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THE N-INTEGRAL

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Abstract. In this paper, we introduced a Henstock-type integral named N- integral of a real valued function f on a closed and bounded interval $[a, b]$. The set N-integrable functions lie entirely between Riemann integrable functions and Henstock-Kurzweil integrable functions. Furthermore, this new integral integrates all improper Riemann integrable functions even if they are not Lebesgue integrable. It was shown that for a Henstock-Kurzweil integrable function f on $[a, b]$, the following are equivalent:

(1) The function f is N-integrable;
(2) There exists a null set S for which given $\epsilon > 0$ there exists a gauge δ such that for any δ-fine partial division $D = \{(\xi, [u, v])\}$ of $[a, b]$ we have

$$(\phi_S(D) \cap \Gamma_\epsilon) \sum |f(v) - f(u)||v - u| < \epsilon$$

where $\phi_S(D) = \{(\xi, [u, v]) \in D : \xi \notin S\}$ and

$$\Gamma_\epsilon = \{(\xi, [u, v]) : |f(v) - f(u)| \geq \epsilon\}$$

and

(3) The function f is continuous almost everywhere.

A characterization of continuous almost everywhere functions was also given.

Key words and Phrases: N-integral, Continuity almost everywhere, Henstock-Kurzweil integral.

1. INTRODUCTION

Recall that a real valued function f on $[a, b]$ is said to be Riemann integrable to A if for every $\epsilon > 0$ there exists a constant $\delta > 0$ such that for any division D of $[a, b]$ given by

$$a = x_0 < x_1 < \cdots < x_n = b \text{ and } \{\xi_1, \xi_2, \ldots, \xi_n\}$$

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with $x_{i-1} \le \xi_i \le x_i$ and $x_i - x_{i-1} < \delta$ for all i, we have

$$\left| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A \right| < \epsilon. \tag{1}$$

One may see [1], [4], [6], or [8] for further details. The Lebesgue criterion for Riemann integrability states that a function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if f is bounded and the set of discontinuities of f has measure 0. In this paper, we define an integral that can integrate bounded and unbounded continuous almost everywhere functions which are Henstock integrable. The idea is to exclude some problematic point-interval pairs of a division in forming the Riemann sum. In turn, this new integral can integrate all improper Riemann integrable functions and even more.

The whole paper is composed of five sections. The second section discusses the basic properties of the N-integral, including Saks-Henstock Lemma. In the third section, characterizations of the N-integrable functions will be given by looking at the points of discontinuity. The development of the characterizations utilizes the idea of Γ_ϵ which was adopted from [2] and [9]. But this time Γ_ϵ is defined based on discontinuity instead of nondifferentiability. In the fourth section, we explicitly identify the set that would optimally identify the point-interval pairs that are to be excluded in forming the Riemann sums. Finally, in the last section, we present examples to strengthen our results.

2. THE N-INTEGRAL

Let $[a,b]$ be a compact interval in $\mathbb{R}$. Given a subset X of $[a,b]$, we denote the closure of X by $\overline{X}$. A partial division $D = \{(\xi, [u,v])\}$ of $[a,b]$ is a finite collection of point-interval pairs $(\xi, [u,v])$ with $\xi \in [u,v]$, $[u,v] \subset [a,b]$, and the subintervals $[u,v]$ are nonoverlapping. If in case the union of the subintervals $[u,v]$ in D is $[a,b]$, then we simply say that D is a division of $[a,b]$. For a partial division $D = \{(\xi, [u,v])\}$ of $[a,b]$, we define $I(D)$ such that

$$I(D) = \{[u,v] : (\xi, [u,v]) \in D\}.$$

Given a subset S of $[a,b]$, a partial division $D = \{(\xi, [u,v])\}$ of $[a,b]$ is said to be S-tagged if for each $(\xi, [u,v]) \in D$, $\xi \in S$. A gauge on a set X is a function from X to the set of positive real numbers. Given a gauge δ on $[a,b]$, a point-interval pair $(\xi, [u,v])$ is said to be δ-fine if $[u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$. A partial division $D = \{(\xi, [u,v])\}$ is said to be δ-fine if every point-interval pair $(\xi, [u,v]) \in D$ is δ-fine. Note that for two gauges δ_1 and δ_2, with $\delta_1(x) \le \delta_2(x)$ for all $x \in [a,b]$, every δ_1-fine partial division of $[a,b]$ is also δ_2-fine. Recall that a function f on $[a,b]$ is said to be Henstock integrable if there exists a real number A for which given $\epsilon > 0$ there exists a gauge δ on $[a,b]$ such that for any δ-fine division $D = \{(\xi, [u,v])\}$ we have

$$\left| (D) \sum f(\xi)(v-u) - A \right| < \epsilon.$$

See for example [1], [4], [8] or [6]. Here A is the integral of f on $[a,b]$; it is unique, and we write

$$(H)\int_a^b f = A.$$

If a function f is Henstock integrable on $[a,b]$ then it is Henstock integrable on any subinterval $[u,v]$ of $[a,b]$. It follows that given a Henstock integrable function f on $[a,b]$, we can define a function F by $F(a)=0$ and

$$F(x) = (H)\int_a^x f \text{ for } x \in (a,b].$$

The function F is called the primitive of f. A Henstock primitive satisfies the Strong Lusin (SL) condition [8]. More precisely, for a Henstock primitive F, given a set S of measure zero and $\epsilon > 0$, there exists a gauge δ on $[a,b]$ such that for all δ-fine S-tagged partial division D of $[a,b]$ we have $(D)\sum |F(u,v)| < \epsilon$ where $F(u,v) = F(v) - F(u) = \int_u^v f$. By a null set, we mean a set of measure zero or empty. Given a null set S in $[a,b]$ and a partial division D of $[a,b]$, we set $\phi_S(D)$ as

$$\phi_S(D) = \{(\xi,[u,v]) \in D : \xi \notin S\}$$

and D_S as

$$D_S = \{(\xi,[u,v]) \in D : \xi \in S\}.$$

For any partial division D of $[a,b]$ define the partial division $D^* = \{(\xi^*,[u,v])\}$ where $[u,v]$ comes from D, $\xi^* \in [u,v]$, and

$$I(D^*) = I(D).$$

In this case, we say that D^* is D-compatible or we simply write $D^* \sim D$. The term $D-$compatible was first used by Lee in her thesis [5]. We shall now state the definition of N-integral.

