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DEGREE SUM EXPONENT DISTANCE ENERGY OF SOME GRAPHS

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Abstract. The degree sum exponent distance matrix $\mathcal{M}\chi_{dist}(G)$ of a graph G is a square matrix whose $(i,j)^{th}$ entry is $(d_i + d_j)^{d_{ij}}$ whenever $i \neq j$, otherwise it is zero,where d_i is the degree of i^{th} vertex of G and $d_{ij} = d(v_i, v_j)$ is distance between v_i and v_j. In this paper, we define degree sum exponent distance energy $E\chi_{dist}(G)$ as sum of absolute eigenvalues of $\mathcal{M}\chi_{dist}(G)$. Also, we obtain some bounds on the degree sum exponent distance energy of some graphs and deduce direct expressions for some graphs.

Key words and Phrases: Eigenvalue, degree sum exponent distance matrix, degree sum exponent distance energy

1. INTRODUCTION

The concept of graph energy was introduced by I.Gutman in 1978[1] having direct correlation with the total π-electron energy of a molecule in the quantum chemistry as calculated with the Huckel molecular orbital method. Here adjacency matrix of a graph is considered. Later Laplacian energy [2, 4], signless Laplacian energy [3], were introduced. Recently several results on energy related with degree of a vertex and distance in a graph were studied such as distance energy [5],degree sum energy of some graphs [6], degree square sum polynomial of some graphs [8], degree sum energy [9],a survey on energy of graphs [7], complementary distance energy[10], degree sum distance energy [11], degree product distance energy[12],degree exponent energy[13] and degree exponent sum energy[16].

For every pair of vertices in a connected graph there are, degree associated each one of them and in addition there is distance between them (length of the

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shortest path). In continuation with this, in order to upgrade, we now introduce concept of degree sum exponent distance energy of connected graph which is slight generalization of degree sum energy since if exponent is made one, it coincides with degree sum energy. The purpose of this paper is to compute the characteristic polynomial, eigenvalues and energy of the new matrix associated with graph, called degree sum exponent distance matrix, and compute bounds for degree sum exponent distance energy and obtain expressions for some standard graphs.

2. Degree Sum Exponent Distance Energy

Let G be a connected graph of order n with vertex set $V(G) = (v_1, v_2, ..., v_n)$. We denote $d(v_i)$ as the degree of a vertex v_i which is the number of edges incident on it and d_{ij} as the distance between two vertices v_i and v_j, the length of the shortest path joining them. We define degree sum exponent distance matrix of G as,
$\mathcal{M}\chi_{dist}(G) = [\chi_{ij}]$ where,

$$\begin{aligned} \chi_{ij} &= (d(v_i) + d(v_j))^{d_{ij}} \text{ if } i \neq j \\ &= 0 \text{ if } i = j \end{aligned} \tag{1}$$

Example: For graph G given below,

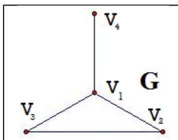
<table>
<tr>
<td>

Fig 2.1</td>
<td>$DSED(G) = \begin{pmatrix} 0 & 5 & 5 & 4 \\ 5 & 0 & 4 & 9 \\ 5 & 4 & 0 & 9 \\ 4 & 9 & 9 & 0 \end{pmatrix}$</td>
<td>Eigenvalues are, -11.1616, $-4, -3.2007, 18.3623$, and energy is $E\chi_{dist}(G) = 36.7246$.</td>
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We note that,

(1) $\mathcal{M}\chi_{dist}(G)$ is real symmetric, so that the eigenvalues of $\mathcal{M}\chi_{dist}(G)$ are real. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the eigenvalues of $\mathcal{M}\chi_{dist}(G)$ then, they can be arranged in a non-increasing order as $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$.
(2) $\sum_1^n \alpha_i = 0$, since trace[$\mathcal{M}\chi_{dist}(G)$]=0.
(3) The highest exponent term corresponds to diam(G).
(4) For any r-regular graph all the entries in the matrix are in powers of $2r$.
(5) Two non-isomorphic graphs having same order, regularity as well as diameter have same largest eigenvalue α_1.

