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ON THE LOCATING-CHROMATIC NUMBERS OF SUBDIVISIONS OF FRIENDSHIP GRAPH

Brilly Maxel Salindeho[1], Hilda Assiyatun[2], Edy Tri Baskoro[3]

[1]**Department of Mathematics, Mulawarman University,**
brillyms@gmail.com
[2,3]**Department of Mathematics, Bandung Institute of Technology**

Abstract. Let c be a k-coloring of a connected graph G and let $\pi = \{C_1, C_2, \ldots, C_k\}$ be the partition of $V(G)$ induced by c. For every vertex v of G, let $c_\pi(v)$ be the coordinate of v relative to π, that is $c_\pi(v) = (d(v, C_1), d(v, C_2), \ldots, d(v, C_k))$, where $d(v, C_i) = min\{d(v, x)|x \in C_i\}$. If every two vertices of G have different coordinates relative to π, then c is said to be a locating k-coloring of G. The locating-chromatic number of G, denoted by $\chi_L(G)$, is the least k such that there exists a locating k-coloring of G. In this paper, we determine the locating-chromatic numbers of some subdivisions of the friendship graph Fr_t, that is the graph obtained by joining t copies of 3-cycle with a common vertex, and we give lower bounds to the locating-chromatic numbers of few other subdivisions of Fr_t.

Keywords: *friendship graph, locating-chromatic number, locating coloring, subdivision*

1. INTRODUCTION

The concept of locating-chromatic number was first studied by Chartrand et al. [1] by combining the concept of graph partition dimension and graph coloring. The locating-chromatic numbers of some classes of graphs were studied, especially recently for certain Barbell graphs in [2], Halin graphs in [3], and graphs resulting from certain operations of other graphs, such as join of graphs in [4] and Cartesian product of graphs in [5]. Trees with certain locating-chromatic number were also studied in [6] and [7]. Bounds for locating-chromatic numbers of trees and subdivisions of graph on one edge were also established in [8] and [9], respectively.

Suppose that $G = (V, E)$ is a simple connected graph. Let c be a k-coloring on G and let $\pi = \{C_1, C_2, ..., C_k\}$ be the partition of $V = V(G)$ induced by c. For every vertex v of G, let $c_\pi(v) = (d(v, C_1), d(v, C_2), \ldots, d(v, C_k))$ be the *coordinate*

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of v relative to π, where $d(v, C_i) = min\{d(v, x)|x \in C_i\}$ is the shortest distance between v and vertices in C_i. If every two vertices of G have different coordinates relative to π, then c is said to be a *locating k-coloring* of G. The *locating-chromatic number* of G, denoted by $\chi_L(G)$, is the least k such that there exists a locating k-coloring of G. As shown by Chartrand et al. in [1], if u and v are vertices of G such that $d(u, w) = d(v, w)$ for every $w \in V - \{u, v\}$, then $c(u) \neq c(v)$.

In [4], Behtoei and Anbarloei studied the locating chromatic number of *friendship graph* Fr_t, which is the graph obtained by joining the complete graph K_1 to the t disjoint copies of K_2. They showed that $\chi_L(Fr_t) = 1 + min\{k|t \leq \binom{k}{2}\}$. In this paper, we study some subdivisions of Fr_t and their locating-chromatic numbers. In general, a *subdivision* of a graph G is a graph obtained by replacing some edges of G, say ϵ_1, ϵ_2, ..., ϵ_r, respectively with paths P_1, P_2, ..., P_r of length one or greater, where these paths may differ in length. In particular, when we say a subdivision of a graph *on some edges* $l \geq 0$ *times*, we are specifying which or how many edges are replaced and ensuring the paths replacing the edges are all of length $l + 1$. Purwasih et al. [9] showed that $\chi_L(G) \leq 1 + \chi_L(H)$ if G is a subdivision of a graph H on one edge. We investigate the case where $H = Fr_t$ by determining the locating-chromatic number of any subdivision of Fr_t on one edge and also the locating-chromatic number of any subdivision of Fr_t once on each of its cycle. We also give a tight upper bound for any subdivision of Fr_t. Throughout this paper, for $t \geq 2$ we denote the center of Fr_t, that is the vertex with the largest degree, by z. For every natural number n, we also denote $[n] = \{1, 2, \ldots, n\}$.

2. MAIN RESULTS

In this section, we determine the locating-chromatic number of any subdivision of Fr_t on one edge. We also determine the locating-chromatic number of any subdivision of Fr_t once on each of its cycle.