Definition 2.1. *A function f on $[a,b]$ is said to be N-integrable if there exists a null set S and a real number A for which given $\epsilon > 0$ there exists a gauge δ on $[a,b]$ such that for any δ-fine division $D = \{(\xi,[u,v])\}$ of $[a,b]$ and any $D^* = \{(\xi^*,[u,v])\} \sim \phi_S(D)$ we have*

$$\left|(D^*)\sum f(\xi^*)(v-u) - A\right| < \epsilon.$$

Here, A is the N-integral of f over $[a,b]$ and we write

$$\int_a^b f = A.$$

A null set S satisfying the N-integrability condition for f, is said to be an *avoided set for f* on $[a,b]$.

Given a measure zero subset S of $[a,b]$ and a division D of $[a,b]$, $\phi_S(D)$ may not be a division of $[a,b]$. But given $\alpha > 0$, there exists a gauge δ such that for any

δ-fine division $D = \{(\xi, [u,v])\}$ of $[a,b]$

$$\left| [a,b] \backslash \left(\bigcup_{(\xi,[u,v]) \in \phi_S(D)} [u,v] \right) \right| < \alpha.$$

This follows from the fact that, for a measure zero set S and $\alpha > 0$, there exists an open set $\mathcal{O}$ containing S whose measure is less than α. From this fact also follows that for a function f, if S is a set of measure zero and $\epsilon > 0$ there exists a gauge δ such that for any δ-fine S-tagged partial division D_S of $[a,b]$ we have $(D_S) \sum |f(\xi)||v-u| < \epsilon$. Given a division $D = \{(\xi, [u,v])\}$ of $[a,b]$ define $\|D\| = \max\{|v-u| : (\xi, [u,v]) \in D\}$. Recall that a function f on $[a,b]$ is said to be Riemann integrable if there exists a real number A for which given $\epsilon > 0$ there exists a positive number δ_ϵ such that for any division $D = \{(\xi, [u,v])\}$ of $[a,b]$ with $\|D\| < \delta_\epsilon$ we have

$$\left| (D) \sum f(\xi)(v-u) - A \right| < \epsilon.$$

See for example [1, 6]. Let $\epsilon > 0$ and choose such positive number δ_ϵ. Define a constant gauge δ on $[a,b]$ such that for each $\xi \in [a,b]$, $\delta(\xi) = \delta_\epsilon/2$. If $D = \{(\xi, [u,v])\}$ is a δ-fine division of $[a,b]$ and $D^* \sim D$ then $\|D\| = \|D^*\| < \delta_\epsilon$. It follows that

$$\left| (D^*) \sum f(\xi^*)(v-u) - A \right| < \epsilon.$$

We conclude this by the following theorem.

Theorem 2.2. *Let f be a function on $[a,b]$. If f is Riemann integrable then f is N-integrable.*

Theorem 2.3. *Let f be an N-integrable function on $[a,b]$. Then f is Henstock integrable on $[a,b]$.*

Proof. Let f be N-integrable on $[a,b]$, A its integral, S be an avoided set for f on $[a,b]$, and $\epsilon > 0$. Then there exists a gauge δ_1 such that for any δ_1-fine division $D = \{(\xi, [u,v])\}$ of $[a,b]$ we have

$$\left| (\phi_S(D)) \sum f(\xi)(v-u) - A \right| < \epsilon.$$

Since S is of measure zero, it follows from the discussion above that there exists a gauge δ_2 such that for any δ_2-fine S-tagged partial division D_S of $[a,b]$ we have $(D_S) \sum |f(\xi)||v-u| < \epsilon$. Now define $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$. Then for any δ-fine division D of $[a,b]$ we have

$$\begin{aligned} \left| (D) \sum f(\xi)(v-u) - A \right| &\leq \left| (\phi_S(D)) \sum f(\xi)(v-u) - A \right| \\ &\quad + (D_S) \sum |f(\xi)||v-u| \\ &< 2\epsilon. \end{aligned}$$

Therefore f is Henstock integrable.

The idea of taking the minimum of two gauges, as seen in the proof of Theorem 2.3, has been very useful throughout this paper, even if there are times that the process of doing it is not mentioned anymore.

Theorem 2.4 (Cauchy-Criterion). *Let f be a function on $[a,b]$. Then f is N-integrable if and only if there exists a null set S for which given $\epsilon > 0$ there exists a gauge δ on $[a,b]$ such that for any δ-fine divisions D and P, for any $D^* \sim \phi_S(D)$ and $P^* \sim \phi_S(P)$, we have*

$$\left|(D^*)\sum f(\xi^*)(v-u) - (P^*)\sum f(\xi^*)(v-u)\right| < \epsilon.$$

The following theorems can be shown.

Theorem 2.5. *Let f be a function on $[a,b]$.*

(1) *If f is N-integrable on $[a,b]$, then f is N-integrable on every subinterval of $[a,b]$.*
(2) *For $c \in (a,b)$, if f is N-integrable on each of the intervals $[a,c]$ and $[c,b]$, then f is N-integrable on $[a,b]$ and $\int_a^c f + \int_c^b f = \int_a^b f$.*

The following theorem states the linearity of the N-integral.

