SOME PROPERTIES OF LEFT WEAKLY JOINTLY PRIME (R, S)-SUBMODULES

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Abstract. Let R and S be commutative rings and M be an (R, S)-module. A proper (R, S)-submodule P of M is called a left weakly jointly prime if for each elements a and b in R and (R, S)-submodule K of M with $abKS \subseteq P$ implies either $aKS \subseteq P$ or $bKS \subseteq P$. In this paper, we present some properties of left weakly jointly prime (R, S)-submodule. We give some necessary and sufficient conditions of left weakly jointly prime (R, S)-submodules. Moreover, we present that every left weakly jointly prime (R, S)-submodule contains a minimal left weakly jointly prime (R, S)-submodule. At the end of this paper, we show that in left multiplication (R, S)-module, every left weakly jointly prime (R, S)-submodule is equal to jointly prime (R, S)-submodules.

Keywords and Phrases: weakly prime submodule, jointly prime submodule, prime submodule

1. INTRODUCTION

Throughout this paper, ring R and ring S will denote commutative rings, and R-module M means an Abelian group under addition. The concept of the R-module has been studied in depth in Adkins [1].

An R-module has been generalized into an (R, S)-bimodule. When R and S are arbitrary rings, Khumprapussorn *et al.* [7] have generalized (R, S)-bimodule into (R, S)-module. An (R, S)-module has an (R, S)-bimodule structure when both rings R and S have central idempotent elements. When R and S are rings with identity, we have an (R, S)-module is also an (R, S)-bimodule.

Moreover, an (R, S)-submodule of an (R, S)-module M is a subgroup N of M such that $rns \in N$ for all $r \in R$, $n \in N$, and $s \in S$. Let P be a proper (R, S)-submodule of M. By Khumprapussorn *et al.* [7], a proper (R, S)-submodule P of M is called jointly prime if for each left ideal I of R, right ideal J of S, and

(R,S)-submodule N of M with $INJ \subseteq P$ implies either $IMJ \subseteq P$ or $N \subseteq P$. If R and S are commutative rings, we have a proper (R,S)-submodule P of M is called jointly prime if for each ideal I of R, ideal J of S, and (R,S)-submodule N of M with $INJ \subseteq P$ implies either $IMJ \subseteq P$ or $N \subseteq P$. The concept of jointly prime (R,S)-submodules when R and S are arbitrary rings have been studied by Khumprapussorn *et al.*[7] and continued by Yuwaningsih and Wijayanti [8].

On module theory, a proper submodule N of an R-module M is called prime if for each element a of R and element $m \in M$ with $am \in N$ implies $m \in N$ or $aM \subseteq N$. Prime submodules have been introduced and studied by Dauns [6]. As time went by, the researchers began to generalize the definition of prime submodules to weakly prime submodules. A proper submodule N of M is called weakly prime if for each submodule P of M and elements a, b of R satisfy $abP \subseteq N$, implies either $aP \subseteq N$ or $bP \subseteq N$. Weakly prime submodules have been introduced by Behboodi and Koohy [4]. Moreover, the studied about weakly prime submodules have been continued by Azizi [2], Behboodi [5], and Azizi [3].

In this paper, we present some properties of left weakly jointly prime (R,S)-submodules as the generalization of jointly prime (R,S)-submodules. A proper (R,S)-submodule P of M is called left weakly jointly prime if for each (R,S)-submodule N of M and elements a, b of R such that $abNS \subseteq P$ implies either $aNS \subseteq P$ or $bNS \subseteq P$. Moreover, we present some properties of left weakly jointly prime (R,S)-submodules. Some of these properties are as follows: a proper (R,S)-submodules is left weakly jointly prime if and only if the annihilator of this quotient (R,S)-module over ring R is a prime ideal of R; when we give a left weakly jointly prime (R,S)-submodule, the set of that annihilator over ring R form a chain of prime ideals; if a left weakly jointly prime (R,S)-submodule is irreducible then it is a jointly prime; any left weakly jointly prime (R,S)-submodule P of M contains a minimal left weakly jointly prime (R,S)-submodule; and every left weakly jointly prime (R,S)-submodule is equal to jointly prime (R,S)-submodule in left multiplication (R,S)-module.

