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CALDERÓN COMPLEX INTERPOLATION OF MORREY SPACES

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Abstract. In this note we will discuss some results related to complex interpolation of Morrey spaces. We first recall the Riesz-Thorin interpolation theorem in Section 1. After that, we discuss a partial generalization of this theorem in Morrey spaces proved in [19]. We also discuss non-interpolation property of Morrey spaces given in [3, 17]. In Section 3, we recall the definition of Calderón's complex interpolation method and the description of complex interpolation of Lebesgue spaces. In Section 4, we discuss the description of complex interpolation of Morrey spaces given in [6, 10, 14, 15]. Finally, we discuss the description of complex interpolation of subspaces of Morrey spaces in the last section. This note is a summary of the current research about interpolation of Morrey spaces, generalized Morrey spaces, and their subspaces in [6, 9, 10, 11, 12, 14, 15].

Key words and phrases: *Morrey spaces, generalized Morrey spaces, complex interpolation.*

1. The Riesz-Thorin interpolation theorem

We first recall the definition of Lebesgue spaces. Let $1 \le p \le \infty$. The Lebesgue space $L^p = L^p(\mathbb{R}^n)$ is defined to be the set of all measurable function f on $\mathbb{R}^n$ such that the norm

$$\|f\|_{L^p} := \begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^p \ dx\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \inf\{M > 0 : |f(x)| \le M \text{a.e } x \in \mathbb{R}^n\}, & p = \infty \end{cases} \tag{1}$$

is finite. An example of L^p-function is the simple function

$$f = \sum_{j=1}^{k} a_j \chi_{A_j},$$

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where $a_j \in \mathbb{C}$ and $\{A_j\}_{j=1}^k$ is a collection of disjoint subsets of $\mathbb{R}^n$ with finite measure. In this case,

$$\|f\|_{L^p} = \begin{cases} \left(\sum_{j=1}^k |a_j|^p |A_j|\right)^{\frac{1}{p}}, & p < \infty, \\ \max\limits_{j=1,2,\ldots,k} |a_j|, & p = \infty. \end{cases}$$

Note that the L^p space is a Banach space with the norm defined in (1) Moreover, we also have the log-convexity property of L^p-norm as follows.

Lemma 1.1. [7, Exercise 1.1.16] *Let* $0 \le \theta \le 1$, $1 \le p_0, p_1 \le \infty$, *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *Then for every* $f \in L^{p_0} \cap L^{p_1}$ *we have*

$$\|f\|_{L^p} \le \|f\|_{L^{p_0}}^{1-\theta} \|f\|_{L^{p_1}}^{\theta}. \tag{2}$$

Proof. Without loss of generality, assume that p_0, $p_1 < \infty$. Since

$$\frac{1}{p_0/((1-\theta)p)} + \frac{1}{p_1/(\theta p)} = 1,$$

we have

$$\|f\|_{L^p}^p = \int_{\mathbb{R}^n} |f(x)|^{p(1-\theta)} |f(x)|^{p\theta}\, dx \le \|f\|_{L^{p_0}}^{p(1-\theta)} \|f\|_{L^{p_1}}^{p\theta}.$$

Taking pth root gives (2). □

Note that, Lemma 1.1 can be viewed as the inclusion $L^{p_0} \cap L^{p_1} \subseteq L^p$. A complement to this result is the following lemma.

Lemma 1.2. *Keep the same assumption as in Lemma 1.1. Then*

$$L^p \subseteq L^{p_0} + L^{p_1}.$$

Here, $L^{p_0} + L^{p_1}$ *is defined to be the set of all functions* f *for which* $f = f_0 + f_1$ *for some* $f_0 \in L^{p_0}$ *and* $f_1 \in L^{p_1}$.

Proof. Without loss of generality, assume that $1 \le p_0 < p_1 < \infty$. Let $f \in L^p$. Define $f_0 := f\chi_{\{|f| \ge 1\}}$ and $f_1 := f - f_0$. Since $p_0 < p_1$, we have $p_0 < p < p_1$, so

$$\int_{\mathbb{R}^n} |f_0(x)|^{p_0}\, dx \le \int_{\mathbb{R}^n} |f_0(x)|^{p_0 - p} |f(x)|^p\, dx \le \|f\|_{L^p}^p < \infty$$

and

$$\int_{\mathbb{R}^n} |f_1(x)|^{p_1}\, dx \le \int_{\mathbb{R}^n} |f_1(x)|^{p_1 - p} |f(x)|^p\, dx \le \|f\|_{L^p}^p < \infty.$$

Therefore, $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$. Since $f = f_0 + f_1$, we conclude that $f \in L^{p_0} + L^{p_1}$. □

As a preparation for proving the Riesz-Thorin interpolation theorem, we prove Hadamard's three lines lemma.

Lemma 1.3. *Let* $S := \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ *and* $\overline{S}$ *be its closure. Let* F *be any continuous function on* $\overline{S}$ *such that* F *is holomorphic in* S *and* F *is bounded on* $\overline{S}$. *Then, for every* $\theta \in (0,1)$ *and* $s \in \mathbb{R}$, *we have*

$$|F(\theta + is)| \leq \left(\sup_{t\in\mathbb{R}} |F(it)|\right)^{1-\theta} \left(\sup_{t\in\mathbb{R}} |F(1+it)|\right)^{\theta}.$$

Proof. Let $M_0 := \sup_{t\in\mathbb{R}} |F(it)|$, $M_1 := \sup_{t\in\mathbb{R}} |F(1+it)|$, and $M := \sup_{z\in\overline{S}} |F(z)|$. Define

$$G(z) := \frac{F(z)}{M_0^{1-z} M_1^z} \quad \text{and} \quad G_n(z) := G(z) e^{\frac{z^2-1}{n}} \ (n \in \mathbb{N}).$$

Since

$$|M_0^{1-z} M_1^z| = M_0^{1-\mathrm{Re}(z)} M_1^{\mathrm{Re}(z)} \geq \min(1, M_0) \cdot \min(1, M_1),$$

we have

$$|G(z)| \leq \frac{|F(z)|}{|M_0^{1-z} M_1^z|} \leq \frac{M}{\min(1, M_0) \cdot \min(1, M_1)} =: C.$$

Consequently, for every $\sigma \in (0,1)$ and $r \in \mathbb{R}$, we have

$$|G_n(\sigma + ir)| = |G(\sigma + ir)| e^{\frac{\sigma^2 - r^2 - 1}{n}} \leq C e^{\frac{\sigma^2-1}{n}} e^{-\frac{r^2}{n}} \leq C e^{-\frac{r^2}{n}}.$$

Therefore, $\lim_{|r|\to\infty} G_n(\sigma + ir) = 0$ unifrormly over $\sigma \in [0,1]$. Hence, we can choose $R = R(n) > 0$ such that for every $r \geq R$ and $\sigma \in [0,1]$, we have

$$|G_n(\sigma + ir)| \leq 1. \tag{3}$$

Observe that for every $r \in \mathbb{R}$, we have

$$|G_n(ir)| = \frac{|F(ir)|}{|M_0^{1-ir} M_1^{ir}|} e^{\frac{-r^2-1}{n}} \leq \frac{M_0}{M_0} = 1 \tag{4}$$

and

$$|G_n(1 + ir)| = \frac{|F(1 + ir)|}{|M_0^{1-ir} M_1^{ir}|} e^{\frac{-r^2}{n}} \leq \frac{M_1}{M_1} = 1. \tag{5}$$

By the maximum modulus principle and (3)-(5), we have

$$|G_n(z)| \leq 1 \tag{6}$$

whenever $0 \leq \mathrm{Re}(z) \leq 1$ and $-R \leq \mathrm{Im}(z) \leq R$. Combining (3) and (6), we have

$$|G_n(z)| \leq 1$$

for every $z \in \overline{S}$. Taking $n \to \infty$, we get

$$|G(z)| \leq 1$$

for every $z \in \overline{S}$. This implies

$$|F(\theta + is)| = |G(\theta + is)| M_0^{1-\theta} M_1^{\theta} \leq M_0^{1-\theta} M_1^{\theta},$$

as desired. □

We now state and prove the Riesz-Thorin interpolation theorem as follows.

Theorem 1.4. *Let $0 \leq \theta \leq 1$ and $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. Suppose that T is a bounded linear operator from L^{p_0} to L^{q_0} and L^{p_1} to L^{q_1}. Then T is bounded from L^p to L^q, where p and q are defined by*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{7}$$

Moreover,

$$\|T\|_{L^p \to L^q} \leq \|T\|_{L^{p_0} \to L^{q_0}}^{1-\theta} \|T\|_{L^{p_1} \to L^{q_1}}^{\theta}.$$

Proof. The proof follows the idea in [7]. We only handle the case where p_0 and p_1 are finite. Let $M_0 := \|T\|_{L^{p_0} \to L^{q_0}}$ and $M_1 := \|T\|_{L^{p_1} \to L^{q_1}}$. Let f be a simple function and write

$$f = \sum_{j=1}^{k} a_j e^{i\alpha_j} \chi_{A_j},$$

where $a_j > 0$, $\alpha_j \in \mathbb{R}$, and $\{A_j\}_{j=1}^k$ is a collection of pairwise disjoint subsets of $\mathbb{R}^n$ with $|A_j| < \infty$. Note that

$$\|Tf\|_{L^q} = \sup_{g \text{ is simple}, \ \|g\|_{L^{q'}} = 1} \left| \int_{\mathbb{R}^n} Tf(x) g(x) \, dx \right|. \tag{8}$$

Now, let g be fixed and write

$$g = \sum_{\ell=1}^{m} b_\ell e^{i\beta_\ell} \chi_{B_\ell}$$

where $b_\ell > 0$, $\beta_\ell \in \mathbb{R}$, and $\{B_\ell\}_{\ell=1}^m$ is a collection of pairwise disjoint subsets of $\mathbb{R}^n$ with finite measure. Then, by linearity of T, we have

$$\int_{\mathbb{R}^n} Tf(x) g(x) \, dx = \sum_{j=1}^{k} \sum_{\ell=1}^{m} a_j e^{i\alpha_j} b_\ell e^{i\beta_\ell} \int_{\mathbb{R}^n} T\chi_{A_j}(x) \chi_{B_\ell}(x) \, dx. \tag{9}$$

Let $S := \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ and $\overline{S}$ be its closure. For every $z \in \overline{S}$, define

$$F(z) := \sum_{j=1}^{k} \sum_{\ell=1}^{m} a_j^{\frac{p(1-z)}{p_0} + \frac{pz}{p_1}} e^{i\alpha_j} b_\ell^{\frac{q'(1-z)}{q_0'} + \frac{q'z}{q_1'}} e^{i\beta_\ell} \int_{\mathbb{R}^n} T\chi_{A_j}(x) \chi_{B_\ell}(x) \, dx,$$

where $q_0' := \frac{q_0}{q_0 - 1}, q_1' := \frac{q_1}{q_1 - 1}$, and $q' := \frac{q}{q-1}$. Since a_j and b_ℓ are positive, we see that F is continuous on $\overline{S}$ and F is holomorphic in S. Morever, by (7) and (9), we have

$$F(\theta) = \int_{\mathbb{R}^n} Tf(x) g(x) \, dx. \tag{10}$$

By Hölder's inequality and the boundedness of T from L^{p_0} to L^{q_0}, for every $z \in \overline{S}$, we have

$$\begin{aligned}|F(z)| &\le \sum_{j=1}^{k}\sum_{\ell=1}^{m} a_j^{\frac{p(1-\mathrm{Re}(z))}{p_0}+\frac{p\mathrm{Re}(z)}{p_1}} b_\ell^{\frac{q'(1-\mathrm{Re}(z))}{q'_0}+\frac{q'\mathrm{Re}(z)}{q'_1}} \left|\int_{\mathbb{R}^n} T\chi_{A_j}(x)\chi_{B_\ell}(x)\,dx\right| \\ &\le \sum_{j=1}^{k}\sum_{\ell=1}^{m}(a_j^{\frac{p}{p_0}}+a_j^{\frac{p}{p_1}})(b_\ell^{\frac{q'}{q'_0}}+b_\ell^{\frac{q'}{q'_1}})M_0|A_j|^{\frac{1}{p_0}}|B_\ell|^{\frac{1}{q'_0}}.\end{aligned}$$