Theorem 2.6. *Let f and g be N-integrable functions on $[a,b]$ and $k \in \mathbb{R}$. Then*

(1) *The function kf is N-integrable on $[a,b]$ and $\int_a^b kf = k\int_a^b f$, and*
(2) *The function $f+g$ is N-integrable and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.*

Theorem 2.7. *Let f be an N-integrable function on $[a,b]$ and X be an avoided set for f. If S is a null set containing X then S is also an avoided set for f.*

Proof. Let X be an avoided set for an N-integrable function f on $[a,b]$, S be a subet of $[a,b]$ with measure zero containing X, and $\epsilon > 0$. Then there exists a gauge δ on $[a,b]$ such that for any δ-fine division D of $[a,b]$, and any $D^* \sim \phi_X(D)$ we have

$$\left|(D^*)\sum f(\xi^*)(v-u) - \int_a^b f\right| < \epsilon.$$

Further since S is of measure zero, we may assume that for any δ-fine S-tagged partial division D_S of $[a,b]$ we have

$$(D_S)\sum |f(\xi)||v-u| < \epsilon.$$

Now let D be any δ-fine division of $[a,b]$, $D^* \sim \phi_S(D)$ and $D_{S\backslash X} = \{(\xi,[u,v]) \in D : \xi \in S\backslash X\}$. Then $(D^* \cup D_{S\backslash X}) \sim \phi_X(D)$ and

$$\begin{aligned}\left|(D^*)\sum f(\xi^*)(v-u) - \int_a^b f\right| &\leq \left|(D^* \cup D_{S\backslash X})\sum f(\xi^*)(v-u) - \int_a^b f\right| \\ &\quad + (D_{S\backslash X})\sum |f(\xi)||v-u| \\ &< 2\epsilon.\end{aligned}$$

Theorem 2.8 (Saks-Henstock Lemma). *Let f be an N-integrable function on $[a,b]$ with primitive F. Then there exists a null set S such that given $\epsilon > 0$, there exists a gauge δ on $[a,b]$ such that for any δ-fine division D of $[a,b]$ and any $D^* \sim \phi_S(D)$ we have*

$$(D^*)\sum |f(\xi^*)(v-u) - F(u,v)| < \epsilon.$$

PROOF. Let f be N-integrable. Then f is Henstock integrable and the primitive F of f satisfies the Strong Lusin Condition (SL). Let S be an avoided set for f on $[a,b]$ and $\epsilon > 0$. Then there exists gauge δ on $[a,b]$ such that for any δ-fine division D of $[a,b]$ and any $D^* \sim \phi_S(D)$ we have

$$\left|(D^*)\sum f(\xi^*)(v-u) - F(a,b)\right| < \frac{\epsilon}{8}.$$

Let $\epsilon > 0$ and choose such δ. Since F is SL we may assume that for any δ-fine S-tagged partial division D we have

$$(D)\sum |F(u,v)| < \frac{\epsilon}{4}.$$

Now, let D be a δ-fine division of $[a,b]$ and $D^* \sim \phi_S(D)$. Split D^* into D_1 and D_2 such that D_1 contains those pairs $(\xi^*, [u,v])$ with $f(\xi^*)(v-u) - F(u,v) < 0$. Let $D_S = \{(\xi, [u,v]) \in D : \xi \in S\}$. We will show that $(D_i)\sum |f(\xi^*)(v-u) - F(u,v)| < \frac{\epsilon}{4}$ for $i = 1, 2$. For D_2, let $D_2^0 = \bigcup_{(\xi^*,[u,v])\in D_2}[u,v]$. For D_S, let $D_S^0 = \bigcup_{(\xi,[u,v])\in D_S}[u,v]$. The subset $\overline{[a,b]\backslash(D_2^0 \cup D_S^0)}$ of $[a,b]$ is a finite union of subintervals of $[a,b]$. We know that for each interval in $\overline{[a,b]\backslash(D_2^0 \cup D_S^0)}$, f is N-integrable. It follows that there exists a gauge $\delta_2 \le \delta$ such that for any δ_2-fine division B of $\overline{[a,b]\backslash(D_2^0 \cup D_S^0)}$ and any $B^* \sim \phi_S(B)$ we have

$$\left|(B^*)\sum f(\xi^*)(v-u) - (B)\sum F(u,v)\right| < \frac{\epsilon}{8}.$$

Let B be a δ_2-fine division of $\overline{[a,b]\backslash(D_2^0 \cup D_S^0)}$, $B^* \sim \phi_S(B)$, and C be the subset of $\phi_S(D)$ such that $D_2 \sim C$. Based on how B is obtained, B and D_2 are nonoverlapping. Furthermore, notice that the union $C \cup B \cup D_S$ forms a δ-fine division of $[a,b]$. Denote the union $C \cup B \cup D_S$ by Q. Since $B^* \sim \phi_S(B)$ and $D_2 \sim C$, it follows that $(D_2 \cup B^*)$ is $\phi_S(Q)$-compatible. Therefore

$$\begin{aligned}
(D_2)\sum |f(\xi^*)(v-u) - F(u,v)| &= \left|(D_2)\sum \{f(\xi^*)(v-u) - F(u,v)\}\right| \\
&= \left|(D_2)\sum f(\xi^*)(v-u) - (C)\sum F(u,v)\right| \\
&\le \left|(D_2 \cup B^*)\sum f(\xi^*)(v-u) - F(a,b)\right| \\
&\quad + \left|(B^*)\sum f(\xi^*)(v-u) - (B)\sum F(u,v)\right| \\
&\qquad + (D_S)\sum |F(u,v)| \\
&< \frac{\epsilon}{8} + \frac{\epsilon}{8} + \frac{\epsilon}{4} \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

A similar argument can be made for D_1.