2.1. Subdivision of Fr_t on one edge.

Throughout this subsection, let $t \geq 2$ and $l \geq 1$ be two natural numbers and let G be a subdivision of Fr_t on one edge l times. For each $n \geq 3$, we define $d_n = \binom{n-1}{2} + 1$. Observe that if $t \geq 3$, we have $d_{k-1} < t \leq d_k$ for some $k \in \{4, 5, 6, \ldots\}$. We begin with the following lemmas.

Lemma 1. If $3 \leq t = d_k$ and $l = 2$, then $\chi_L(G) = k + 1$.

Proof. By definition of G, there are exactly $d_k - 1$ number of 3-cycles and a 5-cycle in G. Consider the collection of 2-subsets of $[k - 1]$, denoted by $[k - 1]^2$. Since $|[k - 1]^2| = \binom{k-1}{2}$, we can denote the elements of $[k - 1]^2$ by $u_1, u_2, \ldots$, and $u_{\binom{k-1}{2}}$.

Now, we start by assigning colors to the vertices of G. We immediately assign the color $k+1$ to the vertex z. To assign colors to other vertices, observe that since there are $d_k - 1 = \binom{k-1}{2}$ number of 3-cycles, there are $2\binom{k-1}{2}$ vertices other than z that lie on a 3-cycle. We denote these vertices by $v_1, v_2, \ldots$, and $v_{2\binom{k-1}{2}}$, where v_{2i-1} and v_{2i} are on the same 3-cycle for each $i = 1, 2, \ldots, \binom{k-1}{2}$. If we write

$u_i = \{a_i, b_i\}$ where $a_i < b_i$, assign the color a_i and b_i respectively to v_{2i-1} and v_{2i}. To finish the color assignment, let the 5-cycle in G be $zw_1w_2w_3w_4z$. Assign the colors $1, k+1, k$, and 1 respectively to w_1, w_2, w_3, and w_4. Let c be the obtained coloring. Clearly, c is a well-defined graph coloring since no two adjacent vertices are assigned the same color.

We show that c is a locating coloring. Let x and y be two vertices with the same color. If x and y are in the same cycle, then the only possibilities are, without loss of generality, either $(x, y) = (z, w_2)$ or $(x, y) = (w_1, w_4)$. However, in both of these cases, the k-th component of the coordinate of x and y differ since $2 = d(x, w_3) \neq d(y, w_3) = 1$ and w_3 is the only vertex colored k.

Let us now assume that x and y are in different cycles. If both are in different 3-cycles, clearly their coordinates differ since their neighbors other than z have different colors by definition of $u_1, u_2, \ldots$, and $u_{\binom{k-1}{2}}$. If, without loss of generality, x is in a 5-cycle and y is in a 3-cycle, then either $x = w_1$ or $x = w_4$ since the colors k and $k+1$ are not assigned to y. In both cases, however, their neighbors other than z also have different colors. Hence, their coordinates differ. Thus, we have shown that c is a locating $(k+1)$-coloring and that $\chi_L(G) \leq k+1$.

We now show that $\chi_L(G) > k$ by contradiction. Suppose that there exists a locating k-coloring c' for G. Suppose that $c'(z) = k$. Hence, without loss of generality, the pair of vertices $\{v_{2i-1}, v_{2i}\}$ has to be assigned by the pair of colors $u_i = \{a_i, b_i\}$ for each $i = 1, 2, \ldots, \binom{k-1}{2}$. Moreover, without loss of generality, let $c'(w_1) = 1$. If $c'(w_2) \neq k$, then we let $c'(w_2) = m \in \{2, 3, ..., k-1\}$. However, there are two vertices v_p and v_{p+1} such that $(c'(v_p), c'(v_{p+1})) = (1, m)$. Observe that $d(w_1, w) = d(v_p, w)$ for any vertex w that is assigned by any color other than 1. Hence, w_1 and v_p have the same coordinate, contradicting the definition of a locating coloring. Thus, we must have $c'(w_2) = k$. However, by the same argument, we must also have $c'(w_3) = k$, which contradicts the definition of coloring. Thus, we have shown that $\chi_L(G) > k$ and we conclude that $\chi_L(G) = k+1$. ■

Lemma 2. If $3 \leq t = d_k$ and $l \neq 2$, then $\chi_L(G) = k$.

