J. Indones. Math. Soc.
Vol. 26, No. 02 (2020), pp. 258–265.

DETERMINATION OF THE SOME RESULTS FOR COUPLED FIXED POINT THEORY IN C^*- ALGEBRA VALUED METRIC SPACES

Saleh OMRAN[1], Özen ÖZER[2,*]

[1] **Mathematics Department, South Valley University, Qena, Egypt,**
salehomran@yahoo.com
[2]**Faculty of Engineering, Kırklareli University, Kırklareli, 39100, TURKEY,**
ozenozer39@gmail.com

Abstract. In the present paper, we introduce the coupled fixed point theorem in C^*-algebra valued metric spaces. We get a C^*-algebra valued metric space which get values in noncomutative operators. We demonstrate existance and uniqeness of coupled fixed point in a such space. Besides, we support our results by giving numerical examples.

Key words and Phrases: Cone Metric Spaces, C^*-Algebras, Hilbert C^*-Modules, Fixed Point Theory

1. INTRODUCTION

Huang and Zhang [4] started to work on the notion of cone metric space and got significant results on the fixed point theorems for mappings satisfying different contractive conditions.After them, some other authors considered and worked on the existence of fixed points of self mappings satisfactory a contractive type condition.The author and et a.l introduced in [12] the notation of the Quaternion valued metric spaces. In [8] and [9], we considered C^*- algebras valued metric spaces and proved certain fixed-point theorem in such spaces.

Here, we study on C^*- algebra valued metric space and give some examples.The idea of this metric is to replace the set of real numbers by the positive cone C^*-algebra.Notation of the set of positive elements on the C^*- algebras was introduced in [5]. There is a new version of the C^*-algebra valued metric space in [6]. Readers also have a look references [1], [7] and [10]-[14] for C^*-algebras and fixed point theory.

2020 Mathematics Subject Classification: 47H10, 55M20.
Received: 20-02-2020, accepted: 22-06-2020.
*Corresponding Author

2. Preliminaries

We begin with the notions and facts about structures of C^*-algebra and Fixed Point Theory.

2.1. **Positive Cone and C*- Algebra.**

Definition 2.1. *Let a conjugate-linear involution $* : \mathcal{A} \longrightarrow \mathcal{A}$ satisfies following conditions*

$$(u^*)^* = u,\ (u.v)^* = v^*u^*,\ ,(u+v)^* = u^* + v^*\ , \|u^*u\| = \|u\|^2$$

for all u, v in $\mathcal{A}$.Then $\mathcal{A}$ is called as a complex Banach algebra. The C^*-condition $\|x^*x\| = \|x\|^2$ necessitates that the involution is an isometry in the case of $\|x^*\| = \|x\|$ for all x in $\mathcal{A}$.

if A is C^*-algebra has a unit, then it is called as a unital and $\|1\| = 1$.Generally, C^*-algebra is non-commutative.

Definition 2.2. *A *- homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is defined between two different C^*-algebras with $\varphi(u^*) = \varphi(u)^*$ for all $u \in \mathcal{A}$. Let $\mathcal{A}$ and $\mathcal{B}$ be unital. if $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, then φ is called as a unital .*

All of $*$-homomorphisms between C^*-algebras are continuous which is very significat fact.

Example 2.3. *$\mathcal{H}$ is a Hilbert space and $\mathcal{B}(\mathcal{H})$ is the set of all bounded operators on $\mathcal{H}$. $\mathcal{B}(\mathcal{H})$ with the operator norm $\|u\| = sup\{\|u\xi\|,\ \xi \in \mathcal{H}, \|\xi\| = 1\}$ for each $u \in \mathcal{B}(\mathcal{H})$ and involution defines $u \in \mathcal{B}(\mathcal{H})$ its adjoint is a C^*-algebra.*