3. THE N-INTEGRABLE FUNCTIONS

In this section, we identify which among the Henstock integrable functions are N-integrable. For a partial division $D = \{(\xi, [u, v])\}$ of $[a, b]$ define the double-tagged partial division $D^{\star\star} = \{(\{s, t\}, [u, v])\}$ where $[u, v]$ comes from D, $s, t \in [u, v]$, and the union of intervals in D and $D^{\star\star}$ are the same. In this case we say that $D^{\star\star}$ is D-double-tagged compatible or we simply write $D^{\star\star} \approx D$. Given a function f on $[a, b]$ and $\epsilon > 0$ we set

$$\Gamma_\epsilon^{\star\star} = \{(\{s, t\}, [u, v]) : |f(s) - f(t)| \geq \epsilon, s, t \in [u, v] \text{ and } [u, v] \subset [a, b]\}.$$

Note that the $\Gamma_\epsilon^{\star\star}$ is dependent only on f and ϵ.

Theorem 3.1. *Let f be a Henstock integrable function on $[a, b]$ and F be its primitive. Then f is N-integrable if and only if there exists a null set S for which given $\epsilon > 0$ there exists a gauge δ on $[a, b]$ such that for any δ-fine partial division $D = \{(\xi, [u, v])\}$ of $[a, b]$, and any $\phi_S(D)$-double-tagged compatible partial division $D^{\star\star} = \{(\{s, t\}, [u, v])\}$ we have*

$$(D^{\star\star} \cap \Gamma_\epsilon^{\star\star}) \sum |f(s) - f(t)||v - u| < \epsilon.$$

Proof. Let f be an N-integrable function on $[a, b]$, S be an avoided set for f on $[a, b]$ and $\epsilon > 0$. Then there exists a gauge δ on $[a, b]$ such that for any δ-fine partial division D of $[a, b]$ and any $D^* \sim \phi_S(D)$ we have

$$(D^*) \sum |f(\xi^*)(v - u) - F(u, v)| < \epsilon.$$

Choose such a gauge δ and let $D = \{(\xi, [u, v])\}$ be a δ-fine partial division of $[a, b]$. If $D^{\star\star} = \{(\{s, t\}, [u, v])\}$ is a $\phi_S(D)$-double-tagged compatible partial division of $[a, b]$, then $D_1^* = \{(s, [u, v]) : (\{s, t\}, [u, v]) \in D^\star\}$ and $D_2^* = \{(t, [u, v]) : (\{s, t\}, [u, v]) \in D^\star\}$ are both $\phi_S(D)$-compatible. It follows that

$$\begin{aligned}
(D^{\star\star} \cap \Gamma_\epsilon^{\star\star}) \sum |f(s) - f(t)||v - u| &\leq (D^{\star\star} \cap \Gamma_\epsilon^{\star\star}) \sum |f(s)(v - u) - F(u, v)| \\
&\quad + (D^{\star\star} \cap \Gamma_\epsilon^{\star\star}) \sum |F(u, v) - f(t)(v - u)| \\
&< \epsilon + \epsilon \\
&= 2\epsilon.
\end{aligned}$$

For the converse, there exists a null set S such that given $\epsilon > 0$, there exists a gauge δ such that for any δ-fine partial division $D = \{(\xi, [u, v])\}$ of $[a, b]$ and any $\phi_S(D)$-double-tagged compatible partial division $D^{\star\star} = \{(\{s, t\}, [u, v])\}$ we have

$$(D^{\star\star} \cap \Gamma_\epsilon^{\star\star}) \sum |f(s) - f(t)||v - u| < \epsilon.$$

Let $\epsilon > 0$ be given and choose such a gauge δ. Since f is Henstock integrable, we may assume that for any δ-fine partial division $D = \{(\xi, [u, v])\}$ of $[a, b]$ we have

$$(D) \sum \left| f(\xi)(v-u) - (H) \int_u^v f \right| < \epsilon \text{ and } (D_S) \sum |f(\xi)||v-u| < \epsilon.$$

Now, let $D = \{(\xi, [u, v])\}$ be a δ-fine division of $[a, b]$ and $D^* = (\xi^*, [u, v]) \sim \phi_S(D)$. Note that the double-tagged partial division $D^{\star\star} = \{(\{\xi, \xi^*\}, [u, v])\}$ is $\phi_S(D)$-double-tagged compatible and

$$\begin{aligned} &|(D^*) \textstyle\sum f(\xi^*)(v-u) - (\phi_S(D)) \sum f(\xi)(v-u)| \\ &\qquad \le (D^{\star\star}) \sum |f(\xi^*) - f(\xi)|\, |v-u| \\ &\qquad \le (D^{\star\star} \backslash \Gamma_\epsilon^{\star\star}) \sum |f(\xi^*) - f(\xi)|\, |v-u| \\ &\qquad\qquad + (D^{\star\star} \cap \Gamma_\epsilon^{\star\star}) \sum |f(\xi^*) - f(\xi)|\, |v-u|. \end{aligned}$$

Then

$$\begin{aligned} &\left|(D^*) \textstyle\sum f(\xi^*)(v-u) - (H) \int_a^b f \right| \\ &\quad \le \left|(D^*) \sum f(\xi^*)(v-u) - (\phi_S(D)) \sum f(\xi)(v-u)\right| \\ &\qquad + \left|(\phi_S(D) \cup D_S) \sum f(\xi)(v-u) - (H) \int_a^b f\right| \\ &\qquad + (D_S) \sum |f(\xi)||v-u| \\ &\quad < \epsilon(b-a) + \epsilon + \epsilon + \epsilon \\ &\quad = \epsilon(b-a+2). \end{aligned}$$

In what follows, given a function f on $[a, b]$ and a positive number ϵ,

$$\Gamma_\epsilon = \{(\xi, [u, v]) : |f(v) - f(u)| \ge \epsilon, \xi \in [u, v] \text{ and } [u, v] \subset [a, b]\}.$$

Similar to $\Gamma_\epsilon^{\star\star}$, Γ_ϵ depends only on f and ϵ.