Proof. Since $t = d_k$ and $l \neq 2$, there are exactly $d_k - 1$ number of 3-cycles and an $(l+3)$-cycle in G. We start by coloring G. Assign the color k to the vertex z and assign colors to the vertices lying in 3-cycles other than z by using the same way used in the proof of the previous lemma. Consider these cases.

a. Suppose that the $(l+3)$-cycle is $zs_1s_2 \ldots s_{4q-1}$z for some q. Assign the color k to $s_2, s_4, \ldots, s_{4q-2}$. Assign the color 1 to $s_1, s_3, \ldots, s_{2q-1}$. Assign the color 2 to $s_{2q+1}, s_{2q+3}, \ldots, s_{4q-1}$. Let c be the resulting coloring. Clearly, c is a well-defined coloring. We show that c is a locating coloring. Let x and y be two vertices with the same color. If x and y are in the same cycle, then both have to be in the $(l+3)$-cycle. If both are colored k and their neighbors are only vertices of color 1, then their coordinates differ by their distances to a vertex colored 3 since $t = 3$, or other colors other than 1 and 2. It is also the case when their neighbors are only vertices of color 2. If their neighbors are vertices of color 1 and 2, one of them is z and the other one is s_{2q}. In this case, their coordinates also differ by their distances to a vertex colored other than 1 and 2. The same argument also applies if the color of x and y are 1 or 2. Moreover, if x is in the $(l+3)$-cycle and y is in a 3-cycle without loss of generality, then x and y are not colored k. However, the neighbors of x are only vertices colored k, while some of the neighbors of y are not colored k. Hence, their coordinates differ. Thus, c is a locating k-coloring.

b. Suppose that the $(l+3)$-cycle is $zs_1s_2\dots s_{4q-3}z$ for some q. Assign the color k to $s_2, s_4, \dots, s_{4q-4}$. Assign the color 1 to $s_1, s_3, \dots, s_{2q-1}$. Assign the color 2 to $s_{2q+1}, s_{2q+3}, \dots, s_{4q-3}$. Let c be the resulting coloring. Clearly, c is a well-defined coloring. We show that c is a locating coloring. Let x and y be two vertices with the same color. By using the same argument as in part a, we see that c is indeed a locating k-coloring.

c. Suppose that the $(l+3)$-cycle is a $(2q-1)$-cycle $zs_1\ s_2\ \dots\ s_{2q-2}z$. Assign the color k to s_2, s_4, …, s_{2q-6} and s_{2q-3}. Assign the color 1 to s_1, s_3, …, s_{2q-5}. Assign the color 2 to s_{2q-2} and s_{2q-4}. Let c be the resulting coloring. Clearly, c is well-defined. The argument to show that c is indeed a locating k-coloring is similar to part a or part b with minor difference, that is if x and y are two vertices of color k other than z, then their neighbors are either only vertices of color 1 or only vertices of color 2.

Thus, we have $\chi_L(G) \leq k$. We now show that $\chi_L(G) > k-1$ by contradiction. Suppose that there exists a $(k-1)$-coloring c' for G. Let $c'(z) = k-1$. Since z is adjacent to all vertices in the 3-cycles, the colors of those vertices have to be in $[k-2]$. However, there are more than $\binom{k-2}{2}$ number of 3-cycles in G, while the cardinality of $[k-2]^2$ is $\binom{k-2}{2}$. Hence, by the pigeon-hole principle, there exist two pairs of vertices lying in 3-cycle, say $\{a_1, b_1\}$ and $\{a_2, b_2\}$, where their elements are different, such that $\{c'(a_1), c'(b_1)\} = \{c'(a_2), c'(b_2)\}$. Let a_1 and b_1 are colored the same as a_2 and b_2, respectively. However, the distances of a_1 and a_2 to a vertex colored other than $c'(a_1) = c'(a_2)$ is equal. This means their coordinates are equal, contradicting the definition of locating coloring. Thus, we have $\chi_L(G) > k-1$ so that $\chi_L(G) = k$. ■

Lemma 3. If $3 \leq t < d_k$, then $\chi_L(G) = k$.

Proof. We start by coloring the graph G. Assign the color k to z. Assign colors to vertices lying in 3-cycles other than z by using the same way used in the proof of the first lemma, that is by taking different elements in the set $[k-1]^2$ as pairs of colors for pairs of vertices in each 3-cycle. Since $t < d_k$, there are less than $\binom{k-1}{2}$ number of 3-cycles, so that there exist elements in $[k-1]^2$ that are not used as a pair of color in any 3-cycle. Denote this element by (g_1, g_2).

Suppose that the $(l+3)$-cycle in G is $zs_1s_2\dots s_{l+2}z$. If $l+3$ is odd, use the colors $g_1, g_2, g_1, g_2, \dots, g_1, g_2$ respectively to color $s_1, s_2, s_3, s_4, \dots, s_{l+2}$. Otherwise, assign the color k to the vertex $s_{\frac{l+3}{2}}$ and use the colors $g_1, g_2, g_1, g_2, \dots$ respectively to color $s_1, s_2, s_3, s_4, \ldots, s_j$, where $j = \frac{l+3}{2} - 1$, and use the colors $g_2, g_1, g_2, g_1, \ldots$ respectively to color $s_{l+2}, s_{l+1}, s_l, s_{l-1}, \ldots, s_{j+2}$.