Gelfand, Naimark in [3] gave following important results.
All of the commutative C^*-algebras are isomorphic to $C_0(Y)$ for some of the locally compact space Y. Besides, every C^*-algebra is isomorphic to a sub-algebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Remark 2.4. In general,if C^*-algebra $\mathcal{A}$ does not have an identity element, it is possible to append one as follows: Consider the algebra $\mathcal{A}^{\Im}$ is written direct sums of $\mathcal{A}$ and $C1$ with multiplication given by

$$(a, \lambda)(b, \vartheta) = (ab + a\vartheta + b\lambda, \lambda\vartheta)$$

We define a norm on $\mathcal{A}^{\Im}$ via $\|(a, \lambda 1)\| = \|a\| + \lambda|$. Then we have

$$\|(a, \lambda)(b, \vartheta)\| = (\|(a, \lambda)\|)(\|(b, \vartheta)\|).$$

It is clear that the embedding of $\mathcal{A}$ into $\mathcal{A}$ is linear and isometric and that $\mathcal{A}$ sits inside of $\mathcal{A}^{\Im}$ as a closed ideal.

2.2. **Positivity.**

Definition 2.5. *$\sigma(a) \in \mathcal{R}^+$ a self adjoint element is called positive in C^*-algebra if c is positive (i.e. $c \geq 0$). The set of all positive elements is defined as $\mathcal{A}_+ = \{c \in \{\mathcal{A},\ c \geq 0\}$*

Theorem 2.6. *Let x be a normal element in a C^*-algebra. If $f \in C(\sigma(a))$, then let $f(x)$ be image of a under the functional calculus, then $f(\sigma(x))$ =$\sigma(f(x))$.*

Lemma 2.7. *Let $\mathcal{A}$ be a C^*-algebra. Then followings are satisfy.*
(a) If $x \in \mathcal{A}$ is normal, $x^*x \geq 0$ holds.
(b) If $x \in \mathcal{A}$ is self-adjoint and $\|x\| \leq 1$, then $x \geq 0$ satisfy.
(c) If $x, y \in \mathcal{A}_+$ then $x + y \in \mathcal{A}_+$
(d)$\mathcal{A}_+$ is closed in $\mathcal{A}$

For a given $a, b \in \mathcal{A}_+$, we denote $a \leq b$ if $a - b \geq 0$, $\mathcal{A}_+$ becomes a partially ordered vector space. Furthermore in a unital C^*-algebra, for all $a, b \in \mathcal{A}$, $0 \leq a \leq b$ implies that $\|a\| \leq \|b\|$, so the positive cone in a C^*-algebra is automatically normal.

Lemma 2.8. *Let $\mathcal{A}$ be unital C^*-algebra with a unit I. Then followings hold.*
(1) If $x \in \mathcal{A}$ with $\|x\| \leq \frac{1}{2}$ then $I - x$ can be invertible and $\|x(I - x)^{-1}\| < 1$ satisfies.
(2)Assume that $x, y \in \mathcal{A}$ such that $x, y \geq \theta$ and $xy = yx$, then $xy \geq \theta$ holds.

2.3. **C^*-algebras valued metric space.** C^*-algebras valued metric space introduced by [5] and studied self mapping fixed point theorem, here we will introduce some other examples and the relation between C^*-algebras valued metric space. Moreover, we will introduce the couple fixed point theory in this setting.

Definition 2.9. *Let X be a non-empty set and $\mathcal{A}$ be a C^*-algebra with $\mathcal{A}_+$ its positive cone. A C^*-algebra valued metric space is a function $d : X \times X \longrightarrow \mathcal{A}_+$ defined on X such that for any $u, v, w \in X$*
(i) $d(\alpha, \beta) = 0 \Leftrightarrow \alpha, \beta$
(ii) $d(\alpha, \beta) = d(\beta, \alpha)$
(iii) $d(\alpha, \gamma) \leq d(\alpha, \beta) + d(\beta, \gamma)$

From the the definition it is clear that that $d(\alpha, \beta) \geq 0,\ d(\alpha, \alpha) \geq 0$

Definition 2.10. *Let $(X, \mathcal{A}, d)$ be a C^*-algebra valued metric space and $b_n \subset X$ is a sequence in X. If $b \in X$ and $\varepsilon > 0$ and N such that for all $n > N$, $\|d(b_n, b)\| \leq \varepsilon$, then b_n is called a convergent sequence in X to b and denote it by $\lim_{n\to\infty} b_n = b$*

Moreover, for any $\varepsilon > 0$, there is N such that for all $n, m > N$, $\|d(a_n, a_m)\| \leq \varepsilon$ then a_n is called a cauchy sequence in X.