Therefore, $\sup_{z\in\overline{S}}|F(z)|<\infty$. Note that $F(z)$ can be rewritten as

$$F(z)=\int_{\mathbb{R}^n} Tf_z(x)g_z(x)\,dx,$$

where

$$f_z := \sum_{j=1}^{k} a_j^{\frac{p(1-z)}{p_0}+\frac{pz}{p_1}} e^{i\alpha_j}\chi_{A_j} \quad \text{and} \quad g_z := \sum_{\ell=1}^{m} b_\ell^{\frac{q'(1-z)}{q'_0}+\frac{q'z}{q'_1}} e^{i\beta_\ell}\chi_{B_\ell}.$$

By Hölder's inequality and the boundedness of T from L^{p_0} to L^{q_0}, for every $t\in\mathbb{R}$, we have

$$|F(it)| \le \|Tf_{it}\|_{L^{q_0}}\|g_{it}\|_{L^{q'_0}} \le M_0\|f_{it}\|_{L^{p_0}}\|g_{it}\|_{L^{q'_0}}. \tag{11}$$

Since A_j's are pairwise disjoint, we have

$$\|f_{it}\|_{L^{p_0}}^{p_0} = \sum_{j=1}^{k} a_j^p|A_j| = \|f\|_{L^p}^p,$$

so $\|f_{it}\|_{L^{p_0}} = \|f\|_{L^p}^{p/p_0}$. Likewise, $\|g_{it}\|_{L^{q'_0}} = \|g\|_{L^{q'}}^{q'/q'_0} = 1$. Combining these calculations with (11), we obtain

$$|F(it)| \le M_0\|f\|_{L^p}^{p/p_0}. \tag{12}$$

Similarly,

$$|F(1+it)| \le M_1\|f\|_{L^p}^{p/p_1}. \tag{13}$$

By the three-lines lemma and (12)-(13), we have

$$|F(\theta)| \le M_0^{1-\theta}M_1^{\theta}\|f\|_{L^p}^{\frac{p(1-\theta)}{p_0}}\|f\|_{L^p}^{\frac{p\theta}{p_1}} = M_0^{1-\theta}M_1^{\theta}\|f\|_{L^p}.$$

Combining this inequality with (8) and (10), we have

$$\|Tf\|_{L^q} \le M_0^{1-\theta}M_1^{\theta}\|f\|_{L^p} \tag{14}$$

for every simple function f. Finally, (14) can be extended for all $f\in L^p$ by using the density of simple functions in L^p. □

2. Interpolation of linear operators in Morrey spaces

Definition 2.1. *For $1 \le q \le p < \infty$, the Morrey space $\mathcal{M}^p_q = \mathcal{M}^p_q(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^q_{\text{loc}}(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathcal{M}^p_q} := \sup_{a\in\mathbb{R}^n, R>0} |B(a,R)|^{\frac{1}{p}} \left(\frac{1}{|B(a,R)|} \int_{B(a,R)} |f(x)|^q \, dx \right)^{\frac{1}{q}}$$

is finite.

Remark:

(1) For $p = q$, we have $\mathcal{M}^p_q = L^q$.
(2) If $1 \le q < p < \infty$, then $f(x) := |x|^{-\frac{n}{p}} \in \mathcal{M}^p_q \setminus L^p$.

G. Stampacchia proved the following extension of the Riesz-Thorin interpolation theorem.

Theorem 2.2. *Let $\theta \in (0,1)$, $1 \le p_0, p_1 < \infty$, $1 \le s_0 \le r_0 < \infty$, and $1 \le s_1 \le r_1 < \infty$. Define p, r, and s by*

$$\left(\frac{1}{p}, \frac{1}{r}, \frac{1}{s}\right) := (1-\theta)\left(\frac{1}{p_0}, \frac{1}{r_0}, \frac{1}{s_0}\right) + \theta\left(\frac{1}{p_1}, \frac{1}{r_1}, \frac{1}{s_1}\right).$$

If T is a bounded linear operator from L^{p_0} to $\mathcal{M}^{r_0}_{s_0}$ and from L^{p_1} to $\mathcal{M}^{r_1}_{s_1}$, then T is bounded from L^p to $\mathcal{M}^r_s$.

Unfortunately, if the domain of the operator T is Morrey spaces, there are some counterexamples given by A. Ruiz and L. Vega [17] for the case $n > 1$ and by O. Blasco et al. in [3] for the case $n = 1$. Let us recall the result in [3].

Theorem 2.3. [3] *Let $n = 1$, $\theta \in (0,1)$, and $1 < q_1 < q_0$. Define*

$$\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \ r_0 := \frac{2}{\min(\frac{1}{q_0} + \frac{2}{q_1}.2)}, \ r_1 := q_1, \text{ and } \frac{1}{r} := \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

Then there exists a bounded linear operator T from $L^{q_0} = \mathcal{M}^{q_0}_{q_0}$ to L^{r_0} and from $\mathcal{M}^{q_0}_{q_1}$ to L^{r_1} such that T is not bounded from $\mathcal{M}^{q_0}_q$ to L^r.

Proof. According to the definition of q, we know that $q_1 < q < q_0$. Hence, we may choose

$$\beta > \frac{\frac{1}{q_0}}{\frac{1}{q} - \frac{1}{q_0}}. \tag{15}$$

Let $N_0 \in \mathbb{N}$ be such that

$$\frac{\beta+1}{\log 2} < \frac{N+1}{\log N}, \tag{16}$$

for every $N \in \mathbb{N} \cap [N_0, \infty)$. Let $N \in \mathbb{N} \cap [N_0, \infty)$ be fixed. We define

$$I^N_j := [N! + jN^\beta, N! + jN^\beta + 1]$$

where $j = 0, 1, \ldots, N-1$ and set $E_N := \cup_{j=0}^{N-1} I_j^N$. Observe that the choice of β allows $\{E_N\}_{N=1}^{\infty}$ to be disjoint. Note that $r_0 < r_1$, so $r_0 < r < r_1$. Therefore, we may choose

$$\gamma \in \left(\frac{2}{r_1}, \frac{2}{r}\right). \tag{17}$$

With this choice of γ, we construct an operator T by the formula

$$Tf(x) := \sum_{N=N_0}^{\infty} N^{-\gamma} \chi_{E_N}(x) f(x)$$

for every measurable function f. By the Hölder inequality, for every $f \in L^{q_0}$ we have

$$\begin{aligned} \|Tf\|_{L^{r_0}} &\le \left(\sum_{N=N_0}^{\infty} N^{-\gamma r_0} |E_N|^{1-\frac{r_0}{q_0}} \left(\int_{E_N} |f(x)|^{q_0} \right)^{\frac{r_0}{q_0}} \right)^{\frac{1}{r_0}} \\ &\le \left(\sum_{N=N_0}^{\infty} N^{-\gamma r_0 + 1 - \frac{r_0}{q_0}} \right)^{\frac{1}{r_0}} \|f\|_{L^{q_0}}. \end{aligned}$$

It follows from (17) that

$$-\gamma r_0 + 1 - \frac{r_0}{q_0} < -\frac{2r_0}{r_1} + 1 - \frac{r_0}{q_0} = 1 - r_0 \left(\frac{2}{q_1} + \frac{1}{q_0} \right) \le -1.$$

Consequently,

$$\|Tf\|_{L^{r_0}} \le C_0 \|f\|_{L^{q_0}}$$

for some constant $C_0 > 0$. We now show that

$$\|Tf\|_{L^{r_1}} \le C_1 \|f\|_{\mathcal{M}_{q_1}^{q_0}} \tag{18}$$

for some $C_1 > 0$ and for every $f \in \mathcal{M}_{q_1}^{q_0}$. Since $\{E_N\}_{N=N_0}^{\infty}$ is a collection of disjoint sets and $q_1 = r_1$, we get

$$\|Tf\|_{L^{r_1}} \le \left(\sum_{N=N_0}^{\infty} N^{-\gamma r_1} \sum_{j=0}^{N-1} \int_{I_j^N} |f(x)|^{r_1} \, dx \right)^{\frac{1}{r_1}}. \tag{19}$$

Combining (19) and

$$\int_{I_j^N} |f(x)|^{q_1} \, dx \le |I_j^N|^{1-\frac{q_1}{q_0}} \|f\|_{\mathcal{M}_{q_1}^{q_0}}^{q_1} = \|f\|_{\mathcal{M}_{q_1}^{q_0}}^{q_1},$$

for each $j = 0, 1, \ldots, N-1$, we get

$$\|Tf\|_{L^{r_1}} \le \left(\sum_{N=N_0}^{\infty} N^{1-\gamma r_1} \right)^{\frac{1}{r_1}} \|f\|_{\mathcal{M}_{q_1}^{q_0}}.$$

According to (17), we have

$$1 - \gamma r_1 < 1 - \frac{2}{r_1} r_1 = -1,$$

so

$$\sum_{N=N_0}^{\infty} N^{1-\gamma r_1} < \infty.$$

This implies (18). The proof of the unboundedness of T from $\mathcal{M}_q^{q_0}$ to L^r goes as follows. Define

$$f_0 := \sum_{N=N_0}^{\infty} \chi_{E_N}.$$

Note that, for every $N \in \mathbb{N}$, we have

$$\begin{aligned}
\|\chi_{E_N}\|_{\mathcal{M}_1^{\frac{q_0}{q}}} &= \sup_{I\subseteq\mathbb{R}} |I|^{\frac{q}{q_0}} \frac{|I \cap E_N|}{|I|} \\
&\lesssim \left((N-1)N^\beta + 1\right)^{\frac{q}{q_0}} \frac{N}{(N-1)N^\beta + 1} \\
&\lesssim \left(N^{\beta+1}\right)^{\frac{q}{q_0}} \frac{1}{N^\beta} = N^{\frac{q(\beta+1)}{q_0}-\beta}.
\end{aligned}$$

Let $J_N := (N_0!, N! + (N-1)N^\beta + 1)$ for every $N \in \mathbb{N} \cap [N_0, \infty)$. Since

$$\|f_0\|^q_{\mathcal{M}_q^{q_0}} = \|f_0^q\|_{\mathcal{M}_1^{\frac{q_0}{q}}} = \|f_0\|_{\mathcal{M}_1^{\frac{q_0}{q}}},$$

we have

$$\begin{aligned}
\|f_0\|^q_{\mathcal{M}_q^{q_0}} &= \sup_{I\in\mathcal{I}(\mathbb{R})} |I|^{\frac{q}{q_0}-1} \int_I \sum_{N=N_0}^{\infty} \chi_{E_N}(y)\, dy \\
&\lesssim \max_{M\in\mathbb{N}} \left\{ |J_M|^{\frac{q}{q_0}-1} \int_{J_M} \sum_{N=N_0}^{M} \chi_{E_N}(y)\, dy, \|\chi_{E_M}\|_{\mathcal{M}_1^{\frac{q_0}{q}}} \right\} \\
&\lesssim \max_{M\in\mathbb{N}} \left\{ \frac{M^2}{(M! + (M-1)M^\beta + 1 - N_0!)^{1-\frac{q}{q_0}}}, M^{\frac{q(\beta+1)}{q_0}-\beta} \right\}.
\end{aligned}$$

It follows from (15) that $\frac{q(\beta+1)}{q_0} - \beta < 0$. This implies

$$\|f_0\|_{\mathcal{M}_q^{q_0}} < \infty.$$

On the other hand, we claim

$$\|Tf_0\|_{L^r} = \infty. \tag{20}$$

Indeed, (20) follows from

$$\|Tf_0\|_{L^r} = \left(\sum_{N=N_0}^{\infty} N^{-\gamma r}|E_N|\right)^{\frac{1}{r}} = \left(\sum_{N=N_0}^{\infty} N^{1-\gamma r}\right)^{\frac{1}{r}}$$

and $1 - \gamma r > -1$. This ends the proof of Theorem 2.3. □

In view of Theorem 2.3, the Riesz-Thorin theorem can not be naturally generalized to Morrey spaces. However, by adding some mild assumptions, there are recent researches about complex interpolation interpolation of Morrey spaces and their generalization (see [6, 9, 10, 11, 12, 13, 14, 15]).

3. Calderón's complex interpolation method

In this section we recall the complex interpolation method introduced by Calderón in [4]. We follow the terminology and presentation in [1, 4, 12]. In Subsections 3.1 and 3.2, we recall the definition of Calderón's first and second complex interpolation method. For the proof of our results in the next section, we shall discuss the Calderón product of Banach spaces in Section 3.3.