It is easy to see, by using the same argument as in the proof of previous lemma, that all vertices colored the same have different coordinates. In this case, vertices colored g_1 and g_2 create the differences. Hence, $\chi_L(G) \leq k$. The proof showing that $\chi_L(G) > k-1$ is similar to the last paragraph of the proof of the last lemma. Thus, we have $\chi_L(G) = k$. ∎

From previous three lemmas, we have the following theorem.

Theorem 1. Let $t \geq 3$ and $l \geq 1$ be two natural numbers. Let G be a subdivision of Fr_t on one edge l times. For each $n \geq 3$, let $d_n = \binom{n-1}{2} + 1$ and $d_{k-1} < t \leq d_k$ for some k. Hence, we have $\chi_L(G) = k+1$ if $t = d_k$ and $l = 2$, and $\chi_L(G) = k$ otherwise.

We treat the case $t = 2$ separately in the next proposition.

Proposition 1. Let $l \geq 1$ be a natural number. If G is a subdivision of Fr_2 on one edge l times, then $\chi_L(G) = 4$.

Proof. We start by coloring G. Assign the color 4 to the vertex z. Assign the color 1 and 2 to the two vertices lying in the only existing 3-cycle. Now, denote the $(l+3)$-cycle in G by $zu_1u_2 \ldots u_{l+2}z$.

Assume first that l is even. Assign the colors $1, 3, 1, 3, \ldots, 1, 3$ respectively to the vertices $u_1, u_2, u_3, u_4, \ldots, u_{l+2}$. By doing this, the vertices colored 3 have their coordinates differed by their distances to the vertex colored 4, and the vertices colored 1 have their coordinates differed by their distances to the vertex colored 2. Thus, we obtain a locating 4-coloring.

Assume now that l is odd. Assign the color 4 to the vertex $u_{\frac{l+3}{2}}$. Assign the color 1 to each vertex of the form $u_1, u_3, \ldots, u_{l_1}$, where $l_1 < \frac{l+3}{2}$, and vertex of the form $u_{l_2}, \ldots, u_{l-1}, u_{l+1}$, where $l_2 > \frac{l+3}{2}$. Assign the color 3 to other remaining vertices. By doing this, the vertices colored 3 have their coordinates differ by their distances to the vertex colored 2, and so do the vertices colored 1. Thus, we obtain a locating 4-coloring.

We have shown that $\chi_L(G) \leq 4$. We now show that $\chi_L(G) > 3$. Suppose that there exists a locating 3-coloring on G. Without loss of generality, assume that the 3-cycle in G is colored by $1, 2$, and 3, where z is assigned the color 3. Suppose that there exists a vertex colored by 2 in the $(l+3)$-cycle in G. Let j be the least index such that u_j is colored by 2. If j is odd, then the vertices $u_1, u_3, \ldots, u_{j-2}$ have to be colored by 1 since the color of z is 3, and we must also have the vertices $u_2, u_4, \ldots, u_{j-1}$ colored by 3. However, the coordinate of u_{j-1} is equal to the coordinate of z, contradicting the definition of locating coloring. If j is even instead, then the vertices $u_1, u_3, \ldots, u_{j-1}$ have to be colored 1 since the color of z is 3, and we must also have the vertices $u_2, u_4, \ldots, u_{j-2}$ colored by 3.

However, the coordinate of u_{j-1} is equal to the coordinate of the vertex colored by 1 on the 3-cycle, contradicting again the definition of locating coloring. Hence, there must not be any vertex colored by 2 on the $(l+3)$-cycle. This means that, since z is colored by 3, $u_1, u_3, \ldots, u_{l+2}$ have to be colored by 1 and $u_2, u_4, \ldots, u_{l+1}$ have to be colored by 3. However, the vertices u_1 and u_{l+2} have the same color and the same coordinate, contradicting the definition of the locating coloring. Thus, we have $\chi_L(G) = 4$. ■

2.2. **Subdivision of Fr_t once on one edge of each cycle.** We now determine the locating-chromatic number of the subdivision of Fr_t once on one edge of each cycle. This means that each cycle of the graph is a 4-cycle. Let G be such graph, where $t \geq 2$.