Definition 2.11. *if every cauchy sequence is convergent squence, $(X, \mathcal{A}, d)$ is a completed C^*-algebra valued metric space.*

Example 2.12. *If X is a Banach space, then $(X, \mathcal{A}, d)$ is a completed C^*-algebra valued metric space with the metric*

$$d(u, v) = \|u - v\|.h$$

for $u, v \in X$ and h is arbitrary positive element in $\mathcal{A}_+$

Definition 2.13. *Let $(X, \mathcal{A}, d)$ be a C^*-algebra valued metric space. The mapping $T : X \longrightarrow X$, is called contractive mapping on x if there is an $c \in \mathcal{A}$ such that $\|c\| < 1$ and satisfy*

$$d(T(\alpha), T(\beta)) \leq c^* d(\alpha, \beta)c$$

for $\alpha, \beta \in X$.

Example 2.14. *Let $X = [-1, 1]$ and $\mathcal{A} = M_{2\times 2}(R)$ with $\|A\| = max\ (|x_1|, |x_2|, |x_3|, |x_4|)$ where $x_i's$ are the inputs of the matrice $A \in M_{2\times 2}(R)$. Then $(X, \mathcal{A}, d)$ is a C^*-algebra valued metric space, where*

$$d(\alpha, \beta) = \begin{pmatrix} |\alpha - \beta| & 0 \\ 0 & |\alpha - \beta| \end{pmatrix}$$

and partial ordering on $\mathcal{A}$ is given as

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \geq \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \Leftrightarrow x_i \geq y_i$$

for $i = 1, .., 4$.

3. MAIN RESULTS

Theorem 3.1. *Assume that $(X, \mathcal{A}, d)$ be a complete C^*-algebra valued metric space.We accept that the mapping $T : X \times X \longrightarrow X$ satisfies*

$$d(T(x, y), T(u, v)) \leq ad(x, u)a^* + bd(y, v)b^* \tag{1}$$

for $x, y, u, v \in X$ and $a, b \in \mathcal{A'}_+$, such that $\|a\| \leq \frac{1}{2}$ and $\|b\| \leq \frac{1}{2}$. Then T has a Unique Coupled Fixed Point.

Proof. Prefer $x_0, y_0 \in\ X$ such that

$$x_1 = T(x_0, y_0), y_1 = T(y_0, x_0) \tag{2}$$
$$x_2 = T(x_1, y_1), y_2 = T(y_1, x_1) \tag{3}$$
$$. \tag{4}$$
$$. \tag{5}$$
$$. \tag{6}$$
$$x_m = T(x_{m-1}, y_{m-1}), y_m = T(y_{m-1}, x_{m-1}) \tag{7}$$
$$x_{m+1} = T(x_m, y_m), y_{m+1} = T(y_m, x_m) \tag{8}$$

From (1) and (2) we have

$$d(x_{n+1}, x_n) = d(T(x_n, y_n), T(x_{n-1}, y_{n-1}) \leq ad(x_n, x_{n-1})a^* + bd(y_n, y_{n-1})b^*$$

and smilarly

$$d(y_{n+1}, y_n) = d(T(y_n, x_n), T(y_{n-1}, x_{n-1}) \leq ad(y_n, y_{n-1})a^* + bd(x_n, x_{n-1})b^*$$

So,we get

$$\begin{aligned} d_{nn} &\leq ad(x_n, x_{n-1})a^* + bd(y_n, y_{n-1})b^* + ad(y_n, y_{n-1})a^* + bd(x_n, x_{n-1})b^* \\ &= (a+b)d(x_n, x_{n-1})(a^* + b^*) + (a+b)d(y_n, y_{n-1})(a^* + b^*) \\ &= (a+b)[d(x_n, x_{n-1}) + d(y_n, y_{n-1})](a^* + b^*) \\ &= (a+b)d_{(n-1)(n-1)}(a^* + b^*), \end{aligned}$$

where $d_{nn} = d(x_{n+1}, x_n) + d(y_{n+1}, y_n)$ for all $n \geq 0$.
In general, we have