3.1. **The first complex interpolation method.** A pair (X_0, X_1) is said to be a compatible couple of Banach spaces if there exists a Hausdorff topological vector space Z such that X_0 and X_1 are subspaces of Z and that the embedding of X_0 and X_1 into Z is continous. From now on, let $\overline{S} := \{z \in \mathbb{C} : 0 \le \mathrm{Re}(z) \le 1\}$ and S be its interior.

Definition 3.1 (Calderón's first complex interpolation functor)**.** *Let (X_0, X_1) be a compatible couple of Banach spaces. Define $\mathcal{F}(X_0, X_1)$ as the set of all continuous functions $F : \overline{S} \to X_0 + X_1$ such that*

(1) $\sup_{z\in\overline{S}} \|F(z)\|_{X_0+X_1} < \infty$*,*
(2) F is holomorphic on S,
(3) the functions $t \in \mathbb{R} \mapsto F(j + it) \in X_j$ are bounded and continuous on $\mathbb{R}$ for $j = 0, 1$.

The norm on $\mathcal{F}(X_0, X_1)$ is defined by

$$\|F\|_{\mathcal{F}(X_0,X_1)} := \max\left\{\sup_{t\in\mathbb{R}} \|F(it)\|_{X_0},\ \sup_{t\in\mathbb{R}} \|F(1 + it)\|_{X_1}\right\}.$$

Definition 3.2 (Calderón's first complex interpolation spaces)**.** *Let $\theta \in (0, 1)$ and (X_0, X_1) be a compatible couple of Banach spaces. The complex interpolation space $[X_0, X_1]_\theta$ with respect to (X_0, X_1) is defined by*

$$[X_0, X_1]_\theta := \{f \in X_0 + X_1 : f = F(\theta) \text{ for some } F \in \mathcal{F}(X_0, X_1)\}$$

The norm on $[X_0, X_1]_\theta$ is defined by

$$\|f\|_{[X_0,X_1]_\theta} := \inf\{\|F\|_{\mathcal{F}(X_0,X_1)} : f = F(\theta) \text{ for some } F \in \mathcal{F}(X_0, X_1)\}.$$

The fact that $[X_0, X_1]_\theta$ is a Banach space can be seen in [4] and [1, Theorem 4.1.2]. When X_0 and X_1 are Lebesgue spaces, Calderón gave the following description of $[X_0, X_1]_\theta$.

Theorem 3.3. [4] *Let $\theta \in (0, 1)$, $1 \le p_0 \le \infty$, and $1 \le p_1 \le \infty$. Then*

$$[L^{p_0}, L^{p_1}]_\theta = L^p$$

where p is defined by

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Note that the Riesz-Thorin complex interpolation theorem can be seen as a corollary of Theorem 3.3 and the following Calderón's result.

Theorem 3.4. [4] *Let $\theta \in (0,1)$. Let (X_0, X_1) and (Y_0, Y_1) be two compatible couples of Banach spaces. If T is a bounded linear operator from X_j to Y_j for $j = 0, 1$, then T is bounded from $[X_0, X_1]_\theta$ to $[Y_0, Y_1]_\theta$.*

We also invoke the following useful lemma.

Lemma 3.5. [4], [1, Theorem 4.2.2] *Let $\theta \in (0,1)$ and (X_0, X_1) be a compatible couple of Banach spaces. Then we have $X_0 \cap X_1$ is dense in $[X_0, X_1]_\theta$.*

3.2. **The second complex interpolation method.** First let us recall the definition of Banach space-valued Lipschitz continuous functions. Let X be a Banach space. Denote by $\mathrm{Lip}(\mathbb{R}, X)$ the set of all functions $f : \mathbb{R} \to X$ such that

$$\|f\|_{\mathrm{Lip}(\mathbb{R},X)} := \sup_{-\infty<s<t<\infty} \frac{\|f(t) - f(s)\|_X}{|t-s|}$$

is finite.

Definition 3.6. [1, 4] *(Calderón's second complex interpolation functor) Let (X_0, X_1) be a compatible couple of Banach spaces. Denote by $\mathcal{G}(X_0, X_1)$ the set of all continuous functions $G : \bar{S} \to X_0 + X_1$ such that:*

(1) $\sup_{z\in\bar{S}} \left\| \frac{G(z)}{1+|z|} \right\|_{X_0+X_1} < \infty$,
(2) G is holomorphic on S,
(3) the functions

$$t \in \mathbb{R} \mapsto G(j + it) - G(j) \in X_j$$

are Lipschitz continuous on $\mathbb{R}$ for $j = 0, 1$.

The space $\mathcal{G}(X_0, X_1)$ is equipped with the norm

$$\|G\|_{\mathcal{G}(X_0,X_1)} := \max\left\{\|G(i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_0)},\ \|G(1 + i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_1)}\right\}. \tag{21}$$

Definition 3.7. [1, 4] *(Calderón's second complex interpolation space) Let $\theta \in (0,1)$. The second complex interpolation space $[X_0, X_1]^\theta$ with respect to (X_0, X_1) is defined to be the set of all $f \in X_0 + X_1$ such that $f = G'(\theta)$ for some $G \in \mathcal{G}(X_0, X_1)$. The norm on $[X_0, X_1]^\theta$ is defined by*

$$\|f\|_{[X_0,X_1]^\theta} := \inf\{\|G\|_{\mathcal{G}(X_0,X_1)} : f = G'(\theta) \text{ for some } G \in \mathcal{G}(X_0, X_1)\}.$$

The relation between the second complex interpolation and the interpolation of linear operators is given as follows.

Theorem 3.8 (Calderón, 1964)**.** *Let $\theta \in (0,1)$ and $j \in \{0,1\}$. Suppose that T is a bounded linear operator from X_j to Y_j. Then, T is bounded from $[X_0, X_1]^\theta$ to $[Y_0, Y_1]^\theta$.*

We now describe the second complex interpolation of Lebesgue spaces.

Theorem 3.9 (Calderón, 1964). *Let $\theta \in (0,1)$, $1 \le p_0, p_1 \le \infty$, and $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then*

$$[L^{p_0}, L^{p_1}]^\theta = L^p.$$

The relation between the inclusion and the second complex interpolation spaces is given as follows.

Lemma 3.10. [11, Lemma 2.8] *If $X_0 \hookrightarrow Y_0$ and $X_1 \hookrightarrow Y_1$, then*

$$[X_0, X_1]^\theta \hookrightarrow [Y_0, Y_1]^\theta.$$

Proof. Let $f \in [X_0, X_1]^\theta$. Then $f = G'(\theta)$ for some $G \in \mathcal{G}(X_0, X_1)$. By using the following inequalities

$$\|x_0\|_{Y_0} \lesssim \|x_0\|_{X_0}, \quad \|x_1\|_{Y_1} \lesssim \|x_1\|_{X_1}, \quad \text{and} \quad \|x\|_{Y_0+Y_1} \lesssim \|x\|_{X_0+X_1},$$

for every $x_0 \in X_0$, $x_1 \in X_1$, and $x \in X_0 + X_1$, we can show that $G \in \mathcal{G}(Y_0, Y_1)$. Thus, $f \in [Y_0, Y_1]^\theta$. □

The relation between the first and second complex interpolation functors is given in the following lemma:

Lemma 3.11. [10, Lemma 2.4] *For $G \in \mathcal{G}(X_0, X_1)$, $z \in \overline{S}$, and $k \in \mathbb{N}$, define*

$$H_k(z) := \frac{G(z + 2^{-k}i) - G(z)}{2^{-k}i}. \tag{22}$$

Then we have $H_k(\theta) \in [X_0, X_1]_\theta$.

Proof. We give a simplified proof of [10, Lemma 2.4]. The proof is adapted from [11]. The continuity and holomorphicity of H_k is a consequence of the corresponding property of G. Let $j \in \{0,1\}$ be fixed. Since $t \in \mathbb{R} \mapsto G(j+it) \in X_j$ is Lipschitz-continuous, we see that $t \in \mathbb{R} \mapsto H_k(j+it) \in X_j$ is bounded and continuous on $\mathbb{R}$. Therefore, $H_k \in \mathcal{F}(X_0, X_1)$. Moreover,

$$\begin{aligned}\|H_k(\theta)\|_{[X_0,X_1]_\theta} &\le \|H_k\|_{\mathcal{F}(X_0,X_1)} \\ &= \max_{j=0,1} \sup_{t\in\mathbb{R}} \left\| \frac{G(j+i(t+2^{-k})) - G(j+it)}{2^{-k}i} \right\|_{X_j} \\ &\le \|G\|_{\mathcal{G}(X_0,X_1)} < \infty,\end{aligned}$$

as desired. □

We shall also use the following useful connection between the first and second complex interpolation, obtained by Bergh [2].

Lemma 3.12. [2] *Let (X_0, X_1) be a compatible couple and $\theta \in (0,1)$. Then we have*

$$[X_0, X_1]_\theta = \overline{X_0 \cap X_1}^{[X_0,X_1]^\theta}. \tag{23}$$

3.3. **Calderón product.** In order to describe the first complex interpolation spaces, sometimes it is easier to calculate the Calderón product of Banach lattices and applying the result of Sestakov in [18]. The definition of the Calderón product and Sestakov's lemma are given as follows.

Definition 3.13. *Let $\theta \in (0,1)$ and (X_0, X_1) be a compatible couple of Banach spaces of measurable functions in $\mathbb{R}^n$. The Calderón product ${X_0}^{1-\theta}{X_1}^{\theta}$ of X_0 and X_1 is defined by*

$$ {X_0}^{1-\theta}{X_1}^{\theta} := \bigcup_{f_0 \in X_0, f_1 \in X_1} \{f : \mathbb{R}^n \to \mathbb{C} \, : \, |f(x)| \le |f_0(x)|^{1-\theta}|f_1(x)|^{\theta} \text{ a.e. } x \in \mathbb{R}^n\}. $$

For $f \in {X_0}^{1-\theta}{X_1}^{\theta}$, we define

$$ \|f\|_{{X_0}^{1-\theta}{X_1}^{\theta}} $$
$$:= \inf\{\|f_0\|_{X_0}^{1-\theta}\|f_1\|_{X_1}^{\theta} : f_0 \in X_0, f_1 \in X_1, |f(x)| \le |f_0(x)|^{1-\theta}|f_1(x)|^{\theta} \text{ a.e. } x \in \mathbb{R}^n\}. $$

Theorem 3.14. *Let $\theta \in (0,1)$ and (X_0, X_1) be a compatible couple of Banach spaces of measurable functions in $\mathbb{R}^n$. Then $X_0^{1-\theta}X_1^{\theta}$ is a Banach space.*

Proof. This result was due to Calderón [4]. For the convenience of the reader, we give the detailed proof. We first prove the triangle inequality in $X_0^{1-\theta}X_1^{\theta}$. Let $f, g \in {X_0}^{1-\theta}{X_1}^{\theta}$. Choose $\lambda \in (\|f\|_{{X_0}^{1-\theta}{X_1}^{\theta}}, \infty)$ and $\rho \in (\|g\|_{{X_0}^{1-\theta}{X_1}^{\theta}}, \infty)$. Then there is a decomposition

$$ |f(x)| \le \lambda |f_0(x)|^{1-\theta}|f_1(x)|^{\theta}, \quad |g(x)| \le \rho g_0(x)|^{1-\theta}|g_1(x)|^{\theta} \tag{24} $$

such that $f_0, g_0 \in X_0$ and $f_1, g_1 \in X_1$ have norm 1. Observe that (24) is equivalent to:

$$ |f(x)| \le (1-\theta)\varphi(x)^{-\theta}\lambda|f_0(x)| + \theta\varphi(x)^{1-\theta}\lambda|f_1(x)| $$

and

$$ |g(x)| \le (1-\theta)\varphi(x)^{-\theta}\rho|g_0(x)| + \theta\varphi(x)^{1-\theta}\rho|g_1(x)| $$

for any measurable function $\varphi : \mathbb{R}^n \to (0, \infty)$. Since

$$ |f(x) + g(x)| \le (1-\theta)\varphi^{-\theta}(\lambda|f_0(x)| + \rho|g_0(x)|) + \theta\varphi^{1-\theta}(\lambda|f_1(x)| + \rho|g_1(x)|), $$

we conclude that

$$ |f(x) + g(x)| \le (\lambda|f_0(x)| + \rho|g_0(x)|)^{1-\theta}(\lambda|f_1(x)| + \rho|g_1(x)|)^{\theta}. $$

This implies that $\|f+g\|_{{X_0}^{1-\theta}{X_1}^{\theta}} \le \|f\|_{{X_0}^{1-\theta}{X_1}^{\theta}} + \|g\|_{{X_0}^{1-\theta}{X_1}^{\theta}}$.