2. LEFT WEAKLY JOINTLY PRIME (R,S)-SUBMODULES

Before we present the definition of left weakly jointly prime (R,S)-submodules, we describe first the jointly prime (R,S)-submodule. As we have already stated earlier, when R and S are commutative rings, a proper (R,S)-submodule P of M is called jointly prime if for each ideal I of R, ideal J of S, and (R,S)-submodule N of M with $INJ \subseteq P$ implies either $IMJ \subseteq P$ or $N \subseteq P$. When R and S are arbitrary rings, Khumprapussorn *et al.* [7] have given some characteristic of jointly prime (R,S)-submodules. In this paper, we modify those characteristics when R and S are commutative rings as follows.

Proposition 2.1. *Let M be an (R,S)-module with $a \in RaS$ for all $a \in M$. Then the following statements are equivalent.*

(1) P is jointly prime.

(2) *For all ideal I of R, $m \in M$, and ideal J of S, $ImJ \subseteq P$ implies $IMJ \subseteq P$ or $m \in P$.*

(3) *For all $a \in R$, $m \in M$, and $b \in S$, $amb \in P$ implies $aMb \subseteq P$ or $m \in P$.*

(4) *For all $a \in R$ and $m \in M$, $amS \subseteq P$ implies $aMS \subseteq P$ or $m \in P$.*

A proper submodule P of R-module M is called weakly prime if for each submodule K of M and elements a, b of R such that $abK \subseteq P$ implies either $aK \subseteq P$ or $bK \subseteq P$. This definition has studied by Azizi [2]. Based on that definition, we present the definition of left weakly jointly prime (R, S)-submodules as follows.

Definition 2.2. *Let M be an (R, S)-module. A proper (R, S)-submodule P of M is called left weakly jointly prime if for each (R, S)-submodule N of M and elements $a, b \in R$ such that $abNS \subseteq P$ implies $aNS \subseteq P$ or $bNS \subseteq P$.*

Note that right weakly jointly prime (R, S)-submodules can be defined and studied analogously. Now, if we have a condition $a \in RaS$ for all $a \in M$, then we give the definition of left weakly jointly prime (R, S)-submodules as follows.

Definition 2.3. *Let M be an (R, S)-module with $a \in RaS$ for all $a \in M$. A proper (R, S)-submodule P of M is called left weakly jointly prime if for each (R, S)-submodule N of M and ideals I and J of R such that $IJNS \subseteq P$ implies either $INS \subseteq P$ or $JNS \subseteq P$.*

We can easily show that the two definitions of left weakly jointly prime (R, S)-submodule above are equivalent. Moreover, we give some example of left weakly jointly prime (R, S)-submodules as follows.

Example 2.4. *Let $\mathbb{Z}$ be an $(2\mathbb{Z}, 2\mathbb{Z})$-module and $2\mathbb{Z}$ be an $(2\mathbb{Z}, 2\mathbb{Z})$-submodule of $\mathbb{Z}$. We can show that $2\mathbb{Z}$ is a left weakly jointly prime $(2\mathbb{Z}, 2\mathbb{Z})$-submodule. Let any $a, b \in 2\mathbb{Z}$ with $a = 2k$ and $b = 2l$ and let any $(2\mathbb{Z}, 2\mathbb{Z})$-submodule $N = x\mathbb{Z}$ of $\mathbb{Z}$, for some $k, l, x \in \mathbb{Z}$. Clearly that $abN(2\mathbb{Z}) = (8klx)\mathbb{Z}^2 \subseteq (8klx)\mathbb{Z} \subseteq 2\mathbb{Z}$. We obtain $aN(2\mathbb{Z}) \subseteq 4kx\mathbb{Z} \subseteq 2\mathbb{Z}$ or $bN(2\mathbb{Z}) \subseteq 4lx\mathbb{Z} \subseteq 2\mathbb{Z}$. Thus, $2\mathbb{Z}$ is a left weakly jointly prime $(2\mathbb{Z}, 2\mathbb{Z})$-submodule of $\mathbb{Z}$.*

Example 2.5. *Let R and S are commutative rings with*

$$R = \left\{ \left(\begin{array}{cc} a & b \\ 0 & 0 \end{array} \right) \middle| a, b \in 2\mathbb{Z} \right\} \text{ and } S = \left\{ \left(\begin{array}{cc} a & 0 \\ b & 0 \end{array} \right) \middle| a, b \in 2\mathbb{Z} \right\}.$$

Let an (R, S)-module M with

$$M = \left\{ \left(\begin{array}{cc} a & 0 \\ b & c \end{array} \right) \middle| a, b, c \in 2\mathbb{Z} \right\}.$$

Easily we can check that an (R, S)-submodule X of M with

$$X = \left\{ \left(\begin{array}{cc} x & 0 \\ 0 & y \end{array} \right) \middle| x, y \in 2\mathbb{Z} \right\}$$

is a left weakly jointly prime (R, S)-submodule of M.