We define the degree sum exponent distance energy of a graph G as,

$$E\chi_{dist}(G) = \sum_{i=1}^{n} |\alpha_i|.$$

3. Bounds on Degree Sum Exponent Distance Energy and Eigenvalues

In this section, we obtain some bounds on degree sum exponent distance energy and largest eigenvalue.

Lemma 3.1. *Let G be a graph of order n, then we have,*

$$\sum_{i=1}^{n} \alpha_i = 0 \quad and \quad \sum_{i=1}^{n} \alpha_i^2 = 2M, \quad here \quad we \ define, \ M = \sum_{i=1, i<j}^{n} ((d_i + d_j)^{d_{ij}})^2$$

Lemma 3.2. [15] *Let $a_1, a_2, .., a_n$ be non negative numbers.Then ,*

$$n\left[\frac{1}{n}\sum_{i=1}^{n} a_i - (\prod_{i=1}^{n} a_i)^{1/n}\right] \leq n\sum_{i=1}^{n} a_i - (\sum_{i=1}^{n} \sqrt{a_i})^2 \leq n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n} a_i - (\prod_{i=1}^{n} a_i)^{1/n}\right]$$

Lemma 3.3. *The CauchySchwartz inequality: Let a_i and b_i , $1 \leq i \leq n$ be any real numbers, then*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

.

Lemma 3.4. [17] *Let A, B,I(identity matrix) and J(matrix of all $1's$ are square matrices of same order n then block determinant of order n,*

$$|AI_n + B(J_n - I_n)| = |A - B|^{n-1}|A + (n-1)B|$$

.

Theorem 3.5. *If α_1 is the index (largest degree sum exponent distance eigenvalue) of a connected graph G of order n, then*

$$\alpha_1 \leq \sqrt{\frac{2M(n-1)}{n}}$$

where M is defined above, with $d_{ij} = d(v_i, v_j)$ the distance between v_i and v_j.

Proof. The trace of $\mathcal{M}\chi_{dist}(G)$ being zero we have

$$\sum_{i=1}^{n} \alpha_i = 0 \ i.e, \ \sum_{i=2}^{n} \alpha_i = -\alpha_1$$

Further $\sum_{i=1}^{n} \alpha_i^2$=trace$\mathcal{M}\chi_{dist}(G)^2 = 2M$, where M is as defined above. Using Lemma 3.3, with $a_i = 1$ and $b_i = \alpha_i$ $i = 2, 3 \ldots, n$ substituting we get,

$$(\sum_{i=2}^{n} \alpha_i)^2 \leq (n-1)\sum_{i=2}^{n} \alpha_i^2 \leq (n-1)(2M - \alpha_1^2)$$

Therefore, $(-\alpha_1)^2 \leq (n-1)(2M - \alpha_1^2)$. Simplifying further,the bound for the index α_1 follows. □

For graph G in Fig 2.1, $\alpha_1 = -11.1616$, $n = 4$ and $M = 244$. We have, $\sqrt{\frac{2M(n-1)}{n}} = 19.1312$

Theorem 3.6. *If G is connected graph of order n and M is defined above,then*

$$\sqrt{2M} \leq E\chi_{dist}(G) \leq \sqrt{2Mn}$$

Proof. With $a_i = 1$ and $b_i = |\alpha_i|$ and using Lemma 3.3 that is, $(\sum_{i=1}^{n} |\alpha_i|)^2 \leq n \sum_{i=1}^{n} (\alpha_i)^2$. That is, $E\chi_{dist}(G)^2 \leq 2nM$.
Hence, $E\chi_{dist}(G) \leq \sqrt{2Mn}$.
Now for the other part,

$$E\chi_{dist}(G)^2 = (\sum_{i=1}^{n} |\alpha_i|)^2 \geq \sum_{i=1}^{n} |\alpha_i|^2 = 2M$$

so that $E\chi_{dist}(G) \geq \sqrt{2M}$. Combining these two,inequality follows. □

For graph G in Fig 2.1, we have $\sqrt{2M} = 22.09072203$ and $\sqrt{2Mn} = 44.18144407$.