The proof of completeness of ${X_0}^{1-\theta}{X_1}^{\theta}$ goes as follows. Let $\{f^j\}_{j=1}^{\infty} \subset {X_0}^{1-\theta}{X_1}^{\theta}$ be a sequence satisfying

$$ \sum_{j=1}^{\infty} \|f^j\|_{{X_0}^{1-\theta}{X_1}^{\theta}} < \infty $$

Let $\lambda_j \in (\|f\|_{{X_0}^{1-\theta}{X_1}^{\theta}}, \infty)$. Then, there exists $f_0^j \in X_0$ and $f_1^j \in X_1$ have norm 1 such that

$$ |f^j(x)| \le \lambda_j |f_0^j(x)|^{1-\theta}|f_1^j(x)|^{\theta}. \tag{25} $$

Then as before,

$$\sum_{j=1}^{\infty}|f_j(x)| \le \left(\sum_{j=1}^{\infty}\lambda_j|f_0^j(x)|\right)^{1-\theta}\left(\sum_{j=1}^{\infty}\lambda_j|f_1^j(x)|\right)^{\theta}.$$

Since X_0 and X_1 are Banach spaces, we see that $\sum_{j=1}^{\infty}\lambda_j|f_0^j|$ and $\sum_{j=1}^{\infty}\lambda_j|f_1^j|$ converge in X_0 and X_1, respectively. Consequently, $\sum_{j=1}^{\infty} f_j(x)$ converges absolutely for almost all $x \in \mathbb{R}^n$ and belongs to ${X_0}^{1-\theta}{X_1}^{\theta}$ together with the estimate

$$\left\|\sum_{j=1}^{\infty} f_j\right\|_{{X_0}^{1-\theta}{X_1}^{\theta}} \le \sum_{j=1}^{\infty}\|f_j\|_{{X_0}^{1-\theta}{X_1}^{\theta}},$$

or more generally

$$\left\|\sum_{j=J}^{\infty} f_j\right\|_{{X_0}^{1-\theta}{X_1}^{\theta}} \le \sum_{j=J}^{\infty}\|f_j\|_{{X_0}^{1-\theta}{X_1}^{\theta}}, (J \in \mathbb{N})$$

which also yields that $\sum_{j=1}^{\infty} f_j$ converges in ${X_0}^{1-\theta}{X_1}^{\theta}$. □

By virtue of the Hölder inequality and factorization, for $1 \le p_0, p_1 \le \infty$

$$(L^{p_0})^{1-\theta}(L^{p_1})^{\theta} = L^p,$$

where p is defined by $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. We now recall the following result by Sestakov.

Lemma 3.15. [18] *Let (X_0, X_1) be a compatible couple of Banach spaces of measurable functions in $\mathbb{R}^n$. Then for every $\theta \in (0,1)$, we have*

$$[X_0, X_1]_\theta = \overline{X_0 \cap X_1}^{X_0^{1-\theta}X_1^{\theta}}.$$

4. The description of complex interpolation of Morrey spaces

In this section, we will discuss the first and second complex interpolation of Morrey spaces. The interpolation by using the first method can be found in [6, 9, 10, 15]. Meanwhile, the result on the second complex interpolation is given by Lemarié-Rieusset [14]. The presentation in this section and Section 5 follows [12].

4.1. The first complex interpolation of Morrey spaces.

The first result about the description of the first complex interpolation of Morrey spaces was due to Cobos et al. [6].

Theorem 4.1. [6] *Let $\theta \in (0,1)$, $1 \le q_0 \le p_0 < \infty$, and $1 \le q_1 \le p_1 < \infty$. Define p and q by*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \tag{26}$$

respectively. Then

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subseteq \mathcal{M}^p_q. \tag{27}$$

Assuming $\frac{p_0}{q_0} = \frac{p_1}{q_1}$, Lu et al. [15] improved the description of $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta$ in Theorem 4.1. Morever, their result are in the setting of Morrey spaces over metric measure space.

Theorem 4.2. [15] *Let $\theta \in (0,1)$, $1 \le q_0 \le p_0 < \infty$, and $1 \le q_1 \le p_1 < \infty$. Assume that $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. Then*

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \overline{\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q}, \tag{28}$$

where p and q are defined by (26).

The key parts of the proof of Theorem 4.2 are Lemma 3.15 and the calculation of the Calderón product between $\mathcal{M}^{p_0}_{q_0}$ and $\mathcal{M}^{p_1}_{q_1}$.

Proposition 4.3. *Let $\theta \in (0,1)$, $1 \le q_0 \le p_0 < \infty$, and $1 \le q_1 \le p_1 < \infty$. Assume that $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. Define p and q by* (26). *Then*

$$(\mathcal{M}^{p_0}_{q_0})^{1-\theta} (\mathcal{M}^{p_1}_{q_1})^{\theta} = \mathcal{M}^p_q. \tag{29}$$

Proof. Let $B = B(a,r)$ be any ball in $\mathbb{R}^n$ and $\varepsilon > 0$. Let $f \in (\mathcal{M}^{p_0}_{q_0})^{1-\theta} (\mathcal{M}^{p_1}_{q_1})^{\theta}$. Then, there exist some functions $f_0 \in \mathcal{M}^{p_0}_{q_0}$ and $f_1 \in \mathcal{M}^{p_1}_{q_1}$ such that

$$|f(x)| \le |f_0(x)|^{1-\theta} |f_1(x)|^{\theta}, \text{ a.e. } x \in \mathbb{R}^n \tag{30}$$

and

$$\|f_0\|^{1-\theta}_{\mathcal{M}^{p_0}_{q_0}} \|f_1\|^{\theta}_{\mathcal{M}^{p_1}_{q_1}} \le (1+\varepsilon) \|f\|_{(\mathcal{M}^{p_0}_{q_0})^{1-\theta}(\mathcal{M}^{p_1}_{q_1})^{\theta}}. \tag{31}$$

By using Hölder's inequality and (30), we have

$$\begin{aligned} \left(\int_B |f(x)|^q \, dx \right)^{\frac{1}{q}} &\le \left(\int_B |f_0(x)|^{q(1-\theta)} |f_1(x)|^{q\theta} \, dx \right)^{\frac{1}{q}} \\ &\le \left(\int_B |f_0(x)|^{q_0} \, dx \right)^{\frac{1-\theta}{q_0}} \left(\int_B |f_1(x)|^{q_1} \, dx \right)^{\frac{\theta}{q_1}}. \end{aligned} \tag{32}$$

Combining $\frac{p_0}{q_0} = \frac{p_1}{q_1}$, (26), and inequalities (31)–(32), we obtain

$$\begin{aligned} |B|^{\frac{1}{p} - \frac{1}{q}} \left(\int_B |f(x)|^q dx \right)^{\frac{1}{q}} &\le \frac{|B|^{\frac{1-\theta}{p_0} + \frac{\theta}{p_1}}}{|B|^{\frac{1-\theta}{q_0} + \frac{\theta}{q_1}}} \|f_0\|^{1-\theta}_{L^{q_0}(B)} \|f_1\|^{\theta}_{L^{q_1}(B)} \\ &\le \|f_0\|^{1-\theta}_{\mathcal{M}^{p_0}_{q_0}} \|f_1\|^{\theta}_{\mathcal{M}^{p_1}_{q_1}} \\ &\le (1+\varepsilon) \|f\|_{(\mathcal{M}^{p_0}_{q_0})^{1-\theta}(\mathcal{M}^{p_1}_{q_1})^{\theta}}. \end{aligned}$$

Since ε is arbitary, we have $f \in \mathcal{M}^p_q$ with $\|f\|_{\mathcal{M}^p_q} \le \|f\|_{(\mathcal{M}^{p_0}_{q_0})^{1-\theta}(\mathcal{M}^{p_1}_{q_1})^{\theta}}$. Thus,

$$(\mathcal{M}^{p_0}_{q_0})^{1-\theta}(\mathcal{M}^{p_1}_{q_1})^{\theta} \subseteq \mathcal{M}^p_q.$$

Conversely, let $f \in \mathcal{M}^p_q$. Define $\tilde{f}_j := |f|^{\frac{q}{q_j}}$ where $j \in \{0,1\}$. It follows from (26) and $\frac{p_0}{q_0} = \frac{p_1}{q_1}$ that

$$\frac{p}{q} = \frac{p_0}{q_0} = \frac{p_1}{q_1}. \tag{33}$$

Then $\tilde{f}_j \in \mathcal{M}^{p_j}_{q_j}$ with $\|\tilde{f}_j\|_{\mathcal{M}^{p_j}_{q_j}} = \|f\|^{\frac{q}{q_j}}_{\mathcal{M}^p_q}$ for $j = 0, 1$. Observe that, we have

$$|\tilde{f}_0|^{1-\theta}|\tilde{f}_1|^{\theta} = |f|^{\frac{q(1-\theta)}{q_0}}|f|^{\frac{q\theta}{q_1}} = |f| \tag{34}$$

and

$$\|f\|_{(\mathcal{M}^{p_0}_{q_0})^{1-\theta}(\mathcal{M}^{p_1}_{q_1})^{\theta}} \le \|\tilde{f}_0\|^{1-\theta}_{\mathcal{M}^{p_0}_{q_0}}\|\tilde{f}_1\|^{\theta}_{\mathcal{M}^{p_1}_{q_1}} = \|f\|^{\frac{q(1-\theta)}{q_0}+\frac{q\theta}{q_1}}_{\mathcal{M}^p_q} = \|f\|_{\mathcal{M}^p_q} < \infty. \tag{35}$$

Consequently, $f \in \left(\mathcal{M}^{p_0}_{q_0}\right)^{1-\theta}\left(\mathcal{M}^{p_1}_{q_1}\right)^{\theta}$. Therefore, $\mathcal{M}^p_q \subseteq (\mathcal{M}^{p_0}_{q_0})^{1-\theta}(\mathcal{M}^{p_1}_{q_1})^{\theta}$. Thus, we have proved (29). □

The description of the right-hand side of (28) and can be refined as follows.

Theorem 4.4. [10] *Keep the same assumption as in Theorem 4.2 and assume also that $q_0 \neq q_1$. Then we have*

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \left\{f \in \mathcal{M}^p_q : \lim_{N\to\infty}\|f - \chi_{\{\frac{1}{N}\le|f|\le N\}}f\|_{\mathcal{M}^p_q} = 0\right\}. \tag{36}$$

Note that Theorem 4.4 is an improvement of Theorems 4.2, in the sense that, $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta$ is now written in terms of the parameters p and q only and this description is more explicit than the right-hand side of (28). In order to prove Theorem 4.4, we need two lemmas. The first one is the fact that the set in the right-hand side of (36) is closed. The second lemma tells us that this set contains $\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}$.

Lemma 4.5. *Let $1 \le p \le q < \infty$. Then the set*

$$A := \left\{f \in \mathcal{M}^p_q : \lim_{N\to\infty}\left\|f - \chi_{\{\frac{1}{N}\le|f|\le N\}}f\right\|_{\mathcal{M}^p_q} = 0\right\} \tag{37}$$

is a closed subset $\mathcal{M}^p_q$.