3. SOME PROPERTIES OF LEFT WEAKLY JOINTLY PRIME (R, S)-SUBMODULES

In this section, we present some properties of left weakly jointly prime (R, S)-submodules. However, before we recall the definition of annihilator of the quotient (R, S)-modules over a ring R as follows.

Definition 3.1. *Let M be an (R, S)-module and N be an (R, S)-submodule of M. We define the annihilator of quotient (R, S)-module M/N over a ring R is the set $(N :_R M) = \{r \in R \mid rMS \subseteq N\}$.*

In general $(N :_R M)$ is only an additive subgroup of R. But if we have the condition $S^2 = S$, clearly that $(N :_R M)$ is an ideal of R.

Now, we give some properties of left weakly jointly prime (R, S)-submodules. In the first property, we show that every jointly prime (R, S)-submodules is a left weakly jointly prime.

Proposition 3.2. *Let M be an (R, S)-module with $a \in RaS$ for all $a \in M$ and P is a jointly prime (R, S)-submodule of M. Then, P is a left weakly jointly prime (R, S)-submodule of M.*

PROOF. Let any ideal I, J of R and an (R, S)-submodule N of M such that $IJNS \subseteq P$. Since R is commutative, then $I(RJNS)S = R(IJNS)S \subseteq IJNS \subseteq P$. Since P is jointly prime then we get $IMS \subseteq P$ or $RJNS \subseteq P$. Since $a \in RaS$ for all $a \in M$, we have $IMS \subseteq P$ or $JNS \subseteq P$. Hence, $INS \subseteq IMS \subseteq P$ or $JNS \subseteq P$. Hence, P is a left weakly jointly prime (R, S)-submodule of M. $\square$

Now, we present some necessary and sufficient conditions of a proper (R, S)-submodule being left weakly jointly prime.

Proposition 3.3. *Let M be an (R, S)-module with $S^2 = S$ and N be a proper (R, S)-submodule of M. Then, N is a left weakly jointly prime (R, S)-submodule of M if and only if for each (R, S)-submodule K of M with $K \nsubseteq N$ satisfy $(N :_R K)$ is a prime ideal of R.*

PROOF. ($\Rightarrow$). Let an (R, S)-submodule K of M with $N \subset K$ and ideal I and ideal J of R such that $IJ \subseteq (N :_R K)$. This means $IJKS \subseteq N$. Since N is a left weakly jointly prime (R, S)-submodule of M, then $IKS \subseteq N$ or $JKS \subseteq N$. Consequently, we obtain $I \subseteq (N :_R K)$ or $J \subseteq (N_R K)$. Hence, $(N :_R K)$ is a prime ideal of R.
($\Leftarrow$). Let an ideal I and J of R and (R, S)-submodule L of M such that $IJLS \subseteq N$. Then, $IJ(L + N)S \subseteq N$, so $IJ \subseteq (N :_R L + N)$. Since $(N :_R L + N)$ is prime ideal, then we obtain $I \subseteq (N :_R L + N)$ or $J \subseteq (N :_R L + N)$. Consequently, we obtain $I(L + N)S \subseteq N$ or $J(L + N)S \subseteq N$. Thus, $ILS \subseteq N$ or $JLS \subseteq N$. Hence, N is a left weakly jointly prime (R, S)-submodule of M. $\square$

Proposition 3.4. *Let M be an (R, S)-module with $S^2 = S$ and N be a proper (R, S)-submodule of M. Then, N is a left weakly jointly prime (R, S)-submodule if and only if for every $a, b \in R$ and $m \in M \setminus N$ satisfy $abmS \subseteq N$ implies $amS \subseteq N$ or $bmS \subseteq N$. On the other word, $(N :_R m)$ is a prime ideal of R.*

PROOF. ($\Rightarrow$). Let any $a, b \in R$ and $m \in M \setminus N$. We construct $\langle m \rangle$ is an (R, S)-submodule of M. Then we get

$$abmS \subseteq ab\langle m \rangle S \subseteq N.$$

Since N is a left weakly jointly prime (R, S)-submodule of M, then we have $amS \subseteq a\langle m \rangle S \subseteq N$ or $bmS \subseteq b\langle m \rangle S \subseteq N$.