Theorem 3.2. *Let f be an N-integrable function on $[a, b]$. Then there exists a null set S for which given $\epsilon > 0$ there exists a gauge δ on $[a, b]$ such that for any δ-fine partial division D of $[a, b]$*

$$(\phi_S(D) \cap \Gamma_\epsilon) \sum |f(v) - f(u)||v-u| < \epsilon.$$

Proof. This follows immediately from Theorem 3.1.

Theorem 3.3. *Let f be a function on $[a, b]$. If there exists a null set S for which given $\epsilon > 0$ there exists a gauge δ on $[a, b]$ such that for any δ-fine partial division $D = \{(\xi, [u, v])\}$ of $[a, b]$ we have*

$$(\phi_S(D) \cap \Gamma_\epsilon) \sum |f(v) - f(u)||v-u| < \epsilon$$

then f is continuous almost everywhere on $[a, b]$.

Proof. Let D_f be the set of discontinuity of f in $[a, b]\backslash S$. Assume that $a, b \in S$. If $x \in D_f$, there exists $\eta(x) > 0$ such that for all $s > 0$ there exists a closed interval $J_{x,s}$ whose one endpoint is x with $J_{x,s} \subset (x - s, x + s)$ and that $|f(J_{x,s})| > \eta(x)$. Here $f(I) = |f(v) - f(u)|$ whenever $I = [u, v]$. Fix a positive integer n and let $E_n = \left\{x \in [a, b] : \eta(x) \geq \frac{1}{n}\right\}$. We will show that for each n, E_n is of Lebesgue measure zero. Given $\epsilon > 0$, there exists a gauge δ mapping $[a, b]$ to $(0, 1)$ such that for any δ-fine partial division D of $[a, b]$ we have

$$(D \cap \Gamma_{\frac{\epsilon}{n}}) \sum |f(I_x)||I_x| < \frac{\epsilon}{n}$$

Let $\mathcal{S}_n = \{J_{x,s} : x \in E_n, 0 < s < \delta(x)\}$. Note that $\mathcal{S}_n$ is a Vitali cover for E_n. By the Vitali covering theorem there are pairwise disjoint intervals $J_1, J_2, \ldots, J_k$ in $\mathcal{S}_n$ and sequence of closed intervals $\{J_i\}_{i=k+1}^{\infty} \subset \mathcal{S}_n$ such that

$$E_n \subset \bigcup_{i=1}^{k} J_i \cup \bigcup_{i=k+1}^{\infty} J_i \text{ and } \sum_{i=k+1}^{\infty} |J_i| \leq \epsilon.$$

Note that $D = \{(x, J_i)\}_{i=1}^{k}$ is a δ-fine partial division of $[a, b]$ and for each $J_i \in D$, $J_i = J_{x,s}$ for some $J_{x,s}$ in $\mathcal{S}_n$. Hence

$$\begin{aligned} \frac{1}{n}(D) \sum |J_i| &= \frac{1}{n}(D) \sum |J_{x,s}| \\ &\leq (D) \sum \eta(x)|J_{x,s}| \\ &< (D\backslash\Gamma_{\frac{\epsilon}{n}}) \sum |f(J_{x,s})||J_{x,s}| + (D \cap \Gamma_{\frac{\epsilon}{n}}) \sum |f(J_{x,s})||J_{x,s}| \\ &\leq \frac{\epsilon}{n}(b - a) + \frac{\epsilon}{n} \end{aligned}$$

It follows that $\mu(E_n) < \epsilon(2 + b - a)$. Since ϵ is arbitary we conclude that E_n is of measure zero. Since D_f is a countable union of measure zero sets E_n, D_f is of measure zero. Since D_f and S are of measure zero, we conclude that f is continuous almost everywhere on $[a.b]$.

Theorem 3.4. *Let f be a Henstock integrable function on $[a, b]$. If f is continuous almost everywhere on $[a, b]$ then f is N-integrable.*

Proof. Let f be continuous almost everywhere on $[a, b]$ and Henstock integrable. Let D_f be the set of discontinuity of f on $[a, b]$. Let $\epsilon > 0$. Since f is Henstock integrable, there exists a gauge δ on $[a, b]$ such that for any δ-fine partial division $D = \{(\xi, [u, v])\}$ of $[a, b]$ we have

$$(D) \sum \left| f(\xi)(v - u) - (H) \int_u^v f \right| < \epsilon \text{ and } (D_{D_f}) \sum \left| (H) \int_u^v f \right| < \epsilon$$

where $D_{D_f} = \{(\xi, [u, v]) \in D : \xi \in D_f\}$. By continuity of f on each element of $[a, b]\backslash D_f$, we may assume that for every δ-fine pair $(\xi, [u, v])$ with $\xi \in [a, b]\backslash D_f$

$$|f(\xi) - f(\xi^*)| < \epsilon$$

whenever $\xi^* \in [u, v]$. So for any δ-fine division D of $[a, b]$ and $D^* \sim \phi_{D_f}(D)$ we have

$$
\begin{aligned}
&\left|(D^*)\sum f(\xi^*)(v-u)-(H)\int_a^b f\right| \\
&\qquad\le \left|(D^*)\sum f(\xi^*)(v-u)-(\phi_{D_f}(D))\sum(H)\int_u^v f\right| \\
&\qquad\qquad + (D_{D_f})\sum\left|(H)\int_u^v f\right| \\
&\qquad\le (\phi_{D_f}(D))\sum|f(\xi^*)-f(\xi)||v-u| \\
&\qquad + (\phi_{D_f}(D))\sum\left|f(\xi)(v-u)-(H)\int_u^v f\right| \\
&\qquad\qquad + (D_{D_f})\sum\left|(H)\int_u^v f\right| \\
&\qquad< \epsilon(b-a)+\epsilon+\epsilon.
\end{aligned}
$$

Combining Theorem 3.2, Theorem 3.3 and Theorem 3.4, we have the following result.

