mathcal{K}^*(\xi,U) \circ R^* = 0$ holds, then the equation $S^2(Y,U) + 9\psi^2 g(Y,U) + \left[(n-1)^2 + 9\psi^2\right]\eta(Y)\,\eta(U) + 6\psi S(Y,\phi U) = 0$, is satisfied on M, where $Y, U \in \chi(M)$ and $\psi = trace(\phi)$.*

Proof. Let us consider an LP-Sasakian manifold M satisfying the condition

$$\left(\mathcal{K}^*(\xi,U).R^*\right)(X,Y)\,Z = 0. \qquad (76)$$

Then, we have

$$\begin{aligned}
0 \; = \; & \mathcal{K}^*(\xi,U)\,R^*(X,Y)\,Z - R^*\left(\mathcal{K}^*(\xi,U)\,X,Y\right)Z \\
& - R^*\left(X,\mathcal{K}^*(\xi,U)\,Y\right)Z - R^*(X,Y)\,\mathcal{K}^*(\xi,U)\,Z. \qquad (77)
\end{aligned}$$

Replacing Z by ξ in (77), we get

$$\begin{aligned}
0 \; = \; & \mathcal{K}^*(\xi,U)\,R^*(X,Y)\,\xi - R^*\left(\mathcal{K}^*(\xi,U)\,X,Y\right)\xi \\
& - R^*\left(X,\mathcal{K}^*(\xi,U)\,Y\right)\xi - R^*(X,Y)\,\mathcal{K}^*(\xi,U)\,\xi. \qquad (78)
\end{aligned}$$

In view of (37), (40), (3) and (78), we have

$$\begin{aligned}
0 \; & = \; R^*(X,Y)\,\mathcal{K}^*(\xi,U)\,\xi \\
& = \; R^*(X,Y)\,Q^* U \\
& = \; R^*(X,Y)\,QU + 3\psi R^*(X,Y)\,\phi U. \qquad (79)
\end{aligned}$$

The inner product of the equation (79) with vector field V gives

$$0 = g\left(R^*(X,Y)\,QU,V\right) + 3\psi g\left(R^*(X,Y)\,\phi U,V\right). \qquad (80)$$

Let $\{e_i\}$ $(1 \leq i \leq n)$ be an orthonormal basis of the tangent space at any point of the manifold M. Setting $X = V = e_i$ and taking summation over $i(1 \leq i \leq n)$ and using (18) in (80), we get

$$\begin{aligned} 0 &= S^2(Y,U) + 9\psi^2 g(Y,U) \\ &\quad + \left[(n-1)^2 + 9\psi^2\right]\eta(Y)\eta(U) + 6\psi S(Y,\phi U). \end{aligned} \tag{81}$$

This gives the theorem. □

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