Theorem 3.7. *If G is any graph of order n and Δ is the absolute value of the determinant of $\chi_{dist}(G)$ then,*

$$\sqrt{2M + n(n-1)\Delta^{\frac{2}{n}}} \leq E\chi_{dist}(G) \leq \sqrt{2Mn}$$

where M is defined as above.

Proof. For lower bound consider,

$$[E\chi_{dist}(G)]^2 = (\sum_{i=1}^{n} |\alpha_i|)^2 = \sum_{i=1}^{n} (\alpha_i)^2 + 2\sum_{i<j} |\alpha_i||\alpha_j|$$

Since Arithmetic Mean (AM) $\geq$ Geometric Mean (GM) we have,

$$\frac{1}{n(n-1)} \sum_{i \neq j} |\alpha_i||\alpha_j| \geq (\prod_{i \neq j} |\alpha_i||\alpha_j|)^{\frac{1}{n(n-1)}} = \prod_{i=1}^{n} (|\alpha_i|^{2n-2})^{\frac{1}{n(n-1)}} = (\prod_{i=1}^{n} |\alpha_i|^{\frac{2}{n}}) = \Delta^{\frac{2}{n}}$$

using Lemma 3.2 .
therefore we have, $\prod_{i \neq j} |\alpha_i||\alpha_j| \geq n(n-1)\Delta^{\frac{2}{n}}$.
Combining we get, $[E\chi_{dist}(G)]^2 \geq 2M + n(n-1)\Delta^{\frac{2}{n}}$
ie, $E\chi_{dist}(G) \geq \sqrt{2M + n(n-1)\Delta^{\frac{2}{n}}}$ (1)

For upper bound define,

$$\begin{aligned} X = \sum_{i=1}^{n}\sum_{j=1}^{n} (|\alpha_i| + |\alpha_j|)^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} (|\alpha_i|^2 + |\alpha_j|^2) + 2(\sum_{i,j=1, i \neq j}^{n} |\alpha_i||\alpha_j|) \\ &= n\sum_{i=1}^{n} (\alpha_i)^2 + n\sum_{i=1}^{n} (\alpha_j)^2 - 2(\sum_{i,j=1, i \neq j}^{n} |\alpha_i||\alpha_j|) \\ &= 2nM + 2nM - 2[E\chi_{dist}(G)]^2 = 4nM - 2[E\chi_{dist}(G)]^2 \end{aligned}$$

Since $X \geq 0$ we get $E\chi_{dist}(G) \leq \sqrt{2Mn}$ (2)

Combining lower bound and upper bound, we arrive at the desired result. □

For the graph G in Fig 2.1, $\Delta = 2624$ and $\sqrt{2M + n(n-1)\Delta^{\frac{2}{n}}} = 26.20495997$.

Theorem 3.8. *Let G be a connected n vertex graph and Δ is the absolute value of the determinant of degree sum exponent distance matrix $\chi_{dist}(G)$, then*

$$\sqrt{2M + n(n-1)\Delta^{2/n}} \leq E\chi_{dist}(G) \leq \sqrt{2(n-1)M + n\Delta^{2/n}}$$

where M is defined as above.

Proof. Let $a_i = \alpha_i^2$, $i = 1, 2, ..., n$.. Then from Lemma 3.1 and Lemma 3.2 we obtain

$$n[\frac{1}{n}\sum_{i=1}^{n}\alpha_i^2 - (\prod_{i=1}^{n}\alpha_i^2)^{1/n}] \leq n\sum_{i=1}^{n}\alpha_i^2 - (\sum_{i=1}^{n}\alpha_i)^2 \leq n(n-1)[\frac{1}{n}\sum_{i=1}^{n}\alpha_i^2 - (\prod_{i=1}^{n}\alpha_i^2)^{1/n}]$$

i.e,

$$2M - n\Delta^{2/n} \leq 2nM - [E\chi_{dist}(G)]^2 \leq 2(n-1)M - n(n-1)\Delta^{2/n}$$

Thus,

$$2M + n(n-1)\Delta^{2/n} \leq [E\chi_{dist}(G)]^2 \leq 2(n-1)M + n\Delta^{2/n}$$

We get the desired result. □

For the graph G in Fig 2.1, $\sqrt{2(n-1)M + n\Delta^{2/n}} = 40.85217223$.