Theorem 3.5. *Let f be a function $[a,b]$. Then the following statements are equivalent:*

(1) *The function f is N-integrable*
(2) *The function f is Henstock integrable on $[a,b]$ and there exists a null set S for which given $\epsilon>0$ there exists a gauge δ such that for any δ-fine partial division $D=\{(\xi,[u,v])\}$ of $[a,b]$ we have*

$$(\phi_S(D)\cap\Gamma_\epsilon)\sum|f(v)-f(u)||v-u|<\epsilon$$

and

(3) *The function f is Henstock integrable on $[a,b]$ and continuous almost everywhere.*

4. THE AVOIDED SET

For a Riemann integrable function f, the avoided set may be empty, even if f has discontinuities. So it is natural to ask what is the optimal avoided set for an N-integrable function. We answer this query in this section. The ideas here will also able us to present N-integral as a generalization of improper Riemann integral. In what follows, for a function f on $[a,b]$, we set

$$S_\infty=\{x\in[a,b]:\text{ for some }\{x_n\}\text{ in }[a,b],x_n\to x\text{ and }f(x_n)\to\pm\infty\}.$$

Note that for any function f on $[a,b]$, S_∞ is closed.

Lemma 4.1. *Let $f:[a,b]\to\mathbb{R}$ be N-integrable and S be an avoided set for f. Then $S_\infty\subset S$.*

Proof. Let x be in S_∞ but not in S and δ be a gauge on $[a,b]$. Choose a δ-fine pair $(x,[u,v])$ such that f is not bounded on $[u,v]$. Note that since $(x,[u,v])$ is δ-fine, it is an element of some δ-fine division D of $[a,b]$ with $(x,[u,v])\in\phi_S(D)$.

The result below can be easily shown.

Lemma 4.2. *Let $f : [a,b] \to \mathbb{R}$ and X be a subset of $[a,b]$ be of measure with $X \cap S_\infty = \emptyset$. Then given $\epsilon > 0$, there exists a gauge δ such that for any δ-fine X-tagged partial division $D = \{(\xi, [u,v])\}$ of $[a,b]$ and any $D^* = \{(\xi^*, [u,v])\}$ with $I(D^*) = I(D)$ we have*

$$(D^*) \sum |f(\xi^*)|(v-u) < \epsilon.$$

Theorem 4.3. *Let $f : [a,b] \to \mathbb{R}$ and suppose that S_∞ is of measure zero. Then f is N-integrable if and only if there exists a real number A for which given $\epsilon > 0$ there exists a gauge δ such that for any δ-fine partial division $D = \{(\xi, [u,v])\}$, and any $D^* = \{(\xi^*, [u,v])\} \sim \phi_{S_\infty}(D)$, we have*

$$\left|(D^*) \sum f(\xi^*)(v-u) - A\right| < \epsilon.$$

PROOF. Suppose f is N-integrable. Then there exist a subset S of $[a,b]$ of measure zero and a real number A for which given $\epsilon > 0$ there exists a gauge δ such that for any δ-fine partial division $D = \{(\xi, [u,v])\}$, and any $D^* = \{(\xi^*, [u,v])\} \sim \phi_S(D)$ we have

$$\left|(\phi_S(D)) \sum f(\xi^*)(v-u) - A\right| < \epsilon.$$

From the previous lemma, we may assume that for any δ-fine partial division $D = \{(\xi, [u,v])\}$ of $[a,b]$ whose tags are in $S \setminus S_\infty$ and any $D^* = \{(\xi^*, [u,v])\} \sim D$, we have

$$(D^*) \sum |f(\xi^*)|(v-u) < \epsilon.$$

Now, let $D = \{(\xi.[u,v])\}$ be a δ-fine division of $[a,b]$ and $D^* = \{(\xi^*, [u,v])\} \sim \phi_{S_\infty}(D)$. Set

$$D_{S\setminus S_\infty} = \{(\xi.[u,v]) \in D : \xi \in S \text{ and } \xi \notin S_\infty\}.$$

Note that $I(\phi_S(D)) = I\left(D^* \setminus D^*_{S\setminus S_\infty}\right)$, where $D^*_{S\setminus S_\infty}$ is the subset of D^* such that

$$I(D^*_{S\setminus S_\infty}) = I(D_{S\setminus S_\infty}).$$

It follows that

$$\begin{aligned} \left|(D^*) \sum f(\xi^*)(v-u) - A\right| &\leq \left|\left(D^* \setminus D^*_{S\setminus S_\infty}\right) \sum f(\xi^*)(v-u) - A\right| \\ &\qquad + \left(D^*_{S\setminus S_\infty}\right) \sum |f(\xi^*)|(v-u) \\ &< 2\epsilon. \end{aligned}$$

The converse follows from the definition.

The importance of stating the result below is that it presents the N-integral as a generalization of improper Riemann integral.