Proof. Let $\{f_j\}_{j=1}^\infty \subset A$ be such that f_j converges to f in $\mathcal{M}^p_q$. Fix $j \in \mathbb{N}$. For every $N \in \mathbb{N}$, we have

$$\left\|\chi_{\{|f|<\frac{1}{N}\}}f\right\|_{\mathcal{M}^p_q} \le \|f - f_j\|_{\mathcal{M}^p_q} + \left\|\chi_{\{|f|<\frac{1}{N}\}\cap\{|f_j|\ge\frac{2}{N}\}}f_j\right\|_{\mathcal{M}^p_q} + \left\|\chi_{\{|f_j|<\frac{2}{N}\}}f_j\right\|_{\mathcal{M}^p_q}$$

and

$$\left\|\chi_{\{|f|>N\}}f\right\|_{\mathcal{M}^p_q} \le \|f - f_j\|_{\mathcal{M}^p_q} + \left\|\chi_{\{|f|>N\}\cap\{|f_j|\le\frac{N}{2}\}}f_j\right\|_{\mathcal{M}^p_q} + \left\|\chi_{\{|f_j|>\frac{N}{2}\}}f_j\right\|_{\mathcal{M}^p_q}.$$

On the set $\{|f| < \frac{1}{N}\} \cap \{|f_j| \geq \frac{2}{N}\}$, we have

$$|f_j| \leq |f_j - f| + |f| < |f_j - f| + \frac{1}{N} \leq |f_j - f| + \frac{1}{2}|f_j|,$$

and hence $|f_j| \leq 2|f - f_j|$. Consequently,

$$\left\|\chi_{\{|f|<\frac{1}{N}\}} f\right\|_{\mathcal{M}^p_q} \leq 3\|f - f_j\|_{\mathcal{M}^p_q} + \left\|\chi_{\{|f_j|<\frac{2}{N}\}} f_j\right\|_{\mathcal{M}^p_q}. \tag{38}$$

Meanwhile, on the set $\{|f| > N\} \cap \{|f_j| \leq \frac{N}{2}\}$, we have

$$|f_j| \leq \frac{N}{2} < \frac{|f|}{2} \leq \frac{|f - f_j|}{2} + \frac{|f_j|}{2},$$

and hence, $|f_j| \leq |f - f_j|$. Therefore,

$$\|\chi_{\{|f|>N\}} f\|_{\mathcal{M}^p_q} \leq 2\|f - f_j\|_{\mathcal{M}^p_q} + \|\chi_{\{|f_j|>\frac{N}{2}\}} f_j\|_{\mathcal{M}^p_q}. \tag{39}$$

By combining (38) and (39), we get

$$\begin{aligned}\left\|f - \chi_{\{\frac{1}{N}\leq|f|\leq N\}} f\right\|_{\mathcal{M}^p_q} &\leq \left\|\chi_{\{|f|<\frac{1}{N}\}} f\right\|_{\mathcal{M}^p_q} + \|\chi_{\{|f|>N\}} f\|_{\mathcal{M}^p_q} \\ &\leq 5\|f - f_j\|_{\mathcal{M}^p_q} + \left\|\chi_{\{|f_j|<\frac{2}{N}\}} f_j\right\|_{\mathcal{M}^p_q} + \left\|\chi_{\{|f_j|>\frac{N}{2}\}} f_j\right\|_{\mathcal{M}^p_q}.\end{aligned}$$

Since $f_j \in A$, we have

$$\limsup_{N\to\infty} \|f - \chi_{\{\frac{1}{N}\leq|f|\leq N\}} f\|_{\mathcal{M}^p_q} \leq 5\|f - f_j\|_{\mathcal{M}^p_q}.$$

By taking $j \to \infty$, we have $\lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N}\leq|f|\leq N\}} f\|_{\mathcal{M}^p_q} = 0$, and hence, $f \in A$. □

Lemma 4.6. [10] *Maintain the same conditions as Proposition* 4.3 *and let* A *be defined by* (37). *Then*

$$\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1} \subseteq A.$$

Proof. Without loss of generality, we assume that $q_1 > q_0$. Then, $q_1 > q > q_0$. Consequently, for every $f \in \mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}$, we have

$$\begin{aligned}\left\|f - \chi_{\{\frac{1}{N}\leq|f|\leq N\}} f\right\|_{\mathcal{M}^p_q} &\leq \|\chi_{\{|f|<\frac{1}{N}\}}|f|^{1-\frac{q_0}{q}}|f|^{\frac{q_0}{q}}\|_{\mathcal{M}^p_q} + \|\chi_{\{|f|>N\}}|f|^{1-\frac{q_1}{q}}|f|^{\frac{q_1}{q}}\|_{\mathcal{M}^p_q} \\ &\leq N^{\frac{q_0-q}{q}} \left\||f|^{\frac{q_0}{q}}\right\|_{\mathcal{M}^p_q} + N^{\frac{q-q_1}{q}} \left\||f|^{\frac{q_1}{q}}\right\|_{\mathcal{M}^p_q} \\ &= N^{\frac{q_0-q}{q}} \|f\|^{\frac{q_0}{q}}_{\mathcal{M}^{p_0}_{q_0}} + N^{\frac{q-q_1}{q}} \|f\|^{\frac{q_1}{q}}_{\mathcal{M}^{p_1}_{q_1}} \to 0\end{aligned}$$

as $N \to \infty$, which implies $f \in A$. □

We are now ready to prove Theorem 4.4.

Proof of Theorem 4.4. By virtue of Theorem 4.2 and Lemmas 4.5 and 4.6, we have

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \overline{\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q} \subseteq A.$$

Conversely, let $f \in A$. For every $N \in \mathbb{N}$, define $f_N := \chi_{\{\frac{1}{N} \le |f| \le N\}} f$. As in the proof of Lemma 4.6, we may assume that $q_0 < q_1$. Then $q_0 < q < q_1$. This implies

$$\|f_N\|_{\mathcal{M}^{p_0}_{q_0}} \le \left\| \chi_{\{\frac{1}{N} \le |f|\}} |f|^{1-\frac{q}{q_0}} |f|^{\frac{q}{q_0}} \right\|_{\mathcal{M}^{p_0}_{q_0}} \le N^{\frac{q-q_0}{q_0}} \|f\|^{q/q_0}_{\mathcal{M}^p_q} < \infty$$

and

$$\|f_N\|_{\mathcal{M}^{p_1}_{q_1}} \le \left\| \chi_{\{|f| < N\}} |f|^{1-\frac{q}{q_1}} |f|^{\frac{q}{q_1}} \right\|_{\mathcal{M}^{p_1}_{q_1}} \le N^{\frac{q_1-q}{q_1}} \|f\|^{q/q_1}_{\mathcal{M}^p_q} < \infty.$$

Therefore, $f \in \overline{\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q}$ by the definition of A. According to Theorem 4.2, we have $f \in [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta$ as desired. □

4.2. **The second complex interpolation of Morrey spaces.** Observe that the function $f(x) := |x|^{-n/p}$ does not belong to the set in the right-hand side of (36), but this function is in $\mathcal{M}^p_q$. From this observation, one may inquire whether we can interpolate Morrey spaces and that the output is also Morrey spaces. The affirmative answer was given by Lemarié-Rieusset [14]. He proved the following result about the second complex interpolation of Morrey spaces.

Theorem 4.7. [14] *Keep the same asssumption as in Theorem 4.2. Then*

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta = \mathcal{M}^p_q.$$

It is written in the book [1, p. 90] that the first complex interpolation space is the main interest in this book and the second complex interpolation method is considered as a technical tool. Therefore, Theorem 4.7 can be seen as an example of the importance of the second complex interpolation method.

In order to prove Theorem 4.7, we prove the following lemmas about the construction of the second complex interpolation functor.

Lemma 4.8. [9, Lemma 4] *Let $q_0 > q_1$ and $f \in L^0$. Define $q : \overline{S} \to \mathbb{C}$, $F : \overline{S} \to L^0$ and $G : \overline{S} \to L^0$ by:*

$$\frac{1}{q(z)} = \frac{1-z}{q_0} + \frac{z}{q_1}, \tag{40}$$

$$F(z) := \operatorname{sgn}(f) \exp\left(\frac{q}{q(z)} \log |f|\right) \quad (z \in \overline{S}), \tag{41}$$

and

$$G(z) := (z-\theta) \int_0^1 F(\theta + (z-\theta)t)\, dt \quad (z \in \overline{S}), \tag{42}$$

respectively. Define $F_0, F_1, G_0, G_1 : \overline{S} \to L^0$ by:

$$F_0(z) := F(z)\chi_{\{|f| \le 1\}}, \quad F_1(z) := F(z)\chi_{\{|f|>1\}}, \tag{43}$$

and

$$G_0(z) := G(z)\chi_{\{|f| \le 1\}}, \quad G_1(z) := G(z)\chi_{\{|f|>1\}}. \tag{44}$$

Then, for any $z \in \overline{S}$, we have

$$|G(z)| \le (1+|z|)(|f|^{q/q_0} + |f|^{q/q_1}). \tag{45}$$

For any $z \in \mathbb{C}$ with $\varepsilon < \mathrm{Re}(z) < 1-\varepsilon$ and $w \in \mathbb{C}$ with $|w| \ll 1$, we have

$$\left|\frac{G_0(z+w)-G_0(z)}{w} - F_0(z)\right| \le C_\varepsilon |w| \cdot |f|^{\frac{q}{q_0}}, \tag{46}$$

$$\left|\frac{G_1(z+w)-G_1(z)}{w} - F_1(z)\right| \le C_\varepsilon |w| \cdot |f|^{\frac{q}{q_1}}, \tag{47}$$

where the constant C_ε depending only on $\varepsilon \in (0,1/2)$.

Proof. For $t \in [0,1]$, define $v := (z-\theta)t + \theta$. Since $\mathrm{Re}(v) \in [0,1]$, we have

$$\begin{aligned} |F(v)| &\le |f|^{\frac{q}{q_0}(1-\mathrm{Re}(v)) + \frac{q}{q_1}\mathrm{Re}(v)} \\ &\le (1-\mathrm{Re}(v))|f|^{\frac{q}{q_0}} + \mathrm{Re}(v)|f|^{\frac{q}{q_1}} \le |f|^{\frac{q}{q_0}} + |f|^{\frac{q}{q_1}}. \end{aligned} \tag{48}$$

By the triangle inequality, we have

$$|G(z)| \le |z-\theta| \left(|f|^{\frac{q}{q_0}} + |f|^{\frac{q}{q_1}}\right) \le (1+|z|)\left(|f|^{\frac{q}{q_0}} + |f|^{\frac{q}{q_1}}\right).$$

Writing out the definitions in full, we obtain

$$\begin{aligned} &\left|\frac{G_0(z+w)-G_0(z)}{w} - F_0(z)\right| \\ &= |F_0(\mathrm{Re}(z))| \left|\frac{\exp\left[q\left(\frac{-w}{q_0} + \frac{w}{q_1}\right)\log|f|\right] - 1}{w(\frac{q}{q_1} - \frac{q}{q_0})\log|f|} - 1\right|. \end{aligned}$$

Since $q_0 > q_1$, we have

$$\begin{aligned} &\left|\frac{G_0(z+w)-G_0(z)}{w} - F_0(z)\right| \\ &= \chi_{\{|f|\le 1\}}|f|^{\frac{q}{q_0}(1-\mathrm{Re}(z)) + \frac{q}{q_1}\mathrm{Re}(z)} \left|\frac{\exp\left[q\left(\frac{-w}{q_0} + \frac{w}{q_1}\right)\log|f|\right] - 1}{w(\frac{q}{q_1} - \frac{q}{q_0})\log|f|} - 1\right| \\ &\le \chi_{\{|f|\le 1\}}|f|^{\frac{q}{q_0}} \cdot |f|^{\left(\frac{q}{q_1} - \frac{q}{q_0}\right)\varepsilon} \left|\frac{\exp\left[q\left(\frac{-w}{q_0} + \frac{w}{q_1}\right)\log|f|\right] - 1}{w(\frac{q}{q_1} - \frac{q}{q_0})\log|f|} - 1\right| \\ &\le |f|^{\frac{q}{q_0}} \sup_{0<t\le 1} t^{\left(\frac{q}{q_1} - \frac{q}{q_0}\right)\varepsilon} \left|\frac{\exp\left[q\left(\frac{-w}{q_0} + \frac{w}{q_1}\right)\log t\right] - 1}{w(\frac{q}{q_1} - \frac{q}{q_0})\log t} - 1\right| \le C_\varepsilon |w| \cdot |f|^{\frac{q}{q_0}}. \end{aligned}$$

By a similar argument, we also have

$$\begin{aligned}
&\left|\frac{G_1(z+w)-G_1(z)}{w}-F_1(z)\right| \\
&=\chi_{\{|f|>1\}}|f|^{\frac{q}{q_1}}\cdot|f|^{\left(\frac{q}{q_0}-\frac{q}{q_1}\right)(1-\mathrm{Re}(z))}\left|\frac{\exp\left[q\left(\frac{-w}{q_0}+\frac{w}{q_1}\right)\log|f|\right]-1}{w(\frac{q}{q_1}-\frac{q}{q_0})\log|f|}-1\right| \\
&\le\chi_{\{|f|>1\}}|f|^{\frac{q}{q_1}}\cdot|f|^{\left(\frac{q}{q_0}-\frac{q}{q_1}\right)\varepsilon}\left|\frac{\exp\left[q\left(\frac{-w}{q_0}+\frac{w}{q_1}\right)\log|f|\right]-1}{w(\frac{q}{q_1}-\frac{q}{q_0})\log|f|}-1\right| \\
&\le|f|^{\frac{q}{q_1}}\sup_{t\ge1}t^{\left(\frac{q}{q_0}-\frac{q}{q_1}\right)\varepsilon}\left|\frac{\exp\left[q\left(\frac{-w}{q_0}+\frac{w}{q_1}\right)\log t\right]-1}{w(\frac{q}{q_1}-\frac{q}{q_0})\log t}-1\right|\le C_\varepsilon|w|\cdot|f|^{\frac{q}{q_1}}
\end{aligned}$$

as desired. □

Lemma 4.9. [9, Lemma 12] *Let* $f\in\mathcal{M}^p_q$. *Via* (40) *define* $F:\overline{S}\to\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}$ *and* $G:\overline{S}\to\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}$ *by* (41) *and* (42), *respectively. Then, the function* G *belongs to* $\mathcal{G}(\mathcal{M}^{p_0}_{q_0},\mathcal{M}^{p_1}_{q_1})$.