($\Leftarrow$). Let any $a, b \in R$ and (R, S)-submodule K of M with $K \nsubseteq N$ such that $abKS \subseteq N$. For each $k \in K$, we have $abkS \subseteq N$. Based on the hypothesis, $akS \subseteq N$ or $bkS \subseteq N$. Thus, $aKS \subseteq N$ or $bKS \subseteq N$. Hence, N is a left weakly jointly prime (R, S)-submodule of M. $\square$

Proposition 3.5. *Let M be an (R, S)-module with $S^2 = S$ and N is a proper (R, S)-submodule of M. If M satisfies $a \in RaS$ for all $a \in M$, then the following statements are equivalent.*

(1) *N is a left weakly jointly prime (R, S)-submodule of M.*
(2) *For any $x, y \in M$, if $(N :_R x) \neq (N :_R y)$, then $N = (N + RxS) \cap (N + RyS)$.*

PROOF. $(1 \Rightarrow 2)$. Let $r \in (N :_R x) \setminus (N :_R y)$, where $r \in R$ i.e. $rxS \subseteq N$ and $ryS \nsubseteq N$. Since by Proposition 3.4, $(N :_R y)$ is a prime ideal of R, it's easy to see that $(N :_R y) = (N :_R ryS)$. If we let $t \in (N + RxS) \cap (N + RyS)$, then $t = n_1 + r_1 x s_1 = n_2 + r_2 y s_2$, where $n_1, n_2 \in N$, $r_1, r_2 \in R$, and $s_1, s_2 \in S$. Note that

$$rtS = rn_1 S + rr_1 x s_1 S = rn_2 S + rr_2 y s_2 S$$

so $rtS = rn_1 S + r_1(rx)s_1 S = rn_2 S + r_2(ry)s_2 S$, where $rn_1 S, rn_2 S, r_1 rxS \subseteq N$, so we obtain $r_2(ry)S \subseteq N$. Since $S^2 = S$, then $r_2(ry)SS \subseteq N$ so we get $r_2 \in (N :_R ryS) = (N :_R y)$. Then, $r_2 yS \subseteq N$ so $t = n_2 + r_2 y s_2 \in N$. Hence, it's proved that $(N + RxS) \cap (N + RyS) \subseteq N$. Furthermore, it's clear that $N \subseteq (N + RxS) \cap (N + RyS)$. Thus, $N = (N + RxS) \cap (N + RyS)$.

$(2 \Rightarrow 1)$. Let $r_1, r_2 \in R$ and $a \in M$ where $r_1 r_2 aS \subseteq N$. Since $S^2 = S$, then $r_1 r_2 aSS \subseteq N$. If $r_1 aS \nsubseteq N$, we will show that $r_2 aS \subseteq N$. From $r_1 r_2 aSS \subseteq N$, we obtain $r_1 \in (N :_R r_2 aS) \setminus (N :_R a)$. Consequently, $(N :_R r_2 aS) \neq (N :_R a)$. Put $x = r_2 aS$ and $y = a$, then by our assumption we get $N = (N + Rr_2 aS^2) \cap (N + RaS)$. Since $r_2 aS \subseteq Rr_2 aS^2 \subseteq N + Rr_2 aS$ and $r_2 aS \subseteq RaS \subseteq N + RaS$, we obtain $r_2 aS \subseteq N$. $\square$

Before we present the next properties, we recall the definition of an irreducible (R, S)-submodule as follow.

Definition 3.6. *An (R, S)-submodule N of M is called irreducibe if for each (R, S)-submodule N_1 and N_2 of M such that $N = N_1 \cap N_2$ implies either $N = N_1$ or $N = N_2$.*

Proposition 3.7. *Let M be an (R, S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$, N be a left weakly jointly prime (R, S)-submodule of M, and $x, y \in M$.*

(1) *If $rxs \in N$ where $r \in R$, $s \in S$, then $N = (N + RxS) \cap (N + RryS)$.*

(2) If N is an irreducible (R, S)-submodule, then N is a jointly prime (R, S)-submodule of M.