Theorem 4.4. *Let $f : [a,b] \to \mathbb{R}$. Then f is N-integrable if and only if there exists a real number A and a subset S of $[a,b]$ of measure zero for which given $\epsilon > 0$ there exists a gauge δ on $[a,b]$ such that for any δ-fine partial division $D = \{(\xi, [u,v])\}$,* *and any* $D^* = \{(\xi^*, [u,v])\} \sim \phi_S(D)$ *we have* $[u,v] \cap S = \emptyset$ *for each* $(\xi, [u,v]) \in \phi_S(D)$*,*

$$(b-a) - (\phi_S(D)) \sum (v-u) < \epsilon$$

and

$$\left| (D^*) \sum f(\xi^*)(v-u) - A \right| < \epsilon.$$

In this case, we may choose $S = S_\infty$*.*

5. SOME EXAMPLES

Recall that the function f on $[0,1]$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

is nowhere continuous. This function is called the Dirichlet function. It follows from Theorem 3.5 that this function, while Henstock integrable, is not N-integrable. Moreover the function

$$f(x) = \begin{cases} 2x\sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right) & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$$

while not Riemann integrable, is N-integrable. Therefore the converses of Theorem 2.3 and Theorem 2.2 do not hold. Before we go to our examples, we shall present a characterization of continuous almost everywhere functions.

Theorem 5.1. *A function* f *on* $[a,b]$ *is continuous almost everywhere if and only if there exists a null set* S *for which given* $\epsilon > 0$ *there exists a gauge* δ *on* $[a,b]$ *such that for any* δ*-fine partial division* $D = \{(\xi, [u,v])\}$ *of* $[a,b]$ *we have*

$$(\phi_S(D) \cap \Gamma_\epsilon) \sum |f(v) - f(u)||v-u| < \epsilon.$$

Proof. Let f be continuous almost everywhere on $[a,b]$. Then there exists a null set S in $[a,b]$ such that f is continuous on every $x \in [a,b]\backslash S$. Let $\epsilon > 0$ be given. It follows that there exists a gauge δ on $[a,b]$ such that for any δ-fine pair $(\xi, [u,v])$ with $\xi \in [a,b]\backslash S$, we have $|f(v) - f(u)| < \epsilon$. Choose such gauge δ. Then for any δ-fine division $D = \{(\xi, [u,v])\}$ of $[a,b]$, we have $\phi_S(D) \cap \Gamma_\epsilon = \emptyset$. The converse follows from Theorem 3.3.

A partial partition $D = \{[u,v]\}$ is a finite collection of nonoverlapping subinervals of $[a,b]$. Given a subset X of $[a,b]$, a partial partition D is said to be X-vertexed if for each $[u,v] \in D$, either $u \in X$ or $v \in X$. In [6], a function f on $[a,b]$ is said to be $BV^*(X)$ if there exists a nonnegative number M such that for any X-vertexed partial partition D of $[a,b]$ we have $(D)\sum |f(v) - f(u)| \leq M$. The function f is BVG^* on $[a,b]$ if $[a,b] = \bigcup_{i=1}^{\infty} X_i$ such that for each i, there is a positive number M_i such that for any X_i-vertexed partial partition D of $[a,b]$ we have $(D)\sum |f(v) - f(u)| \leq M_i$. In this case we say that f is BVG^* on $[a,b]$ using $\{(X_i, M_i)\}$.

Theorem 5.2. *If a function f on $[a,b]$ is BVG^* on $[a,b]$ then it is continuous almost everywhere.*

Proof. Let $\epsilon > 0$ and suppose that f is BVG^* using $\{(X_i, M_i)\}$. For each i, put $\delta(x) \leq \eta_i$ for all $x \in X_i$ where $\eta_i = \frac{\epsilon}{M_i 2^{i+1}}$. Let D be δ-fine a partial division of $[a,b]$. For each i, let

$$D_i = \{(\xi, [u,v]) \in D : \xi \in X_i\}.$$

Then

$$\begin{aligned}
(D \cap \Gamma_\epsilon) \sum |f(v) - f(u)|(v-u) &= \sum_{i=1}^{\infty} (D_i \cap \Gamma_\epsilon) \sum |f(v) - f(u)|(v-u) \\
&< \sum_{i=1}^{\infty} \eta_i (D_i \cap \Gamma_\epsilon) \sum |f(v) - f(u)| \\
&\leq \sum_{i=1}^{\infty} \eta_i (D_i \cap \Gamma_\epsilon) \sum |f(v) - f(\xi)| \\
&\qquad + \sum_{i=1}^{\infty} \eta_i (D_i \cap \Gamma_\epsilon) \sum |f(\xi) - f(u)| \\
&\leq \sum_{i=1}^{\infty} \eta_i M_i + \sum_{i=1}^{\infty} \eta_i M_i \\
&< \epsilon.
\end{aligned}$$

The next result follows from Theorem 3.5 and Theorem 5.2.

Theorem 5.3. *Let f be a Henstock integrable function on $[a,b]$. If f is BVG^* on $[a,b]$ then f is N-integrable.*

A function f on $[a,b]$ is Baire class one or of first Baire class or simply Baire one if it is a pointwise limit of a sequence of continuous functions on $[a,b]$. Recently, in [3], a Baire one function f was characterized by topologically describing its set of discontinuity (D_f) and how it behaves on this set. As a consequence, it follows from Theorem [3, Theorem 3.4] that if for a function f, $f(x) = 0$ for $x \in D_f$, then f is Baire one. Therefore we have the following result showing the closeness of N-integrable functions to Baire one functions.

Theorem 5.4. *Let f be a function on $[a,b]$ and suppose that D_f is of measure zero. If $\overline{D_f}$ is of measure zero then there exists a Baire one function g on $[a,b]$ such that $g = f$ almost everywhere on $[a,b]$.*

Finally, we present an example whose set of discontinuity is of measure zero but not countable. Consider the function F defined by

$$F(x) = \begin{cases} (x-a)^2 \sin\left(\frac{1}{x-a}\right)^2 & \text{if } x \in (a,b], \\ 0 & \text{if } x = a. \end{cases}$$