Proof. It follows from (45) that $G(z)\in\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}$ and

$$\sup_{z\in\overline{S}}\frac{\|G(z)\|_{\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}}}{1+|z|}\le\|f\|^{q/q_0}_{\mathcal{M}^p_q}+\|f\|^{q/q_1}_{\mathcal{M}^p_q}.$$

Now let $z_1,z_2\in\overline{S}$. Then, by inequality (48), we get

$$\|G(z_1)-G(z_2)\|_{\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}}\le|z_1-z_2|\left(\|f\|^{q/q_0}_{\mathcal{M}^p_q}+\|f\|^{q/q_1}_{\mathcal{M}^p_q}\right).$$

This shows the continuity of $G:\overline{S}\to\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}$. The proof of holomorphicity of $G:S\to\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}$ goes as follows. Let $\varepsilon\in(0,\frac{1}{2})$ and define

$$S_\varepsilon:=\{z\in S:\varepsilon<\mathrm{Re}(z)<1-\varepsilon\}.$$

According to (46) and (47), we have

$$\begin{aligned}
&\left\|\frac{G(z+w)-G(z)}{w}-F(z)\right\|_{\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}} \\
&\le\left\|\frac{G_0(z+w)-G_0(z)}{w}-F_0(z)\right\|_{\mathcal{M}^{p_0}_{q_0}}+\left\|\frac{G_1(z+w)-G_1(z)}{w}-F_1(z)\right\|_{\mathcal{M}^{p_1}_{q_1}} \\
&\le C_\varepsilon|w|\left(\|f\|^{q/q_0}_{\mathcal{M}^p_q}+\|f\|^{q/q_1}_{\mathcal{M}^p_q}\right).
\end{aligned}$$

Taking $w\to0$, we see that $G:S_\varepsilon\to\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}$ is holomorphic. Since $\varepsilon>0$ is arbitrary, we conclude that $G:S\to\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}$ is holomorphic.

We now verify that $G(j+it_1)-G(j+it_2)\in\mathcal{M}^{p_j}_{q_j}$ for every $t_1,t_2\in\mathbb{R}$ and $j\in\{0,1\}$ and also

$$\|G(j+i\cdot)\|_{\mathrm{Lip}(\mathbb{R},\mathcal{M}^{p_j}_{q_j})}\le(\|f\|_{\mathcal{M}^p_q})^{q/q_j}. \tag{49}$$

for every $j \in \{0,1\}$. Combining $|F(j+it)| = |f|^{\frac{q}{q_j}}$ and

$$G(j+it_1) - G(j+it_2) = -i\int_{t_1}^{t_2} F(j+it)\,dt,$$

we get

$$\|G(j+it_1) - G(j+it_2)\|_{\mathcal{M}^{p_j}_{q_j}} \le |t_1 - t_2|\|f\|^{\frac{q}{q_j}}_{\mathcal{M}^p_q}.$$

This implies (50). Thus, $G \in \mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$ with

$$\|G\|_{\mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})} \le \|f\|^{q/q_j}_{\mathcal{M}^p_q}, \tag{50}$$

as desired. □

Note that we can not use the function F defined by (41) as the first complex interpolation functor because F does not belong to $\mathcal{F}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$ when $f(x) := |x|^{-n/p}$. This fact is a consequence of the following proposition.

Proposition 4.10. [9, Proposition 4] *Let $f(x) := |x|^{-n/p}$ and define F by (41). Then the mapping $t \in \mathbb{R} \mapsto F(it) \in \mathcal{M}^{p_0}_{q_0}$ is not continuous at $t = 0$.*

Proof. Assume that $p_0 > p_1$ and define $Q := \frac{1}{p_1} - \frac{1}{p_0}$. Using $\frac{p_0}{q_0} = \frac{p}{q} = \frac{p_1}{q_1}$, for every $0 < t < \frac{1}{Q}$, we have

$$|F(it) - F(0)| = |x|^{-\frac{n}{p_0}}\left||x|^{-Qit} - 1\right| = 2|x|^{-\frac{n}{p_0}}\left|\sin\left(\frac{Qt\log|x|}{2}\right)\right|. \tag{51}$$

Using (51) and letting $R := \exp((Qt)^{-1})$, we get

$$\begin{aligned}
&\|F(it) - F(0)\|_{\mathcal{M}^{p_0}_{q_0}} \\
&\ge 2|B(0,2R)|^{\frac{1}{p_0}-\frac{1}{q_0}}\left(\int_{B(0,2R)\setminus B(0,R)} |x|^{-\frac{nq_0}{p_0}}\left|\sin\left(\frac{Qt\log|x|}{2}\right)\right|^{q_0} dx\right)^{\frac{1}{q_0}} \\
&\gtrsim R^{\frac{n}{p_0}-\frac{n}{q_0}}\left(\int_{B(0,2R)\setminus B(0,R)} |x|^{-\frac{nq_0}{p_0}}\,dx\right)^{\frac{1}{q_0}} \gtrsim 1,
\end{aligned} \tag{52}$$

where we use

$$\left|\sin\left(\frac{Qt\log|x|}{2}\right)\right| > \sin\left(\frac{1}{2}\right)$$

for every $R < |x| < 2R$. Thus, (52) implies

$$\lim_{t\to 0^+} \|F(it) - F(0)\|_{\mathcal{M}^{p_0}_{q_0}} \ne 0,$$

as desired. □

Now we arrive at our main result in this section.

Theorem 4.11. [9, p. 316] *Keep the same assumption as in Proposition 4.3. Then*

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta = \mathcal{M}^p_q. \tag{53}$$

Proof. Let $f \in \mathcal{M}^p_q$. By a normalization, we may suppose $\|f\|_{\mathcal{M}^p_q} = 1$, for the purpose of proving $f \in [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta$. For every $z \in \overline{S}$, define $F(z)$ and $G(z)$ as we did in Lemma 4.8. Thanks to Lemma 4.9, we have $G \in \mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$. Since $G'(\theta) = F(\theta) = f$, we have

$$\|f\|_{[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta} \le \|G\|_{\mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})} = \max_{j=0,1} \|G(j+i\cdot)\|_{\mathrm{Lip}\left(\mathbb{R}, \mathcal{M}^{p_j}_{q_j}\right)} = 1.$$

This shows that $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta \supset \mathcal{M}^p_q$. Conversely, let $f \in [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta$ with

$$\|f\|_{[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta} = 1.$$

Suppose f is realized as $G'(\theta)$, where $G \in \mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$ and $\|G\|_{\mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})} \le 2$. For every $k \in \mathbb{N}$ and $z \in \overline{S}$, we define $H_k(z)$ by (22). According to Lemma 3.11 and Theorem 4.4, we obtain

$$\|H_k(\theta)\|_{\mathcal{M}^p_q} \lesssim \|H_k(\theta)\|_{[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta} \le \|G\|_{\mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})} \le 2. \tag{54}$$

Meanwhile, since $f = G'(\theta) = \lim_{k \to \infty} H_k(\theta)$ in $\mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$, there exists a subsequence $\{H_{k_j}\}_{j=1}^\infty$ such that $f(x) = \lim_{j \to \infty} H_{k_j}(\theta)(x)$ for almost every $x \in \mathbb{R}^n$. Consequently, by virtue of the Fatou lemma and (54), we have

$$\|f\|_{\mathcal{M}^p_q} \lesssim \liminf_{j \to \infty} \|H_{k_j}(\theta)\|_{\mathcal{M}^p_q} \le 2.$$

This implies $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta \hookrightarrow \mathcal{M}^p_q$. □

5. The description of complex interpolation of subspaces of Morrey spaces

The first result on the complex interpolation of subspaces of Morrey spaces can be traced back to [20]. They investigated the space $\overset{\circ}{\mathcal{M}}{}^p_q := \overline{C^\infty_{\mathrm{c}}}^{\mathcal{M}^p_q}$, where C^∞_{c} is the set of all infinitely differentiable functions with compact support. Their result is given as follows.

Theorem 5.1. *Let $\theta \in (0,1)$, $1 < q_0 \le p_0 < \infty$, and $1 < q_1 \le p_1 < \infty$. Assume that $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. Define p and q by*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then

$$[\overset{\circ}{\mathcal{M}}{}^{p_0}_{q_0}, \overset{\circ}{\mathcal{M}}{}^{p_1}_{q_1}]_\theta = [\mathcal{M}^{p_0}_{q_0}, \overset{\circ}{\mathcal{M}}{}^{p_1}_{q_1}]_\theta = [\overset{\circ}{\mathcal{M}}{}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \overset{\circ}{\mathcal{M}}{}^p_q.$$

Let $1 \le q \le p < \infty$. We now consider the following subspaces:

(1) $\overline{\mathcal{M}}{}^p_q := \overline{L^\infty \cap \mathcal{M}^p_q}^{\mathcal{M}^p_q}$ (see [5]).
(2) Denote by L^0_{c} the set of all measurable functions with compact support and define $\overset{*}{\mathcal{M}}{}^p_q := \overline{L^0_{\mathrm{c}} \cap \mathcal{M}^p_q}^{\mathcal{M}^p_q}$ (see [21]).
(3) $\widetilde{\mathcal{M}}{}^p_q := \overline{L^\infty_{\mathrm{c}}}^{\mathcal{M}^p_q}$ (see [9]).

Observe that $\widetilde{\mathcal{M}}^p_q$ coincides with $\overline{C^\infty_{\rm c}}^{\mathcal{M}^p_q}$. These subspaces can be unified by introducing the following definition.