PROOF.

(1) If $rys \in N$, then clearly that $N = (N + RxS) \cap (N + RrysS)$. Since $S^2 = S$, then $N = (N + RxS) \cap (N + RryS)$. Moreover, if $rys \notin N$, then $(N :_R x) \neq (N :_R y)$. By Proposition 3.5, then $N = (N+RxS)\cap(N+RyS)$, so we have

$$N \subseteq (N + RxS) \cap (N + RryS) \subseteq (N + RxS) \cap (N + RyS) = N.$$

Thus, $N = (N + RxS) \cap (N + RryS)$.

(2) Let any $r \in R$, $s \in S$, and $x \in M$ such that $rxs \in N$. By part (a), for each $y \in M$ we have $N = (N + RxS) \cap (N + RryS)$. Since N is irreducible then $N = N + RxS$ or $N = N + RryS = N + NRrrysS$. We have $x \in N$ or $rys \in N$, so $x \in N$ or $rMs \subseteq N$. Thus, by Proposition 2.1 we have N is a jointly prime (R, S)-submodule of M. $\square$

Let M be an (R, S)-module and N be an (R, S)-submodule of M. For every $a \in R$ we consider $(N :_M a)$ to be:

$$(N :_M a) = \{m \in M \mid amS \subseteq N\}.$$

We can show that $(N :_M a)$ is an (R, S)-submodule of M containing N.

Proposition 3.8. *Let M be an (R, S)-module with $S^2 = S$ and N be a left weakly jointly prime (R, S)-submodule of M. Then, for every $a, b \in R$ satisfy*

$$(N :_M ab) = (N :_M a) \cup (N :_M b).$$

PROOF. Let any $x \in (N :_M ab)$, then $abxS \subseteq N$. Since N is a left weakly jointly prime (R, S)-submodule, then $axS \subseteq N$ or $bxS \subseteq N$. So, we get $x \in (N :_M a)$ or $x \in (N :_M b)$, then $(N :_M ab) \subseteq (N :_M a) \cup (N :_M b)$. Next, let any $y \in (N :_M a) \cup (N :_M b)$. Then, $ayS \subseteq N$ or $byS \subseteq N$. Since $S^2 = S$ and N is an (R, S)-submodule of M, then we get

$$abyS = a(byS)S \subseteq aNS \subseteq N.$$

Hence, we obtain $y \in (N :_M ab)$. Thus, $(N :_M a) \cup (N :_M b) \subseteq (N :_M ab)$. Therefore, it has shown that $(N :_M ab) = (N :_M a) \cup (N :_M b)$. $\square$

The following lemma gives us a property about the necessary and sufficient conditions of left weakly jointly prime (R, S)-submodules.

Lemma 3.9. *Let M be an (R, S)-module with $S^2 = S$ and N be a proper (R, S)-submodule of M. Then, N is a left weakly jointly prime (R, S)-submodule if and only if for each $a, b \in R$ satisfy $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$.*

PROOF. ($\Rightarrow$). Let N is a left weakly jointly prime (R, S)-submodule of M. Since $(N :_M ab) = (N :_M a) \cup (N :_M b)$ is an (R, S)-submodule of M, then $(N :_M a) \subseteq (N :_M b)$ or $(N :_M b) \subseteq (N :_M a)$. Therefore, we have $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$.

($\Leftarrow$). Let any $a, b \in R$ and $m \in M$ with $abmS \subseteq N$. So we get $m \in (N :_M ab)$. Based on the hypothesis, $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$. So $m \in (N :_M a)$ or $m \in (N :_M b)$. Consequently, $amS \subseteq N$ or $bmS \subseteq N$. Hence, N is a left weakly jointly prime (R, S)-submodule of M. □

A weakly jointly prime (R, S)-submodule P of M is called minimal if it is minimal in the class of weakly jointly prime (R, S)-submodules of M. In the next proposition, we present that any left weakly jointly prime (R, S)-submodule contains a minimal left weakly jointly prime (R, S)-submodule.