Theorem 2. For each $n \geq 3$, let $e_n = \left\lfloor \frac{n-1}{2} \right\rfloor + (n-1)\left\lfloor \frac{n-2}{2} \right\rfloor$ and $e_{k-1} < t \leq e_k$ for some k. Hence, we have $\chi_L(G) = k$.

Proof. We define a locating $\chi_L(G)$-coloring $c : V \to [k]$ on G. We first set $c(z) := k$. Assume that $C(1), C(2), \ldots, C(t)$ are all of the 4-cycles in G and denote $C(i)$ by $zu_{i,1}u_{i,2}u_{i,3}z$ for each i. Clearly, $k \geq 4$ since $2 \leq t \leq e_k$.

Define a 3-tuple $W'_i := (w'_{i,1}, w'_{i,2}, w'_{i,3})$ with $w'_{i,1} := 2i-1$,$w'_{i,2} := k$,$w'_{i,3} := 2i$ for $i = 1, 2, \ldots, \left\lfloor \frac{k-1}{2} \right\rfloor$. Next, for $j = 1, 2, \ldots, (n-1)\left\lfloor \frac{k-2}{2} \right\rfloor$, define a 3-tuple $W''_j := (w''_{i,1}, w''_{i,2}, w''_{i,3})$ with

$(w''_{i,1}, w''_{i+1,1}, \ldots, w''_{i+\left\lfloor \frac{k-2}{2} \right\rfloor-1,1}) := (i+1, i+3, \ldots, i+2\left\lfloor \frac{k-2}{2} \right\rfloor - 1)$,

$(w''_{i,2}, w''_{i+1,2}, \ldots, w''_{i+\left\lfloor \frac{k-2}{2} \right\rfloor-1,2}) := (i, i, \ldots, i)$,

$(w''_{i,3}, w''_{i+1,3}, \ldots, w''_{i+\left\lfloor \frac{k-2}{2} \right\rfloor-1,3}) := (i+2, i+4, \ldots, i+2\left\lfloor \frac{k-2}{2} \right\rfloor)$

for $i = 1, 2, \ldots, k-1$, by noting that the components are calculated under modulo $k-1$. Observe that in W''_j, there is no entry that is equal to k. Observe also that W'_i and W''_j never equal to each other since their second entries differ for any i and j. By definition, we also see that W'_{i_1} and W'_{i_2} differ for any different i_1 and i_2, and that W''_{j_1} and W''_{j_2} differ for any different j_1 and j_2. Hence, if we write $W := \{W'_i | i = 1, 2, \ldots, \left\lfloor \frac{k-1}{2} \right\rfloor\} \cup \{W''_j | j = 1, 2, \ldots, (k-1)\left\lfloor \frac{k-2}{2} \right\rfloor\}$, we have $|W| = e_k$. We can then write $W = \{W_1, W_2, \ldots, W_{e_k}\}$. Now, for each $i \in [t]$, define the coloring $c[C(i)] := (c(u_{i,1}), c(u_{i,2}), c(u_{i,3})) := W_i$.

From the definition of W, clearly c is a k-coloring. We now show that c is a locating coloring. Let x and y be two different vertices with the same color in G. If $x = z$, then $y = u_{i,2}$ for some i such that $c(u_{i,2}) = k$, if it exists. However, since $t \geq 2$ and by definition of c, the vertex x is adjacent to vertices with colors other than $c(u_{i,1})$ and $c(u_{i,3})$, while y is only adjacent to vertices with these colors. Hence, the coordinates of x and y differ, so we assume that x and y are not z.

Now, let our x and y be in the 4-cycles $C(i_1)$ and $C(i_2)$, respectively, where i_1 and i_2 are two different elements of $[t]$.

Let $c[C(i_1)]$ and $c[C(i_2)]$ both be in $\{W_i'|i = 1, 2, \ldots, \lfloor \frac{k-1}{2} \rfloor\}$. Hence, we must have $x = u_{i_1,2}$ and $y = u_{i_2,2}$, or vice-versa. However, the colors of the neighbors of x clearly differ from the colors of the neighbors of y. Thus, their coordinates differ.

Let $c[C(i_1)]$ and $c[C(i_2)]$ both be in $\{W_j''|j = 1, 2, \ldots, (k-1)\lfloor \frac{k-2}{2} \rfloor\}$. There are some cases to consider. For the first case, if $x = u_{i_1,2}$ and $y = u_{i_2,2}$, then clearly they have different coordinates by looking at the colors of their neighbors. For the next case, if $x = u_{i_1,2}$ and $y = u_{i_2,1}$ (or $y = u_{i_2,3}$ without loss of generality), then y is adjacent to z, which is a vertex colored k, but x is not adjacent to any vertex colored k, so we know that their coordinates differ. For the last case, if $x = u_{i_1,1}$ and $y = u_{i_2,1}$ (without loss of generality), then, by definition of c, the colors of $u_{i_1,2}$ and $u_{i_2,2}$ differ, so that the colors of the neighbors of x and y also differ, and hence x and y have different coordinates.