$$\begin{aligned} d_{nn} &\leq (a+b)d_{(n-1)(n-1)}(a^* + b^*) \\ &\leq (a+b)^2 d_{(n-2)(n-2)}(a^* + b^*)^2 \\ &\leq \ldots \\ &\leq (a+b)^n d_{00}(a^* + b^*)^n, \end{aligned}$$

where $d_{00} = d(x_1, x_0) + d(y_1, y_0) = A$. Put $P = (a+b)$ and $P^* = (a+b)^*$, so we obtain

$$\begin{aligned} d_n &\leq P^n A (P^*)^n = P^n A^{\frac{1}{2}} A^{\frac{1}{2}} (P^*)^n \\ &= (P^n A^{\frac{1}{2}})(A^{\frac{1}{2}} (P^*)^n) = (P^n A^{\frac{1}{2}})(P^n A^{\frac{1}{2}})^*. \end{aligned}$$

Since A positive element in $\mathcal{A}$, so $A^* = A$ and $(A^{\frac{1}{2}})^* = A^{\frac{1}{2}}$ hold.
For $n+1 > m$ we get

$$\begin{aligned} d_{nm} &= d(x_{n+1}, x_m) + d(y_{n+1}, y_m) \\ &\leq (P^n A^{\frac{1}{2}})(P^n A^{\frac{1}{2}})^* + (P^{n-1} A^{\frac{1}{2}})(P^{n-1} A^{\frac{1}{2}})^* \\ &\quad + ... + (P^m A^{\frac{1}{2}})(P^m A^{\frac{1}{2}})^* \\ &= \sum_{k=m}^{n} (P^k A^{\frac{1}{2}})(P^k A^{\frac{1}{2}})^* \\ &= \sum_{k=m}^{n} |(P^k A^{\frac{1}{2}})|^2. \end{aligned}$$

So, we obtain

$$\begin{aligned}\|d_{nm}\| &\leq \| \sum_{k=m}^{n} |(P^k A^{\frac{1}{2}})|^2 \| \\ &\leq \sum_{k=m}^{n} \|P^k\|^2 \|A^{\frac{1}{2}}\|^2 I \\ &= \|A^{\frac{1}{2}}\|^2 \sum_{k=m}^{n} \|P^k\|^2 I \\ &\leq \|A^{\frac{1}{2}}\|^2 \sum_{k=m}^{n} \|P\|^{2k} I, (\|P^2\| \leq \|P\|^2 \\ &\leq \|A^{\frac{1}{2}}\|^2 \frac{\|P\|^{2m}}{1-\|P\|} I.\end{aligned}$$

Since $\|P\| \leq 1$ due to $\|a\| \leq \frac{1}{2}$. and $\|b\| \leq \frac{1}{2}$.

So, we can see easily $\|d(x_{n+1}, x_m) + d(y_{n+1}, y_m)\| \leq \|A^{\frac{1}{2}}\|^2 \frac{\|P\|^{2m}}{1-\|P\|} I$ and

$$\lim_{m\to\infty} \|A^{\frac{1}{2}}\|^2 \frac{\|P\|^{2m}}{1-\|P\|} I = 0_{\mathcal{A}}$$

Therefore, x_n and y_n are cauchy sequences in X with respect to $\mathcal{A}$. So, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y$$

since $(\mathrm{X}, \mathcal{A}, d)$ is a completed. By using inequalities (1), we get

$$\begin{aligned}0 \leq d(T(x,y), x) &\leq d(T(x,y), x_{n+1}) + d(x_{n+1}, x) \\ &= d(T(x,y), T(x_n, y_n)) + d(x_{n+1}, x) \\ &\leq a d(x, x_n) a^* + b d(y, y_n) b^* + d(x_{n+1}, x).\end{aligned}$$

We obtain

$$\lim_{n\to\infty} d(x_n, x) = 0, \lim_{n\to\infty} d(y_n, y) = 0, \lim_{n\to\infty} d(x_{n+1}, x) = 0$$

Thus,

$$\lim_{n\to\infty} T(x,y) = x, \lim_{n\to\infty} T(y,x) = y$$

in a similar way. So, (x, y) is coupled fixed point. Now, we assume that (x', y') is another coupled fixed point of T then we obtain some of equalities as follows:

$$d(x', x) = d(T(x', y'), T(x, y)) \leq a d(x', x) a^* + b d(y', y) b^*$$
$$d(y', y) \leq a d(y', y) a^* + b d(x', x) b^*$$