Definition 5.2. *Assume that a linear subspace of measurable functions U satisfies the condition: $g \in U$ whenever $f \in U$ and $|g| \le |f|$. For $1 \le q \le p < \infty$, define*

$$U\mathcal{M}^p_q := \overline{U \cap \mathcal{M}^p_q}^{\mathcal{M}^p_q}.$$

Example 5.3. *If $U := L^\infty_{\rm c}, L^0_{\rm c}, L^\infty$, then $U\mathcal{M}^p_q = \widetilde{\mathcal{M}}^p_q, \overset{*}{\mathcal{M}}{}^p_q, \overline{\mathcal{M}}^p_q$.*

Theorem 5.4. *Let $\theta \in (0,1)$, $1 \le q_0 \le p_0 < \infty$, $1 \le q_1 \le p_1 < \infty$, $\frac{p_0}{q_0} = \frac{p_1}{q_1}$, and $q_0 \neq q_1$. Define*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then

$$[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta = \left\{ f \in U\mathcal{M}^p_q : \lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N} \le |f| \le N\}} f\|_{\mathcal{M}^p_q} = 0 \right\}.$$

In order to prove Theorem 5.4, we need to prove the following lemmas:

Lemma 5.5. [9, Lemma 4.2] *Assume the same paramaters as in Theorem 5.4. Let E be a measurable set such that $\chi_E \in U\mathcal{M}^p_q$. Then*

$$\chi_E \in U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}.$$

Proof. Let $\chi_E \in U\mathcal{M}^p_q$ and choose $\{g_k\}_{k=1}^\infty \subseteq U \cap \mathcal{M}^p_q$ for which

$$\lim_{k\to\infty} \|\chi_E - g_k\|_{\mathcal{M}^p_q} = 0.$$

Define $h_k := \chi_{\{g_k \neq 0\} \cap E}$. Then, for each $k = 0, 1$, we have

$$\|\chi_E - h_k\|_{\mathcal{M}^{p_j}_{q_j}} = \|\chi_E - h_k\|^{q/q_j}_{\mathcal{M}^p_q} \le \|\chi_E - g_k\|^{q/q_j}_{\mathcal{M}^p_q} \to 0$$

as $k \to \infty$. Thus, $\chi_E \in U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}$. □

Lemma 5.6. [9, Lemma 4.1] *Assume the same paramaters as in Theorem 5.4. Then $U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1} \subseteq U\mathcal{M}^p_q$.*

Proof. Without loss of generality assume that $q_1 > q_0$. Let $f \in U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}$. In view of Lemma 4.6, we may assume $f = \chi_{\{1/N \le |f| \le N\}} f$ for some $N \in \mathbb{N}$. By the lattice property of the spaces $U\mathcal{M}^{p_0}_{q_0}$, $U\mathcal{M}^{p_1}_{q_1}$ and $U\mathcal{M}^p_q$, we may assume $f = \chi_E$ for some measurable set E. Choose a sequence $\{g_j\}_{j=1}^\infty \subseteq U \cap \mathcal{M}^{p_1}_{q_1}$ such that

$$\lim_{j\to\infty} \|f - g_j\|_{\mathcal{M}^{p_1}_{q_1}} = 0.$$

Define $F_j := \{g_j \neq 0\} \cap E$. Hence $|f - \chi_{F_j}| \le 2$ and $|f - \chi_{F_j}| \le |f - g_j|$. Consequently,

$$\|f - \chi_{F_j}\|_{\mathcal{M}^p_q} = \left\| |f - \chi_{F_j}|^{1-\frac{q_1}{q}} |f - \chi_{F_j}|^{\frac{q_1}{q}} \right\|_{\mathcal{M}^p_q} \le 2^{1-\frac{q_1}{q}} \|f - g_j\|^{\frac{q_1}{q}}_{\mathcal{M}^{p_1}_{q_1}}.$$

This shows that $f \in U\mathcal{M}^p_q$. □

The proof of Theorem 5.4 is given as follows:

Proof of Theorem 5.4. We assume that $q_1 > q_0$. By using Lemma 5.6, the inclusions

$$[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta \subseteq [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subseteq \mathcal{M}^p_q$$

and the fact that $X_0 \cap X_1$ is a dense subset of $[X_0, X_1]_\theta$, we have $[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta \subseteq U\mathcal{M}^p_q$. Consequently,

$$\begin{aligned}[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta &\subseteq U\mathcal{M}^p_q \cap [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \\ &= \left\{ f \in U\mathcal{M}^p_q : \lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N} \le |f| \le N\}} f\|_{\mathcal{M}^p_q} = 0 \right\}.\end{aligned}$$

Conversely, let $f \in U\mathcal{M}^p_q$ be such that

$$\lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N} \le |f| \le N\}} f\|_{\mathcal{M}^p_q} = 0$$

Then, $f \in U\mathcal{M}^p_q \cap [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta$. Note that, for any $0 < b < c < \infty$, we have a pointwise estimate:

$$\chi_{\{b \le |f| \le c\}} \le \frac{1}{b} \chi_{\{b \le |f| \le c\}} |f| \le \frac{|f|}{b}, \tag{55}$$

so $\chi_{\{b\le|f|\le c\}} \in U\mathcal{M}^p_q$. From Lemma 5.5, it follows that $\chi_{\{b\le|f|\le c\}} \in U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}$. For every $N \in \mathbb{N}$ and $z \in \overline{S}$, define

$$F_N(z) = \operatorname{sgn}(f)|f|^{q\left(\frac{1-z}{q_0} + \frac{z}{q_1}\right)} \chi_{\{\frac{1}{N} \le |f| \le N\}}.$$

Decompose $F_N(z) := F_{N,0}(z) + F_{N,1}(z)$ where $F_{N,0}(z) := F_N(z)\chi_{\{|f|\le 1\}}$. Since

$$|F_{N,0}(z)| \le \chi_{\{\frac{1}{N} \le |f| \le 1\}} \quad \text{and} \quad |F_{N,1}(z)| \le \left(N^{\frac{q}{q_0}} + N^{\frac{q}{q_1}}\right) \chi_{\{1 \le |f| \le N\}},$$

we have $F_N(z) = F_{N,0}(z) + F_{N,1}(z) \in U\mathcal{M}^{p_0}_{q_0} + U\mathcal{M}^{p_1}_{q_1}$. Moreover, we also have

$$\sup_{z\in\overline{S}} \|F_N(z)\|_{U\mathcal{M}^{p_0}_{q_0} + U\mathcal{M}^{p_1}_{q_1}} \le \|\chi_{\{\frac{1}{N} \le |f| \le 1\}}\|_{U\mathcal{M}^{p_0}_{q_0}} + \left(N^{\frac{q}{q_0}} + N^{\frac{q}{q_1}}\right) \|\chi_{\{1 \le |f| \le N\}}\|_{U\mathcal{M}^{p_1}_{q_1}}.$$

Observe that for every $w \in \overline{S}$, we have

$$|F_N'(w)| \le \left(\frac{q}{q_0} - \frac{q}{q_1}\right) \left(N^{\frac{q}{q_0}} + N^{\frac{q}{q_1}}\right) (\log N) \times \chi_{\{\frac{1}{N} \le |f| \le N\}}. \tag{56}$$

Then we have

$$\begin{aligned}&\|F_N(z) - F_N(z')\|_{U\mathcal{M}^{p_0}_{q_0} + U\mathcal{M}^{p_1}_{q_1}} \\ &= \left\| \int_{z'}^{z} F_N'(w)\, dw \right\|_{U\mathcal{M}^{p_0}_{q_0} + U\mathcal{M}^{p_1}_{q_1}} \\ &\le \left(\frac{q}{q_0} - \frac{q}{q_1}\right) \left(N^{\frac{q}{q_0}} + N^{\frac{q}{q_1}}\right) (\log N) \times \left(\|\chi_{\{\frac{1}{N} \le |f| \le N\}}\|_{U\mathcal{M}^{p_0}_{q_0} + U\mathcal{M}^{p_1}_{q_1}} \right) |z - z'| \\ &\le \left(\frac{q}{q_0} - \frac{q}{q_1}\right) \left(N^{\frac{q}{q_0}} + N^{\frac{q}{q_1}}\right) \log N \\ &\quad \times \left(\|\chi_{\{\frac{1}{N} \le |f| \le 1\}}\|_{U\mathcal{M}^{p_0}_{q_0}} + \|\chi_{\{1 < |f| \le N\}}\|_{U\mathcal{M}^{p_1}_{q_1}} \right) |z - z'|\end{aligned}$$

for all $z, z' \in \overline{S}$. Thus, $F_N : \overline{S} \to U\mathcal{M}^{p_0}_{q_0} + U\mathcal{M}^{p_1}_{q_1}$ is a continuous function. Likewise we can check that $F_N|_S : S \to U\mathcal{M}^{p_0}_{q_0} + U\mathcal{M}^{p_1}_{q_1}$ is a holomorphic function. Note that, for all $t \in \mathbb{R}$ and $j = 0, 1$, we have

$$|F_N(j+it)| = |f|^{\frac{q}{q_j}} \chi_{\{\frac{1}{N} \le |f| \le N\}} \le N^{\frac{q}{q_j}} \chi_{\{\frac{1}{N} \le |f| \le N\}},$$

so, $F_N(j+it) \in U\mathcal{M}^{p_j}_{q_j}$. Furthermore, by using (56), we get

$$\begin{aligned} \|F_N(j+it) - F_N(j+it')\|_{U\mathcal{M}^{p_j}_{q_j}} &= \left\| \int_{j+it'}^{j+it} F_N'(w)\, dw \right\|_{U\mathcal{M}^{p_j}_{q_j}} \\ &\le \left(\frac{q}{q_0} - \frac{q}{q_1} \right) \left(N^{\frac{q}{q_0}} + N^{\frac{q}{q_1}} \right) \log N \\ &\quad \times \|\chi_{\{1/N \le |f| \le N\}}\|_{U\mathcal{M}^{p_j}_{q_j}} |t - t'| \end{aligned}$$

for all $t, t' \in \mathbb{R}$. This shows that $t \in \mathbb{R} \mapsto F_N(j+it) \in U\mathcal{M}^{p_j}_{q_j}$ are continuous functions. In total, we have showed that $F_N \in \mathcal{F}(U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1})$. Note that, for $M, N \in \mathbb{N}$ with $N < M$, we have

$$\begin{aligned} \|F_M(\theta) - F_N(\theta)\|_{[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta} &\le \|F_M - F_N\|_{\mathcal{F}(U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1})} \\ &= \max_{j=0,1} \sup_{t \in \mathbb{R}} \|F_M(j+it) - F_N(j+it)\|_{U\mathcal{M}^{p_j}_{q_j}} \\ &= \max_{j=0,1} \sup_{t \in \mathbb{R}} \||f|^{q/q_j} \chi_{\{\frac{1}{M} \le |f| \le \frac{1}{N}\} \cup \{N \le |f| \le M\}}\|_{\mathcal{M}^{p_j}_{q_j}} \\ &= \max_{j=0,1} \|f \chi_{\{\frac{1}{M} \le |f| \le \frac{1}{N}\} \cup \{N \le |f| \le M\}}\|^{q/q_j}_{\mathcal{M}^p_q} \\ &\le \max_{j=0,1} \|f - \chi_{\{\frac{1}{N} \le |f| \le N\}} f\|^{q/q_j}_{\mathcal{M}^p_q}. \end{aligned}$$

Since $\lim_{N \to \infty} \|f - \chi_{\{\frac{1}{N} \le |f| \le N\}} f\|_{\mathcal{M}^p_q} = 0$, we see that

$$\|F_M(\theta) - F_N(\theta)\|_{[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta} \to 0$$

whenever $M, N \to \infty$. Thus, $F_N(\theta)$ converges to $g \in [U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta$. Hence, $\lim_{N \to \infty} F_N(\theta) = g$ in $\mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$. Meanwhile, by combining $\mathcal{M}^p_q \subseteq \mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$ and

$$\lim_{N \to \infty} \|f - \chi_{\{\frac{1}{N} \le |f| \le N\}} f\|_{\mathcal{M}^p_q} = 0,$$

we have $\lim_{N \to \infty} F_N(\theta) = f$ in $\mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$, which implies $f = g$. Thus, $f \in [U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta$ as desired. □

Theorem 5.7. *Let $\theta \in (0,1)$, $1 \le q_0 \le p_0 < \infty$, $1 \le q_1 \le p_1 < \infty$, and $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. Define*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then we have

$$\begin{aligned} &[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]^\theta \\ &= \{f \in \mathcal{M}^p_q : \chi_{\{a \le |f| \le b\}} f \in U\mathcal{M}^p_q \text{ for all } 0 < a < b < \infty\}. \end{aligned}$$

From now on, we shall always use the assumption of Theorem 5.7. To prove Theorem 5.7, we shall invoke and prove several lemmas.