Let $c[C(i_1)] \in \{W_j''|j = 1, 2, \ldots, (k-1)\lfloor \frac{k-2}{2} \rfloor\}$ and $c[C(i_2)] \in \{W_j''|j = 1, 2, \ldots, (k-1)\lfloor \frac{k-2}{2} \rfloor\}$ (without loss of generality). Thus, we have, say $x = u_{i_1,1}$, and $y = u_{i_2,2}$ or $y = u_{i_2,1}$. However, both neighbors of x are colored k, but only one of the neighbors of k is colored k. Hence, their coordinates differ.

Thus, we have shown that c is a locating k-coloring, so that $\chi_L(G) \leq k$.

We now show that $\chi_L(G) > k-1$. Suppose that there exists a locating $(k-1)$-coloring on G, which we denote by c'. Assume that $c'(z) = k-1$. We divide all of the 4-cycles into $k-1$ types. Type a consists of all 4-cycles $C(i) = zu_{i,1}u_{i,2}u_{i,3}z$ with $c'(u_{i,2}) = a$. Observe that if there exist two 4-cycles of type $k-1$, say $C(i)$ and $C(j)$, where $c'(u_{i,1}) = c'(u_{j,1})$ without loss of generality, then the coordinates of $u_{i,1}$ and $u_{j,1}$ must be the same since both are adjacent only to two vertices colored $k-1$ and $d(u_{i,1}, u_{i,3}) = d(u_{i,1}, z) + d(z, u_{i,3}) = d(u_{j,1}, z) + d(u_{j,1}, u_{i,3}) = 2$. Moreover, $2 = d(u_{j,1}, u_{j,3}) = d(u_{i,1}, u_{j,3})$ and $d(u_{i,1}, x) = d(u_{i,1}, z) + d(z, x) = d(u_{j,1}, z) + d(z, x) = d(u_{j,1}, x)$ for each vertex x that is not $u_{i,1}, u_{i,2}, u_{i,3}, u_{j,1}, u_{j,2}, u_{j,3}$. Hence, since $u_{i,1}$ and $u_{j,1}$ must not be colored $k-1$, there are at most $\lfloor \frac{k-2}{2} \rfloor$ number of 4-cycles of type $k-1$ by the pigeon-hole principle.

Let $C(i)$ be a 4-cycle of type b where $c'(u_{i,2}) = b \in [k-2]$. By the similar observation to the previous paragraph, there are at most $\lfloor \frac{k-3}{2} \rfloor$ number of 4-cycles of type b. Hence, there are at most $(k-2)\lfloor \frac{k-3}{2} \rfloor$ number of 4-cycles of type other than $k-1$. Thus, by combining with the previous paragraph, there are at most e_{k-1} number of 4-cycles in G. This contradicts the assumption on t. We conclude that $\chi_L(G) > k-1$, so that $\chi_L(G) = k$. ■

2.3. **Upper bound for arbitrary subdivision of Fr_t.** We now study the upper bound for arbitrary subdivision of Fr_t. It is known that $\chi_L(G) \leq 1 + \chi_L(H)$ if G is a subdivision of a graph H on one edge. For $H = Fr_t$, this bound is strengthened.

Theorem 3. If G is a subdivision of Fr_t where $t \geq 2$, then we have $\chi_L(G) \leq \chi_L(Fr_t)$. Precisely, if $\binom{k-2}{2} < t \leq \binom{k-1}{2}$, we have $\chi_L(G) \leq k$.

Proof. Clearly, $k \geq 4$. We construct a locating k-coloring $c : V(G) \to [k]$ on G. We start by setting $c(z) := k$. Let $C(1), C(2), \ldots, C(t)$ denote all the cycles in G. We start with the first case where $t = \binom{k-1}{2}$.

