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A NOTE ON INCLUSION PROPERTIES OF WEIGHTED ORLICZ SPACES

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Abstract. In this paper we present sufficient and necessary conditions for the inclusion relation between two weighted Orlicz spaces which complete the Osançliol result in 2014. One of the keys to prove our results is to use the norm of the characteristic functions of the balls in $\mathbb{R}^n$.

Key words and phrases: *Inclusion property, weighted Lebesgue spaces, weighted Orlicz spaces.*

1. INTRODUCTION

Orlicz spaces are generalization of Lebesgue spaces which were firstly introduced by Z. W. Birnbaum and W. Orlicz in 1931 (see [5, 14]). Let us first recall the definition of Orlicz spaces. Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function [that is, Φ is convex, $\lim_{t\to 0} \Phi(t) = 0 = \Phi(0)$, left-continuous and $\lim_{t\to\infty} \Phi(t) = \infty$]. The **Orlicz space** $L_\Phi(\mathbb{R}^n)$ is the set of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} \Phi(a|f(x)|)dx < \infty$ for some $a > 0$. The space $L_\Phi(\mathbb{R}^n)$ is a Banach space equipped with the norm

$$\|f\|_{L_\Phi(\mathbb{R}^n)} := \inf\left\{b > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{b}\right) dx \le 1\right\}.$$

Meanwhile, for Φ is a Young function, the **weak Orlicz space** $wL_\Phi(\mathbb{R}^n)$ is the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\|f\|_{wL_\Phi(\mathbb{R}^n)} := \inf\left\{b > 0 : \sup_{t>0} \Phi(t)\mu\left(\left\{x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t\right\}\right) \le 1\right\} < \infty.$$

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Now, we move to the weighted Orlicz spaces and weighted weak Orlicz spaces. Let Φ be a Young function and u is a weight on $\mathbb{R}^n$ (i.e $u : \mathbb{R}^n \to (0,\infty)$ is a measurable function). The **weighted Orlicz space** $L^u_\Phi(\mathbb{R}^n)$ is the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $uf \in L_\Phi(\mathbb{R}^n)$. Note that, the space $L^u_\Phi(\mathbb{R}^n)$ is a Banach space equipped with the norm

$$\|f\|_{L^u_\Phi(\mathbb{R}^n)} := \|uf\|_{L_\Phi(\mathbb{R}^n)} = \inf\left\{b > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|u(x)f(x)|}{b}\right) dx \le 1\right\}.$$

Similar with weighted Orlicz spaces, for a Young function Φ and a weight u on $\mathbb{R}^n$, the **weighted weak Orlicz space** $wL^u_\Phi(\mathbb{R}^n)$ is the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\|f\|_{wL^u_\Phi(\mathbb{R}^n)} := \|uf\|_{wL_\Phi(\mathbb{R}^n)} < \infty$.

For $u_1, u_2 : \mathbb{R}^n \to (0,\infty)$, we denote $u_1 \preceq u_2$ if there exists a constant $C > 0$ such that $u_1(x) \le Cu_2(x)$ for all $x \in \mathbb{R}^n$. Note that, if $u_1 \preceq u_2$, then $\|f\|_{L^{u_1}_\Phi(\mathbb{R}^n)} \le C\|f\|_{L^{u_2}_\Phi(\mathbb{R}^n)}$ and $\|f\|_{wL^{u_1}_\Phi(\mathbb{R}^n)} \le C\|f\|_{wL^{u_2}_\Phi(\mathbb{R}^n)}$.

The study of Lebesgue spaces and Orlicz spaces has been carried out by many researchers in the last few decades (see [3, 5, 6, 7, 12, 15], etc.). In 1989, Maligranda [6] discussed inclusion properties of Orlicz spaces. Later in 2016, Masta *et al.* [7] obtained sufficient and necessary conditions for the inclusion relation between two Orlicz spaces and between two weak Orlicz spaces by using a different technique from Maligranda. Moreover, they have found that two Orlicz spaces and two weak Orlicz spaces are comparable with respect to Young functions for any measurable set, although the Lebesgue space L_p are not comparable with respect to the number p.

On the other hand, Osançliol [13] has obtained sufficient and necessary conditions for the inclusion relation between two weighted Orlicz spaces, as in the following theorem.