4. Degree Sum Exponent Distance Energy of some graphs

Theorem 4.1. *The degree sum exponent distance energy of K_n is,* $E\chi_{dist}(K_n)$=$4(n-1)^2$.

Proof. The complete graph K_n is of diameter 1 and hence every pair of vertices are at distance 1 so the degree sum exponent distance matrix of K_n is a matrix with zero diagonal and all non diagonal entries $2(n-1)$ i.e,the degree sum exponent distance matrix of K_n is $2(n-1)$ times the adjacency matrix of K_n. Since the adjacency energy of K_n is $2(n-1)$, the degree sum exponent distance energy of K_n will be $4(n-1)^2$. □

Theorem 4.2. *The degree sum exponent distance energy of $CP(n)$ is,* $E\chi_{dist}(CP(n)) = 32n(n-1)^2$.

Proof. The cocktail party graph $CP(n)$ denotes the $(2n)$-vertex regular graph of degree $(2n-2)$ (obtained by deleting n independent edges from the complete graph K_{2n}). Using Lemma 3.4, where $A = (2n-2)^2 A(K_2)$, $B = (2n-2)J_{2\times 2}$, J is matrix of all 1's and A is the adjacency matrix. The degree sum exponent distance polynomial of $CP(n)$ is then given by,
$|\alpha I - \mathcal{M}\chi_{dist}(CP(n))| = [\alpha + 16(n-1)^2]^n[\alpha - 8(n-1)(2n-3)]^{n-1}[\alpha - 24(n-1)^2]$
which gives, $E\chi_{dist}(CP(n)) = 32n(n-1)^2$. □

For example, in case of $CP(4)$, eigenvalues are $-64(3times), 48(2times)$ and 96 giving energy, $E\chi_{dist}(CP(3)) = 384$.

Theorem 4.3. *The degree sum exponent distance energy of crown graph S_n^0 is,* $E\chi_{dist}(S_n^0) = 16n(n-1)^3$.

Proof. The crown graph is the graph obtained by removing a matching from the complete equi-bipartite graph $K_{n,n}$. So the structure of the degree product distance matrix of S_n^0 is,
$\mathcal{M}\chi_{dist}(S_n^0) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ where A is a matrix of order n with zero diagonal and all non-diagonal entries as $4(n-1)^2$ and B is the matrix of order n with diagonal entry $8(n-1)^3$ and off diagonal entry $2(n-1)$.The eigenvalues of this matrix are given by eigenvalues of $A+B$ and eigenvalues of $A-B$ see[18].
Separately evaluating characteristic polynomials of $A+B$ and $A-B$ and then multiplying we get,degree sum exponent distance polynomial of crown graph,
$|\alpha I - \mathcal{M}\chi_{dist}(S_n^0)| =$
$[\alpha + 2(n-1)^2(6n-5)][\alpha - 2(n-1)^2(2n-1)][\alpha + 2(n-1)(4n^2-6n+1)]^{n-1}[\alpha - 2(n-1)(4n^2-10n+5)]^{n-1}$.
Adding all the absolute eigenvalues, we get the theorem.

□

Lemma 4.4. [14] *If a,b,c and d are real numbers, then the determinant of the form,*

$$\begin{vmatrix} (\alpha+a)I_{n_1} - aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (\alpha+b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

of order n_1+n_2 can be expressed in the simplified form as,

$$(\alpha+a)^{n_1-1}(\alpha+b)^{n_2-1}([\alpha-(n_1-1)a][\alpha-(n_2-1)b]-n_1n_2cd)$$

Theorem 4.5. *The degree sum exponent distance energy of the complete bipartite graph $K_{m,n}$ is,* $E\chi_{dist}(K_{m,n})$ $=8n^2(m-1)+8m^2(n-1)$.