Suppose that there is no 4-cycle in G. Write $C(i) = w_{i,1}w_{i,2}\ldots w_{i,s(i)}w_{i,1}$ where $s(i) \neq 4$ and $w_{i,1} = z$ for each i. Assume that $[k-1]^2 = \{u_1, u_2, \ldots, u_t\}$. If $s(i)$ is odd, then set $c(w_{i,2}) := c(w_{i,4}) := \ldots := c(w_{i,s(i)-1}) := a_i$ and $c(w_{i,3}) := c(w_{i,5}) := \ldots := c(w_{i,s(i)}) := b_i$, where $\{a_i, b_i\} := u_i$. If $s(i)$ is even, then set $c(w_{i,2}) := c(w_{i,4}) := \ldots := c(w_{i,s_1(i)}) := a_i$, $c(w_{i,s(i)-1}) := c(w_{i,s(i)-3}) := \ldots := c(w_{i,s_2(i)}) := a_i$, $c(w_{i,3}) := c(w_{i,5}) := \ldots := c(w_{i,s_3(i)}) := b_i$, $c(w_{i,s(i)}) := c(w_{i,s(i)-2}) := \ldots := c(w_{i,s_4(i)}) := b_i$, and $c(w_{i,\frac{s(i)}{2}+1}) := k$, where $\{a_i, b_i\} := u_i$, $s_1(i) < \frac{s(i)}{2}+1$, $s_2(i) > \frac{s(i)}{2}+1$, $s_3(i) < \frac{s(i)}{2}+1$, and $s_4(i) > \frac{s(i)}{2}+1$. Observe that two adjacent vertices in G lie in a $C(i)$. By definition of c, those two vertices have different colors. Thus, c is a k-coloring.

Next, to show that c is locating coloring, let z_1 and z_2 be two different vertices having the same color in G, that is $c(z_1) = c(z_2) = a_0$. If one of z_1 or z_2 is z, say z_1, then we know that, by definition of c and the fact that $t \geq 2$, z_1 is adjacent to at least 4 vertices which are two vertices in the cycle where z_2 belongs and two other vertices in another cycle, and that three of these four vertices have different colors. However, z_2 is only adjacent to at most two vertices with different colors. Hence, the coordinates of z_1 and z_2 differ. Now, let z_1 and z_2 be vertices other than z. Assume that both are in different cycles, say $C(i_1)$ and $C(i_2)$, respectively. If $a_0 = k$, then $C(i_1)$ and $C(i_2)$ are cycles of even length by definition of c. Again, by definition of c, z_1 and z_2 are vertices that have their distances to z the greatest in the cycles containing them, so that z_1 is adjacent to two vertices colored a_{i_1} and b_{i_1}, and z_2 is adjacent to two vertices colored a_{i_2} and b_{i_2}, but $u_{i_1} \neq u_{i_2}$. Thus, the coordinates of z_1 and z_2 are different. If $a_0 \neq k$, then, since no cycle is a 4-cycle and each of z_1 and z_2 has a neighbor z_1' and z_2', respectively, that $c(z_1') \neq c(z_2')$ by definition, the coordinates of z_1 and z_2 are different.

Now, let z_1 and z_2 be in the same cycle $C(i)$, and both are not z. By definition of c, we have $a_0 \neq k$. Again by definition of c and the fact that $t \geq 2$, there exists a vertex z' colored a_0' outside of $C(i)$ and no vertex in $C(i)$ is colored a_0'. By the numbering of $C(i)$, we have $d(z_1, z') = d(z_1, z) + d(z, z') \neq d(z_2, z) + d(z, z') = d(z_2, z')$. Hence, the coordinates of z_1 and z_2 differ. Thus, we have shown that c is a locating coloring and that $\chi_L(G) \leq k$.

For the next case, suppose that there are q number of 4-cycles in G. We show that $\chi_L(G) \leq k$. Write $q = (k-1)m + r$ where r and m are unique integers satisfying $0 \leq r < k-1$ and $m \geq 0$ by the division algorithm. Let the 4-cycles be denoted by $Q_1, Q_2, \ldots, Q_q$. Consider the complete graph H on the set $[k-1]$.

Assume that k is odd. We must have $m \leq \frac{k-1}{2} - 1$, otherwise we would have $q > \binom{k-1}{2}$, which is a contradiction. Since $k-1$ is even, by decomposing H to obtain its Hamiltonian cycles and its 1-factors, there exist subgraphs $H_1, H_2, \ldots, H_{\frac{k-1}{2}-1}, E_1,\ E_2,\ \ldots, E_{\frac{k-1}{2}}$ of H. We continue by noting that H_i is a Hamiltonian cycle and H_i and H_j are edge-disjoint subgraphs for each different i and j, and that E_i is a complete graph on two vertices and E_i and E_j are edge-disjoint subgraphs for each different i and j.