The function F is differentiable at every point in $[a,b]$, $F'(a)=0$ and

$$F'(x)=2(x-a)\sin\left(\frac{1}{(x-a)^2}\right)-\frac{2}{x-a}\cos\left(\frac{1}{(x-a)^2}\right)$$

for $x\in(a,b]$. Now, consider the Cantor set $\mathcal{C}$. Recal that the Cantor set is nowhere dense, uncountable, and of measure zero. Let $[a,b]\setminus\mathcal{C}=\cup_{n=1}^{\infty}(a_n,b_n)$. For each n, consider

$$F_{1,n}(x)=\begin{cases}(x-a_n)^2\sin\left(\frac{1}{x-a_n}\right)^2 & \text{if } x\in(a_n,b_n],\\ 0 & \text{if } x=a_n\end{cases}$$

and

$$F_{2,n}(x)=\begin{cases}(b_n-x)^2\sin\left(\frac{1}{b_n-x}\right)^2 & \text{if } x\in[a_n,b_n),\\ 0 & \text{if } x=b_n.\end{cases}$$

Note that both $F_{1,n}$ and $F_{2,n}$ are differentiable on $[a_n,b_n]$ and that the graph of $F_{1,n}$ is just the reflection of $F_{2,n}$ with respect to $\left(\frac{a_n+b_n}{2}+\frac{a_n+b_n}{2}\right)$. From $\left(a_n,\frac{a_n+b_n}{2}\right)$, choose c_n such that $F'_{1,n}(c_n)=0$. Then choose d_n such that $c_n-a_n=b_n-d_n$. Hence

$$F_{1,n}(c_n)=(c_n-a_n)^2\sin\left(\frac{1}{c_n-a_n}\right)^2=(b_n-d_n)^2\sin\left(\frac{1}{d_n-d_n}\right)^2=F_{2,n}(d_n).$$

Note also that, by symmetry, $F'_{2,n}(d_n)=0$. Now define

$$F_n(x)=\begin{cases}F_{1,n}(x) & \text{if } x\in(a_n,c_n]\\ F_{1,n}(c_n) & \text{if } x\in(c_n,d_n)\\ F_{2,n}(x) & \text{if } x\in[d_n,b_n).\end{cases}$$

If we set $F_n(a_n)=F_n(b_n)=0$ then F_n is differentiable at all points in $[a_n,b_n]$. At this point, it is important to note that the the graph of F_n is symmetric with respect to the midpoint of a_n and b_n. Define a function G on $[a,b]$ by

$$G(x)=\begin{cases}F_n(x) & \text{if } x\in(a_n,b_n),\\ 0 & \text{if } x\in\mathcal{C}.\end{cases}$$

As constructed, for each n, G is differentiable on (a_n,b_n). Furthermore, it can be shown that $G'(\xi)=0$ for $\xi\in\mathcal{C}$.

At this point we already know that G is differentiable at any point in $[a,b]$. So its derivative G' is Henstock integrable. Note also that G' is continuous on (a_n,b_n) for any n. We will show that G is not continuous at any $\xi\in\mathcal{C}$. For $\xi\in\mathcal{C}$, choose $\{(a_{n_k},b_{n_k})\}$ such that

$$a_{n_k}\to\xi \text{ as } k\to\infty.$$

For each k, choose $m_k>k$ such that

$$a_{n_k}+\frac{1}{\sqrt{2m_k\pi}}\in(a_{n_k},c_{n_k}]$$

and set
$$x_k = a_{n_k} + \frac{1}{\sqrt{2m_k\pi}}.$$
So $x_k \to \xi$ but
$$G'(x_k) = -2\sqrt{2m_k\pi}.$$
We will now show that G' is N-integrable. Let $\epsilon > 0$. Since G' is the derivative of G there exists a gauge δ such that for any δ-fine pair $(\xi, [u, v])$ we have
$$|G'(\xi)(v-u) - G(u, v)| < \epsilon(v-u).$$
We may assume that for any δ-fine pair $(\xi, [u, v])$ with $\xi \in [a, b] \setminus \mathcal{C}$, we have
$$|G'(\xi) - G'(\xi^*)| < \epsilon \text{ whenever } \xi^* \in [u, v].$$

Now let $D = \{(\xi, [u, v])\}$ be a δ-fine division of $[a, b]$ and $D^* \sim \phi_{\mathcal{C}}(D)$. Split D into D_1 and D_2 such that $D_1 = \{(\xi, [u, v]) \in D : \xi \in \mathcal{C}\}$ and $D_2 = D \setminus D_1$. It follows that
$$\begin{aligned}
\left|(D^*)\sum G'(\xi^*)(v-u) - G(a,b)\right| &\leq (D^*)\sum |G'(\xi^*)(v-u) - G(u,v)| \\
&\qquad + (D_1)\sum |G(u,v)| \\
&\leq (\phi_{\mathcal{C}}(D))\sum |G'(\xi^*) - G'(\xi)|(v-u) \\
&\qquad + (\phi_{\mathcal{C}}(D)\sum |G'(\xi)(v-u) - G(u,v)| \\
&\qquad + (D_1)\sum |G(u,v) - G'(\xi)(v-u)| \\
&\qquad + (D_1)\sum |G'(\xi)||v-u| \\
&\leq \epsilon(b-a) + \epsilon(b-a) \\
&< 2\epsilon(b-a).
\end{aligned}$$
Therefore G' is N-integrable on $[a, b]$. Furthermore, it can be shown that for the function G, $S_\infty = \mathcal{C}$. The function G' is neither Lebesgue nor improper Riemann integrable.

Excluding the point-interval pairs which are tagged in the avoided set S in forming the Riemann sum enables us to integrate not just all Riemann integrable functions but even all improper Riemann integrable functions which are not Lebesgue integrable. In fact, there are N-integrable functions that are neither Lebesgue nor improper Riemann integrable. Since the discontinuity of an N-integrable function is F_σ and of measure zero, it follows from [10, p. 273] that it is of first category. A natural question that arises from this paper is whether it is possible to develop an integral that integrates functions whose sets of discontinuity are of the first category.

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