Proof. In $K_{m,n}$, m vertices have degree n and n vertices have degree m.The diameter being 2, the structure of the degree sum exponent distance matrix is,

$$\mathcal{M}\chi_{dist}(K_{m,n}) = \begin{bmatrix} 4n^2A(K_m) & (m+n)J_{m\times n} \\ (m+n)J_{n\times m} & 4m^2A(K_n) \end{bmatrix}$$

where J is matrix of all 1's and A is the adjacency matrix. The degree sum exponent distance polynomial is then given by,

$$|\alpha I - \mathcal{M}\chi_{dist}(K_{m,n})| = \begin{vmatrix} \alpha I_m - 4n^2A(K_m) & -(m+n)J_{m\times n} \\ -(m+n)J_{n\times m} & \alpha I_n - 4m^2A(K_n) \end{vmatrix}.$$

Using Lemma 4.4 we get the degree sum exponent distance polynomial,
$|\alpha I - \mathcal{M}\chi_{dist}(K_{m,n})| =$
$[\alpha+4n^2]^{m-1}[\alpha+4m^2]^{n-1}[\alpha^2 - 4(n^2(m-1)+m^2(n-1))\alpha + 16m^2n^2(n-1)(m-$

$1) - (m+n)^2 mn]$.
The quadratic equation above has $[4(n^2(m-1)+m^2(n-1)]^2 > 4 \times [16m^2n^2(n-1)(m-1)-(m+n)^2mn]$ hence sum of absolute roots is $4(n^2(m-1)+m^2(n-1))$ and on adding all absolute eigenvalues the theorem follows. □

For example, in case of $K_{3,4}$, eigenvalues are $-64(2times), 36(3times), 91.7702$ and 144.2298 giving the energy , $E\chi_{dist}(K_{3,4}) = 472$.

Corollary 4.6. *The degree sum exponent distance energy of the star graph $K_{1,n}$ is,*
$E\chi_{dist}(K_{1,n}) = 8(n-1)$.

Proof. Put $m = 1$ in Theorem 4.5. □

Corollary 4.7. *The degree sum exponent distance energy of the equi-bipartite graph $K_{n,n}$ is,*
$E\chi_{dist}(K_{n,n}) = 16n^2(n-1)$.

Proof. Put $m = n$ in Theorem 4.5. □

Theorem 4.8. *If B_n $(n \geq 3)$ is a book graph of order $(n+2)$ with triangular pages and size $(2n+1)$, then $E\chi_{dist}$ of B_n is, $E\chi_{dist}(B_n)$=$36n - 28$*

Proof. The book graph B_n with triangular pages has two sets of vertices, a set with n vertices of degree 2 and the remaining 2 vertices of degree $(n+1)$. The structure of the degree sum exponent distance matrix is,

$$\mathcal{M}\chi_{dist}(B_n) = \begin{bmatrix} 2(n+1)A(K_2) & (n+3)J_{2\times n} \\ (n+3)J_{n\times 2} & 16A(K_n) \end{bmatrix}$$

where J is matrix of all 1's and A is the adjacency matrix. The degree sum exponent distance polynomial is then given by,

$$|\alpha I - \mathcal{M}\chi_{dist}(B_n)| = \begin{vmatrix} \alpha I_2 - 2(n+1)A(K_2) & -(n+3)J_{2\times n} \\ -(n+3)J_{n\times 2} & \alpha I_n - 16A(K_n) \end{vmatrix}.$$

Using Lemma 4.4 we get the degree sum exponent distance polynomial,
$|\alpha I - \mathcal{M}\chi_{dist}(B_n)| = [\alpha+16]^{n-1}[\alpha+2(n+1)][\alpha^2 - 2(9n-7)\alpha + 2(16(n-1)(n+1) - n(n+3)^2)]$.
The quadratic equation above has $[2(9n-7)]^2 > 4\times 2(16(n-1)(n+1) - n(n+3)^2)$ hence sum of absolute roots is $2(9n-7)$ and the theorem follows on adding all absolute eigenvalues. □

Let $K_n - e$ and $K_n + e$ denote the graph obtained from complete graph K_n by deleting an edge, adding an edge respectively.