For each $p \in [m]$, consider the $k-1$ number of 4-cycles $Q_{(k-1)(p-1)+1}$, $Q_{(k-1)(p-1)+2}, \ldots, Q_{(k-1)p}$. We define the coloring c on these cycles that is associated with the subgraph H_p. Let us write $H_p = h_{p,1}, h_{p,2}, \ldots, h_{p,k-1}, h_{p,1}$. There exist three vertices $v_{p,j,1}, v_{p,j,2}, v_{p,j,3}$ that are not z on $Q_{(k-1)(p-1)+j}$ where $j \in [k-1]$. Set $c(v_{p,j,1}) := h_{p,j}, c(v_{p,j,2}) := h_{p,j+1}, c(v_{p,j,3}) := h_{p,j+2}$, where j, $j+1$, and $j+2$ are calculated under modulo $k-1$. By this definition, adjacent vertices on these cycles have different colors.

For the case $r > \frac{k-1}{2}+1$, we have $m < \frac{k-1}{2}-1$, so that there exist a Hamiltonian cycle H_{m+1} that have not yet been associated with the 4-cycles on the previous paragraph. We define the coloring c on the 4-cycles $Q_{(k-1)m+1}$, $Q_{(k-1)m+2}$, …, $Q_{(k-1)m+r}$ associated with the subgraph H_{m+1}. Similar to the previous paragraph, by changing the role of p with $m+1$ and $j \in [r]$, and when $j = r$, we set $c(v_{m+1,j,2}) := k$, we see that adjacent vertices on these cycles have different colors.

For the case $r \leq \frac{k-1}{2}$, consider the r number of 4-cycles $Q_{(k-1)m+1}$, $Q_{(k-1)m+2}$, …, $Q_{(k-1)m+r}$. We define a coloring c on the cycle $Q_{(k-1)m+p}$ for each $p \in [r]$ associated with the subgraph E_p. Assume that $E_p = e_{p,1}e_{p,2}$. There are three vertices $x_{p,1}, x_{p,2}, x_{p,3}$ that are not z on $Q_{(k-1)m+p}$. Set $c(x_{p,1}) := e_{p,1}, c(x_{p,2}) := k, c(x_{p,3}) := e_{p,2}$. Hence, adjacent vertices on these cycles have different colors.

Note that for the case that k is even, we obtain Hamilton cycles H_1, H_2, …, $H_{\frac{k-2}{2}}$ of H. Again, this time we must have $m \leq \frac{k-2}{2}$. The coloring is done by using the similar way to the case that k is odd, except that the case $r > \frac{k-1}{2}+1$ is replaced with the case $r \neq 0$, and the case $r \leq \frac{k-1}{2}$ is not needed. Next, color the remaining $t-q$ cycles by using the similar coloring used to color cycles before there was any 4-cycle, by noting that the pair $\{a_i, b_i\}$ that is used is the label of two adjacent vertices that have not been used to color the 4-cycles on the above decomposition. Observe that there are exactly $t-q$ pairs of such labels. Thus, we have shown that c is a k-coloring.

We show that c is indeed a locating coloring. Let x and y be two different vertices in G with $c(x) = c(y)$. The cases for x and y that must be considered are:

(1) One of them is z
(2) Both are not z and are in the same 4-cycle
(3) Both are not z and are in the same cycle that is not a 4-cycle
(4) Both are not z, x is in a 4-cycle, and y is in a cycle that is not a 4-cycle
(5) Both are not z and are in different 4-cycles
(6) Both are not z and are in different cycles, but these cycles are not 4-cycles.

The first four cases are easily verified. For the fifth case, if both x and y are colored k, then, since both are not z, their neighbors have to be only two vertices that have the color pair from some labels E_i and E_j, respectively, that are edge-disjoint subgraphs. Hence, the colors of the neighbors of x and y differ. If both are not colored k, then the colors of the neighbors of x and y also differ since each two

4-cycles have their vertices colored based on the labels of the subgraphs of H that are edge-disjoint subgraphs and since x and y are different vertices. In fact, if the colors of the neighbors of x and y are only k, then, since they belong to different 4-cycles, the colors in both of these cycles must be based on E_i and E_j that are edge-disjoint subgraphs. This is impossible if x and y are different vertices.

For the sixth case, the same argument also applies by observing the possibilities of the position of x and y and the labels used. Thus, we have shown that c is a locating k-coloring so that $\chi_L(G) \leq k$.

Lastly, for the case $t < \binom{k-1}{2}$, the $(t+1)$-th, $(t+2)$-th, $\ldots$, and so on that have been colored from the coloring on the case $t = \binom{k-1}{2}$ before is removed so that there are $t < \binom{k-1}{2}$ cycles remaining, and the above cases can be verified again the similar way. Thus, the theorem is proved. ■

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