So,we get

$$d(x', x) + d(y', y) = (a+b)[d(x', x) + d(y', y)](a^* + b^*) = P[d(x', x) + d(y', y)]P^*$$

Hence,we obtain following equality by using $\|P^*\| \leq \|P\|$ since $P \in \mathcal{A}$

$$\|d(x', x) + d(y', y)\| \leq \|P\| \|d(x', x) + d(y', y)\| \|P^*\| \leq \|d(x', x) + d(y', y)\|$$

This is a contradiction, so we get

$$d(x', x) + d(y', y) = 0_{\mathcal{A}}$$

Moreover, we obtain

$$d(x', x) = 0, d(y', y) = 0$$

since $d(x', x)$ and $d(y', y)$ are positive elements. So, we can easily seee that $x' = x$ and $y' = y$ satisfy. Therefore, T has a Unique Coupled Fixed Point. □

It is very important that our results have generalized the Theorem 2.1 in [14] when $a = b$ as the following corollary, and also have Theorem 2.3 in [11].

Corollary 3.2. *Let $(X, \mathcal{A}, d)$ be a complete C^*-algebra valued metric space. We suppose that the mapping $T : X \times X \longrightarrow X$ satisfies*

$$d(T(\alpha, \beta), T(m, n)) \leq c[d(\alpha, m) + d(\beta, n)]c^*$$

for $x, y, u, v \in X$ and $c \in \mathcal{A}'_+$, such that $\|c\| \leq \frac{1}{2}$. Then T has a Unique Coupled Fixed Point.

Acknowledgement. Authors thanks the reviewers for the valuable comments.

REFERENCES

[1] Blackadar, B., *K-theory for operator Algebras* MSRI Publ. 5, Cambridge University Press, 1998.
[2] Davidson, K. R., *C^*-algebras by Example*, Fields Institute monographs, Amer. Math. Soc., Providence, 1996.
[3] I. Gelfand and M. Naimark, On the embedding of normed rings into the ring of operators in Hilbert space, *Math. Sb*, **12** (1943), 197-213.
[4] Huang, L. G. and Zhang, X., Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332** (2007), 1468-1476.
[5] Zhenhua Ma, Lining Jiang and Hongkai Sun, C^*- algebra valued metric spaces and realated fixed point theorems, *Fixed Point Theory and Their Applications*, **206** (2014), 1-11.
[6] Zhenhua Ma and Lining Jiang, C^*- algebra valued b-metric spaces and realated fixed point theorems, *Fixed Point Theory and Their Applications*, (2015), 222. DOI 10.1186/s13663-015-0471-6.
[7] Murphy, G. J., *C^*-algebras and Operator Theory*, Academic Press, 1990.
[8] Özer, Ö. and Omran, S., Common fixed point in C^*-algebra b-valued Metric Space, In the Application of mathematics n technical and natural scences: 8th International Conference for Promoting the Application of Mathematics in Technical and Natural Sciences (AMiTaNS16), 1773(1). 050005. AIP Publishing, 2016. https:/doi.org/10.1063/1.4964975.
[9] Özer, Ö. and Omran, S., On the Generalized C^*-valued Metric Spaces Related with Banach Fixed Point Theory, *International Journal of Advanced and Applied Sciences*, **4(2)** (2017), 35-37. https://doi.org/10.21833/ijaas.2017.02.006.
[10] Pedersen, G. K., *C^*-algebras and Their Automorphism Groups*, Academic Press, London, 1979.
[11] Qiaoling, X., Lining, J., Zhenhua, M., *Common fixed point theorems in C^* algebra valued metric spaces*, arXiv : 1560.05545v2 [math.FA] 30 Jun 2015.

[12] Rezapour, Sh. and Hamlbarani, R., Some notes on the paper "cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **345** (2008), 719-724.

[13] Ahmed El-Sayed, Saleh Omran and Abdalla Asad, Fixed point Theorems in Quaternion -Valued metric spaces, *Abstract and Applied Analysis* (2014).

[14] Tianqing, C., *Some coupled fixed point theorems in C^*-algebra valued metric spaces*, arXiv:1601.07168v1 [math.OA] 26 Jan 2016.