Theorem 4.9.

$$\begin{aligned} E\chi_{dist}(K_n - e) &= 44.3606 \ \textit{if} \ n = 4 \\ &= 4[(n-1)(n-3) + 2(n-2)^2] \ \textit{if} \ n > 4 \end{aligned} \quad (2)$$

Proof. The graph $K_n - e$ is of diameter 2 and has two vertices with distance two and remaining at distance one.
For $n = 4$, using Matlab we have $E\chi_{dist}(K_4 - e) = 44.3606$
The degree sum exponent distance matrix of $K_n - e$ has the form,

$$\mathcal{M}\chi_{dist}(K_n - e) = \begin{bmatrix} 0 & 4(n-2)^2 & (2n-3)J_{1\times n-2} \\ 4(n-2)^2 & 0 & (2n-3)J_{1\times n-2} \\ (2n-3)J_{n-2\times 1} & (2n-3)J_{n-2\times 1} & 2(n-1)A(K_{n-2}) \end{bmatrix}.$$

So that the degree sum exponent distance polynomial of $K_n - e$ is given by,
$|\alpha I - \mathcal{M}\chi_{dist}(K_n - e)|$

$$= \begin{vmatrix} \alpha & -4(n-2)^2 & -(2n-3)J_{1\times n-2} \\ -4(n-2)^2 & \alpha & -(2n-3)J_{1\times n-2} \\ -(2n-3)J_{n-2\times 1} & -(2n-3)J_{n-2\times 1} & \alpha I_{n-2} - 2(n-1)A(K_{n-2}) \end{vmatrix}$$

Using Lemma 4.4 we get the degree sum exponent distance polynomial,
$|\alpha I - \mathcal{M}\chi_{dist}(K_n - e)| = [\alpha + 4(n-2)^2][\alpha + 2(n-1)]^{n-3}[\alpha^2 - 2((n-1)(n-3) + 2(n-2)^2)\alpha + 2(n-2)(4(n-1)(n-3)(n-2) - (2n-3)^2)]$ for $n > 4$. Since $(2((n-1)(n-3) + 2(n-2)^2)^2 > 4 \times 2(n-2)(4(n-1)(n-3)(n-2) - (2n-3)^2)$, the sum of absolute roots of the quadratic is $(2((n-1)(n-3) + 2(n-2)^2)$.
Hence the theorem. □

Theorem 4.10. $E\chi_{dist}(K_n + e)$=$2(n-1)(n-2) + |\alpha_1| + |\alpha_2| + |\alpha_3|$, *where* α_1, α_2 *and* α_3 *are roots of the equation,*
$[\alpha^3 - 2(n-1)(n-2)\alpha^2 - ((2n-1)^2(n-1) + n^4(n-1) + (n+1)^2)\alpha + 2(n+1)(n-1)((n+1)(n-2) - n^2(2n-1))] = 0.$

Proof. In $K_n + e$ there is one vertex with degree n, one vertex with degree 1 and remaining $n-1$ have degree $n-1$. Thus we get the degree sum exponent distance matrix with suitable labeling as,

$$\mathcal{M}\chi_{dist}(K_n + e) = \begin{bmatrix} 0 & (n+1) & (2n-1)J_{1\times n-1} \\ (n+1) & 0 & n^2 J_{1\times n-1} \\ (2n-1)J_{n-1\times 1} & n^2 J_{n-1\times 1} & 2(n-1)A(K_{n-1}) \end{bmatrix}.$$

So that the degree sum exponent distance polynomial of $K_n + e$ is given by,
$|\alpha I - \mathcal{M}\chi_{dist}(K_n + e)|$

$$= \begin{vmatrix} \alpha & -(n+1) & -(2n-1)J_{1\times n-1} \\ -(n+1) & \alpha & -n^2 J_{1\times n-1} \\ -(2n-1)J_{n-1\times 1} & -n^2 J_{n-1\times 1} & \alpha I_{n-1} - 2(n-1)A(K_{n-1}) \end{vmatrix}$$