Lemma 5.8. [10] *Keep the assumption in Theorem 5.7. Then*

$$U \bowtie \mathcal{M}^p_q \subseteq [U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]^\theta. \tag{57}$$

Proof. Without loss of generality, assume that $q_0 > q_1$. Let $f \in U \bowtie \mathcal{M}^p_q$. Since $\chi_{\{a \le |f| \le b\}} \le \frac{1}{a}\chi_{\{a \le |f| \le b\}}|f|$, we have $\chi_{\{a \le |f| \le b\}} \in U\mathcal{M}^p_q$. From Lemma 5.5, we have $\chi_{\{a \le |f| \le b\}} \in U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}$. For $z \in \overline{S}$, define

$$F(z) := \operatorname{sgn}(f)|f|^{\frac{qz}{q_0}+\frac{q(1-z)}{q_1}} \text{ and } G(z) := (z-\theta)\int_0^1 F(\theta+(z-\theta)t)\,dt. \tag{58}$$

Decompose $G(z) = G_0(z) + G_1(z)$ where $G_0(z) := \chi_{\{|f|\le 1\}}G(z)$. Let $0 < \varepsilon < 1$. Since $\chi_{\{\varepsilon \le |f| \le 1\}} \in U\mathcal{M}^{p_0}_{q_0}$ and

$$\chi_{\{\varepsilon \le |f| \le 1\}}|G_0(z)| \le (1+|z|)(|f|^{q/q_0}+|f|^{q/q_1})\chi_{\{\varepsilon \le |f| \le 1\}} \le 2(1+|z|)\chi_{\{\varepsilon \le |f| \le 1\}}, \tag{59}$$

we have $\chi_{\{\varepsilon \le |f| \le 1\}}G_0(z) \in U\mathcal{M}^{p_0}_{q_0}$. Observe that

$$\begin{aligned}\|G_0(z) - \chi_{\{\varepsilon \le |f| \le 1\}}G_0(z)\|_{\mathcal{M}^{p_0}_{q_0}} &= \left\| \chi_{\{|f|\le\varepsilon\}}\frac{F(z)-F(\theta)}{\left(\frac{q}{q_1}-\frac{q}{q_0}\right)\log|f|}\right\|_{\mathcal{M}^{p_0}_{q_0}} \\ &\le \left\| \frac{2|f|^{q/q_0}}{\left(\frac{q}{q_1}-\frac{q}{q_0}\right)\log(\varepsilon^{-1})}\right\|_{\mathcal{M}^{p_0}_{q_0}} \\ &= \frac{2\|f\|^{q/q_0}_{\mathcal{M}^p_q}}{\left(\frac{q}{q_1}-\frac{q}{q_0}\right)\log\varepsilon^{-1}} \to 0 \end{aligned} \tag{60}$$

as $\varepsilon \to 0^+$. Hence $G_0(z) \in U\mathcal{M}^{p_0}_{q_0}$. Similarly, $G_1(z) \in U\mathcal{M}^{p_1}_{q_1}$. Thus $G(z) \in U\mathcal{M}^{p_0}_{q_0} + U\mathcal{M}^{p_1}_{q_1}$. Let $t \in \mathbb{R}$ and $R > 1$. Since $\chi_{\{R^{-1} \le |f| \le R\}} \in U\mathcal{M}^{p_0}_{q_0}$ and

$$|(G(it)-G(0))|\chi_{\{R^{-1}\le|f|\le R\}} \le (2+|t|)(R^{q/q_0}+R^{q/q_1})\chi_{\{R^{-1}\le|f|\le R\}}, \tag{61}$$

we have $[G(it)-G(0)]\chi_{\{R^{-1}\le|f|\le R\}} \in U\mathcal{M}^{p_0}_{q_0}$. Note that

$$\|[G(it)-G(0)]\chi_{\mathbb{R}^n\setminus\{R^{-1}\le|f|\le R\}}\|_{\mathcal{M}^{p_0}_{q_0}} \le \frac{2\|f\|^{q/q_0}_{\mathcal{M}^p_q}}{\left(\frac{q}{q_1}-\frac{q}{q_0}\right)\log R} \to 0 \tag{62}$$

as $R \to \infty$. Thus $G(it) - G(0) \in U\mathcal{M}^{p_0}_{q_0}$. Similarly, $G(1+it) - G(1) \in U\mathcal{M}^{p_1}_{q_1}$. Since $G \in \mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$, we have $G \in \mathcal{G}(U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1})$. From $f = G'(\theta)$, it follows that $f \in [U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]^\theta$. □

Lemma 5.9. [10] *Let $G \in \mathcal{G}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$ and $\theta \in (0,1)$. For $z \in \overline{S}$ and $k \in \mathbb{N}$, define $H_k(z)$ by (22). Then $H_k(\theta) \in \overline{U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q}$.*

Proof. It follows from Lemma 3.11, that $H_k(\theta) \in [U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta$. Let $\varepsilon > 0$. Since $U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}$ is dense in $[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta$, we can find $J_k(\theta) \in U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}$ such that

$$\|H_k(\theta) - J_k(\theta)\|_{[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta} < \varepsilon.$$

Since $[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta \subseteq [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subseteq \mathcal{M}^p_q$, we have

$$\|H_k(\theta) - J_k(\theta)\|_{\mathcal{M}^p_q} \lesssim \|H_k(\theta) - J_k(\theta)\|_{[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta} < \varepsilon.$$

This shows that $H_k(\theta) \in \overline{U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q}$. □

Lemma 5.10. [10] *Under the assumption of Theorem 5.7, we have*

$$\mathcal{M}^p_q \cap \overline{U\mathcal{M}^p_q}^{\mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}} \subseteq U \bowtie \mathcal{M}^p_q.$$

Proof. Let $f \in \mathcal{M}^p_q \cap \overline{U\mathcal{M}^p_q}^{\mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}}$. Choose $\{f_j\}_{j=1}^\infty \subseteq U\mathcal{M}^p_q$ such that

$$\lim_{j\to\infty} \|f - f_j\|_{\mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}} = 0.$$

Then, we can find $\{k_j\}_{j=1}^\infty \subset U\mathcal{M}^{p_0}_{q_0}$ and $\{h_j\}_{j=1}^\infty \subset U\mathcal{M}^{p_1}_{q_1}$ convergent to 0 in $U\mathcal{M}^{p_0}_{q_0}$ and $U\mathcal{M}^{p_1}_{q_1}$, respectively, such that $f - f_j = k_j + h_j$ for all j. Assume $0 < a < 1 < b < \infty$ as before. Let $\Theta \in C_c(\mathbb{R})$ be a piecewise linear function defined by

$$\Theta'(t) := \frac{2}{a}\chi_{(a/2,a)}(t) - \frac{1}{b}\chi_{(b,2b)}(t) \tag{63}$$

except at $t = \frac{a}{2}, a, b, 2b$. Let $C_{a,b} = \frac{2}{a} + \frac{1}{b}$. Since

$$|\Theta(t) - \Theta(s)| \le C_{a,b}|t-s| \text{ and } |\Theta(t) - \Theta(s)| \le 2,$$

we have

$$|\Theta(|f|) - \Theta(|f_j|)| \le C_{a,b}\min(1, ||f| - |f_j||) \le C_{a,b}\min(1, |f - f_j|).$$

Let $B = B(x_0, r)$ be any ball in $\mathbb{R}^n$. Then,

$$\int_B \chi_{[a,b]}(|f(x)|)|\Theta(|f(x)|) - \Theta(|f_j(x)|)|^q\,dx \lesssim \int_B \chi_{[a,b]}(|f(x)|)\min(1, |f(x) - f_j(x)|^q)\,dx.$$

By using the decomposition $f = f_j + k_j + h_j$, we obtain

$$\int_B \chi_{[a,b]}(|f(x)|)|\Theta(|f(x)|) - \Theta(|f_j(x)|)|^q\,dx \lesssim \int_B \chi_{[a,b]}(|f(x)|)\min(1, |k_j(x)|^q)\,dx + \int_B \chi_{[a,b]}(|f(x)|)\min(1, |h_j(x)|^q)\,dx.$$

Keeping in mind, $q_0 > q > q_1$ and $\frac{q_0}{p_0} = \frac{q_1}{p_1} = \frac{q}{p}$. Then

$$\begin{aligned}
&|B|^{\frac{q}{p}-1} \int_B \chi_{[a,b]}(|f(x)|)|\Theta(|f(x)|) - \Theta(|f_j(x)|)|^q \, dx \\
&\lesssim |B|^{\frac{q}{p}-1} \int_B |h_j(x)|^{q_1} \, dx \\
&\quad + |B|^{\frac{q}{p}-1} \left(\int_B \chi_{[a,b]}(|f(x)|) \, dx \right)^{1-\frac{q}{q_0}} \left(\int_B \min(1, |k_j(x)|^{q_0}) \, dx \right)^{\frac{q}{q_0}} \\
&\lesssim (\|h_j\|_{\mathcal{M}^{p_1}_{q_1}})^{q_1} + \left(|B|^{\frac{q}{p}-1} \int_B |f(x)|^q \, dx \right)^{1-\frac{q}{q_0}} \left(\|k_j\|_{\mathcal{M}^{p_0}_{q_0}} \right)^q \\
&\lesssim (\|h_j\|_{\mathcal{M}^{p_1}_{q_1}})^{q_1} + \left(|B|^{\frac{q}{p}-1} \int_B |f(x)|^q \, dx \right)^{1-\frac{q}{q_0}} \left(\|k_j\|_{\mathcal{M}^{p_0}_{q_0}} \right)^q \\
&\lesssim (\|h_j\|_{\mathcal{M}^{p_1}_{q_1}})^{q_1} + \left(\|f\|_{\mathcal{M}^p_q} \right)^{q-\frac{q^2}{q_0}} \left(\|k_j\|_{\mathcal{M}^{p_0}_{q_0}} \right)^q .
\end{aligned}$$

Thus, it follows that

$$\lim_{j\to\infty} \|\chi_{\{a\le|f|\le b\}}\Theta(|f_j|) - \chi_{\{a\le|f|\le b\}}\Theta(|f|)\|_{\mathcal{M}^p_q} = 0.$$

Since $\chi_{\{a\le|f|\le b\}}\Theta(|f_j|) \le a^{-1}|f_j|$, we have $\chi_{\{a\le|f|\le b\}}\Theta(|f|) \in U\mathcal{M}^p_q$. From the equality

$$\chi_{\{a\le|f|\le b\}}|f| = b\chi_{\{a\le|f|\le b\}}\Theta(|f|),$$

it follows that $\chi_{\{a\le|f|\le b\}}f \in U\mathcal{M}^p_q$. □

Now, we are ready to prove Theorem 5.7.

Theorem 5.7. In view of Lemma 5.8, we only need to show that

$$[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]^\theta \subseteq U \bowtie \mathcal{M}^p_q.$$

Let $f \in [U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]^\theta$. Then there exists $G \in \mathcal{G}(U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1})$ such that $G'(\theta) = f$. For $z \in \overline{S}$ and $k \in \mathbb{N}$, define $H_k(z)$ by (22). By virtue of Lemmas 5.6 and 5.9, we have $H_k(\theta) \in U\mathcal{M}^p_q$. Since $H_k(\theta)$ converges to $G'(\theta) = f$ in $\mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$, we see that $f \in U \bowtie \mathcal{M}^p_q$. □

By substituting $U := L^\infty_{\rm c}, L^0_{\rm c}, L^\infty$, we have the following result.

Corollary 5.11. *Keep the same assumption as in the previous theorems. Then*

$$[\widetilde{\mathcal{M}}^{p_0}_{q_0}, \widetilde{\mathcal{M}}^{p_1}_{q_1}]_\theta = [\overset{*}{\mathcal{M}}{}^{p_0}_{q_0}, \overset{*}{\mathcal{M}}{}^{p_1}_{q_1}]_\theta = \widetilde{\mathcal{M}}^p_q,$$

$$[\overline{\mathcal{M}}^{p_0}_{q_0}, \overline{\mathcal{M}}^{p_1}_{q_1}]_\theta = \{f \in \overline{\mathcal{M}}^p_q : \lim_{N\to\infty} \|f\chi_{\{|f|<\frac{1}{N}\}}\|_{\mathcal{M}^p_q} = 0\},$$

$$\begin{aligned}
[\widetilde{\mathcal{M}}^{p_0}_{q_0}, \widetilde{\mathcal{M}}^{p_1}_{q_1}]^\theta &= [\overset{*}{\mathcal{M}}{}^{p_0}_{q_0}, \overset{*}{\mathcal{M}}{}^{p_1}_{q_1}]^\theta \\
&= \bigcap_{0<a<b<\infty} \{f \in \mathcal{M}^p_q : \chi_{\{a\le|f|\le b\}}f \in \widetilde{\mathcal{M}}^p_q\},
\end{aligned}$$

and $[\overline{\mathcal{M}}^{p_0}_{q_0}, \overline{\mathcal{M}}^{p_1}_{q_1}]^\theta = \mathcal{M}^p_q$.

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