Proposition 3.10. *Let M be an (R, S)-module with $a \in RaS$ for all $a \in M$. Any left weakly jointly prime (R, S)-submodule P of M contains a minimal left weakly jointly prime (R, S)-submodule.*

PROOF. Let any weakly jointly prime (R, S)-submodule P of M and let $\mathfrak{J}$ be the set of all weakly jointly prime (R, S)-submodules of M that contained in P. Clearly, $\mathfrak{J} \neq \emptyset$ since $P \in \mathfrak{J}$. By using Zorn's Lemma, we will show that $\mathfrak{J}$ contains a minimal element. Equivalently, we show that every nonempty chain in $\mathfrak{J}$ has a lower bound in $\mathfrak{J}$. Let any nonempty chain $\mathfrak{G} \subseteq \mathfrak{J}$. We can construct the set $Q = \bigcap_{K \in \mathfrak{G}} K$. Then, clearly Q is an (R, S)-submodule of M and $Q \subseteq P$. We claim that Q is a weakly jointly prime (R, S)-submodule of M. Let any ideal I, J of R and an (R, S)-submodule N of M such that $IJNS \subseteq Q$ but $JNS \nsubseteq Q$. We will show that $INS \subseteq Q$. Let any element $n \in JNS \setminus Q$. Then, there exist $K' \in \mathfrak{G}$ such that $n \notin K'$. Since K' is a left weakly jointly prime (R, S)-submodule of M, then from $IJNS \subseteq Q \subseteq K'$ implies $INS \subseteq K'$. Moreover, let any $L \in \mathfrak{G}$. Since $\mathfrak{G}$ is a chain of $\mathfrak{J}$, then $K' \subseteq L$ or $L \subseteq K'$. If $K' \subseteq L$, then we obtain $INS \subseteq K' \subseteq L$. If $L \subseteq K'$, then we get $n \notin L$. Since L is a left weakly jointly prime (R, S)-submodule of M, then from $IJNS \subseteq Q \subseteq L$ implies $INS \subseteq L$. Thus, we obtain $INS \subseteq L$ for any $L \in \mathfrak{G}$ and so $INS \subseteq Q$. Hence, proved that Q is a left weakly jointly prime (R, S)-submodule of M. Since $Q \subseteq P$, then $Q \in \mathfrak{J}$ and Q is a lower bound of $\mathfrak{G}$. Thus, it's proved that every nonempty chain of $\mathfrak{J}$ has a lower bound in $\mathfrak{J}$. Based on Zorn's Lemma, there exist a left weakly jointly prime (R, S)-submodule $P^* \in \mathfrak{J}$ that minimal among the left weakly jointly prime (R, S)-submodules in $\mathfrak{J}$. Thus, it's proved that any left weakly jointly prime (R, S)-submodules P contain minimal left weakly jointly prime (R, S)-submodule P^* of M. □

From Khumprapussorn *et al.*[7], we know that an (R, S)-module M is called left multiplication (R, S)-module provided that for each (R, S)-submodule N of M there exists an ideal I of R such that $N = IMS$. We have the characterization of jointly prime (R, S)-submodule of left multiplication (R, S)-modules as follow.

Theorem 3.11. *Let M be a left multiplication (R, S)-module with $S^2 = S$. Then, P is a jointly prime (R, S)-submodule of M if and only if $(P :_R M)$ is a prime ideal of R.*

In the following proposition, we present that every left weakly jointly prime (R, S)-submodules is equal to jointly prime (R, S)-submodules in left multiplication (R, S)-modules.

Proposition 3.12. *Let M be a left multiplication (R, S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$. An (R, S)-submodule N of M is jointly prime if and only if N is left weakly jointly prime (R, S)-submodules of M.*

PROOF. ($\Rightarrow$). it's obvious.
($\Leftarrow$). Let N is a left weakly jointly prime (R, S)-submodule of M. Then $(N :_R M)$ is a prime ideal of R. Since M is a left multiplication (R, S)-module and ring S satisfy $S^2 = S$, then based on Theorem 3.11 we have N is a jointly prime (R, S)-submodule of M. $\square$

4. CONCLUDING REMARKS

Further work on the properties of left weakly jointly prime (R, S)-submodules can be carried out. For example, the investigation of properties of left weakly jointly prime radicals of an (R, S)-module.

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