Using Lemma 4.4 we get the degree sum exponent distance polynomial,
$|\alpha I - \mathcal{M}\chi_{dist}(K_n + e)| = [\alpha + 2(n-1)]^{n-2}[\alpha^3 - 2(n-1)(n-2)\alpha^2 - ((2n-1)^2(n-1) + n^4(n-1) + (n+1)^2)\alpha + 2(n+1)(n-1)((n+1)(n-2) - n^2(2n-1))]$.
On extracting eigenvalues and taking the absolute sum, we get the theorem. □

For example, in case of K_5+e, eigenvalues are -40.5244, $-8(3times)$, -3.5991, 68.1235 giving the energy as, $E\chi_{dist}(K_5 + e) = 136.247$.

Definition 4.11 (Vertex Coalescence). *If G_1 and G_2 are any two graphs then the graph obtained by gluing G_1 and G_2 at a point is v called vertex coalescence denoted by $G_1 O_v G_2$.*

Definition 4.12 (Edge Coalescence). *If G_1 and G_2 are any two graphs then the graph obtained by merging G_1 and G_2 on an edge e is called edge coalescence denoted by $G_1 O_e G_2$.*

Now we consider the degree sum exponent distance energy of vertex coalescence and edge coalescence of complete graphs of same order. Let K_n be a complete graph of order n then the vertex coalescence of K_n with K_n will be denoted by $K_n O_v K_n$ and the edge coalescence by $K_n O_e K_n$.

$K_n O_v K_n$ has $2n-1$ vertices and $2 \times ({}^nC_2)$ edges whereas $K_n O_e K_n$ has $2n-2$ vertices and $2 \times ({}^nC_2 - 1)$ edges.

Lemma 4.13. [19] *Let a and b be two arbitrary constants, I is the identity matrix and J is $n \times n$ matrix whose all entries $1's$.If $A = (a-b)I + bJ$ then the characteristic polynomial of A is, $|\lambda I - A| = [\lambda - a + b]^{n-1}[\lambda - a - (n-1)b]$.*

Theorem 4.14. *The degree sum exponent distance energy of the vertex coalescence of two complete graphs K_n for $n \geq 3$ is given by,*
$E\chi_{dist}(K_n O_v K_n) = 2(n-1)(2n^2-5n+8) + 2(n-1)\sqrt{(2n^2-5n+8)^2 + 18(n-1)}.$

Proof. The graph $K_n O_v K_n$ has two sets of vertices one at a distance 2 from each other and other at 1, being of diameter 2. With suitable labeling the degree sum exponent distance matrix of $K_n O_v K_n$ takes the form,

$$\mathcal{M}\chi_{dist}(K_n O_v K_n) = \begin{bmatrix} 0 & 3(n-1)J_{1\times n-1} & 3(n-1)J_{1\times n-1} \\ 3(n-1)J_{n-1\times 1} & 2(n-1)A(K_{n-1}) & 4(n-1)^2 J_{n-1\times n-1} \\ 3(n-1)J_{n-1\times 1} & 4(n-1)^2 J_{n-1\times n-1} & 2(n-1)A(K_{n-1}) \end{bmatrix}$$

So that the degree sum exponent distance polynomial of $K_n O_v K_n$

$$|\alpha I - \mathcal{M}\chi_{dist}(K_n O_v K_n)| = \begin{vmatrix} \alpha & -3(n-1)J_{1\times n-1} & -3(n-1)J_{1\times n-1} \\ -3(n-1)J_{n-1\times 1} & \alpha I_{n-1} - 2(n-1)A(K_{n-1}) & -4(n-1)^2 J_{n-1\times n-1} \\ -3(n-1)J_{n-1\times 1} & -4(n-1)^2 J_{n-1\times n-1} & \alpha I_{n-1} - 2(n-1)A(K_{n-1}) \end{vmatrix}$$

Using Lemma 4.11 we get the degree sum exponent distance polynomial,
$|\alpha I - \mathcal{M}\chi_{dist}(K_n O_v K_n)| =$
$[\alpha + 2(n-1)(2(n-1)^2 - n + 2)][\alpha + 2(n-1)]^{2n-4}[\alpha^2 - 2(n-1)(2n^2 - 5n + 8)\alpha - 18(n-1)^3]$.
On extracting eigenvalues and taking the absolute sum, we get the theorem. □

For example, in case of $K_5 O_v K_5$, eigenvalues are -232, $-8(6times)$, -4.0555, 284.0555 giving the energy as, $E\chi_{dist}(K_5 O_v K_5) = 568.111$.

Theorem 4.15. *The degree sum exponent distance energy of the edge coalescence of two complete graphs K_n for $n \geq 3$ is given by, $E\chi_{dist}(K_n O_e K_n)$=$4(2n-3) + 4(n-1)(2n^2 - 7n + 7) + 8(n-1)(n-3)$.*

Proof. The graph $K_nO_eK_n$ has two sets of vertices one at a distance 2 from each other and other at 1, being of diameter 2.There are two vertices of degree $(2n-3)$ and remaining $(2n-4)$ of degree $(n-1)$. With suitable labeling the degree sum exponent distance matrix of $K_nO_eK_n$ takes the form,

$$\mathcal{M}\chi_{dist}(K_nO_eK_n)= \begin{bmatrix} 2(2n-3)A(K_2) & (3n-4)J_{2\times n-2} & (3n-4)J_{2\times n-2} \\ (3n-4)J_{n-2\times 2} & 2(n-1)A(K_{n-2}) & 4(n-1)^2J_{n-2\times n-2} \\ (3n-4)J_{n-2\times 2} & 4(n-1)^2J_{n-2\times n-2} & 2(n-1)A(K_{n-2}) \end{bmatrix}$$

So that the degree sum exponent distance polynomial of $K_nO_eK_n$ is given by,
$|\alpha I - \mathcal{M}\chi_{dist}(K_nO_eK_n)|=$

$$\begin{vmatrix} \alpha I_2 - 2(2n-3)A(K_2) & -(3n-4)J_{2\times n-2} & -(3n-4)J_{2\times n-2} \\ -(3n-4)J_{n-2\times 2} & \alpha I_{n-2} - 2(n-1)A(K_{n-2}) & -4(n-1)^2J_{n-2\times n-2} \\ -(3n-4)J_{n-2\times 2} & -4(n-1)^2J_{n-2\times n-2} & \alpha I_{n-2} - 2(n-1)A(K_{n-2}) \end{vmatrix}$$

So that the degree sum exponent distance polynomial of $K_nO_eK_n$ is given by,
$|\alpha I - \mathcal{M}\chi_{dist}(K_nO_eK_n)|= [\alpha + 2(2n-3)][\alpha + 2(n-1)(2n^2-7n+7)][\alpha + 2(n-1)]^{2n-6}[\alpha^2 - 2(2n^3-7n^2+8n-4)\alpha + 4(2n-3)(n-1)((2n-1)(n-3)+2n-2) - 4(3n-4)^2(n-2)]$.
On extracting eigenvalues and taking the absolute sum, we get the following theorem. □

For example, in case of $K_5O_eK_5$, eigenvalues are $-176, -14, -8(4times)$, $-6.7839, 215.2161$ giving the energy as, $E\chi_{dist}(K_5O_eK_5) = 444.003$.

From Theorem 4.2 and Theorem 4.3 we see that both S_3^0 and $CP(3)$ have same $E\chi_{dist} = 384$, although $CP(3)$ has 12 edges and S_3^0 has 6 edges. We call such graphs as degree sum exponent distance equi-energeic.

Definition 4.16. *Two non isomorphic graphs on same number of vertices are said to be degree sum exponent distance equi-energeic if they have same degree sum exponent distance energy.*

5. Conclusion

We discussed the degree sum exponent distance energy of graphs. Also, we discussed bounds on the energy of degree sum exponent distance energy. There is scope to investigate degree sum exponent distance energy of graphs with higher diameter,trees,unicyclic graphs etc and also to construct degree sum exponent distance equi-energetic graphs.

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