Theorem 1.1. [13] *Let Φ be a continuous Young function satisfying the $\triangle_2$ condition [that is, there exists $K > 0$ such that $\Phi(2t) \le K\Phi(t)$ for all $t \ge 0$], and u_1, u_2 are measurable functions such that $u_i(x + y) \le u_i(x) \cdot u_i(y)$ for every $x, y \in \mathbb{R}^n$, where $i = 1, 2$. Then the following statements are equivalent:*
(1) $u_1 \preceq u_2$.
(2) $L^{u_2}_\Phi(\mathbb{R}^n) \subseteq L^{u_1}_\Phi(\mathbb{R}^n)$.
(3) *There exists a constant $C > 0$ such that $\|f\|_{L^{u_1}_\Phi(\mathbb{R}^n)} \le C\|f\|_{L^{u_2}_\Phi(\mathbb{R}^n)}$, for every $f \in L^{u_2}_\Phi(\mathbb{R}^n)$.*

Related results for weak type of Orlicz spaces can be found in [10].

In this paper, we are interested in studying the inclusion properties of weighted Orlicz spaces. In connection with Theorem 1.1, we shall prove the inclusion relation between weighted Orlicz spaces with respect to Young functions Φ_1, Φ_2 and weights u_1, u_2.

To achieve our purpose, we will use the similar methods as in [1, 7, 8, 9, 13], which pay attention to the characteristic functions of open balls in $\mathbb{R}^n$. Next, we recall some lemmas which will be used later in the next section.

Lemma 1.2. [11] *Suppose that Φ is a Young function and $\Phi^{-1}(s) := \inf\{r \geq 0 : \Phi(r) > s\}$. We have*
(1) $\Phi^{-1}(0) = 0$.
(2) $\Phi^{-1}(s_1) \leq \Phi^{-1}(s_2)$ *for* $s_1 \leq s_2$.
(3) $\Phi(\Phi^{-1}(s)) \leq s \leq \Phi^{-1}(\Phi(s))$ *for* $0 \leq s < \infty$.

Lemma 1.3. [9] *Let Φ_1, Φ_2 be Young functions. For any $s > 0$, if there exists $C_1, C_2 > 0$ such that $\Phi_2^{-1}(s) \leq C_1\Phi_1^{-1}(C_2 s)$, then we have $\Phi_1(\frac{t}{C_1}) \leq C_2\Phi_2(t)$ for $t = \Phi_2^{-1}(s)$.*

In this paper, the letter C will be used for constants whose values may change from line to line, while constants with subscripts, such as C_1, C_2, do not change their values.

2. RESULTS

First, we will investigate the inclusion properties of weighted Orlicz spaces with respect to distinct Young functions Φ_1 and Φ_2. To get the result, we need to estimate the norm of the characteristic function of an open ball in $\mathbb{R}^n$ as in the following lemma.

Lemma 2.1. [4, 7] *Let Φ be a Young function, $a \in \mathbb{R}^n$, and $r > 0$ be arbitrary. Then we have $\left\|\frac{\chi_{B(a,r)}}{u}\right\|_{L^u_\Phi(\mathbb{R}^n)} = \|\chi_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}\left(\frac{1}{|B(a,r)|}\right)}$ where $|B(a,r)|$ denotes the volume of the open ball $B(a,r)$ centered at $a \in \mathbb{R}^n$ with radius $r > 0$.*

Now we come to the inclusion relation between $L^u_{\Phi_1}(\mathbb{R}^n)$ and $L^u_{\Phi_2}(\mathbb{R}^n)$ with respect to Young functions Φ_1, Φ_2. Given two Young functions Φ_1, Φ_2, we write $\Phi_1 \prec \Phi_2$ if there exists a constant $C > 0$ such that $\Phi_1(t) \leq \Phi_2(Ct)$ for all $t > 0$.

Theorem 2.2. *Let Φ_1, Φ_2 be Young functions and $u : \mathbb{R}^n \to (0, \infty)$ be a measurable function. Then the following statements are equivalent:*
(1) $\Phi_1 \prec \Phi_2$.
(2) $L^u_{\Phi_2}(\mathbb{R}^n) \subseteq L^u_{\Phi_1}(\mathbb{R}^n)$.
(3) *There exists a constant $C > 0$ such that $\|f\|_{L^u_{\Phi_1}(\mathbb{R}^n)} \leq C\|f\|_{L^u_{\Phi_2}(\mathbb{R}^n)}$, for every $f \in L^u_{\Phi_2}(\mathbb{R}^n)$.*

Proof. Assume that (1) holds. Suppose that $f \in L^u_{\Phi_2}(\mathbb{R}^n)$. Observe that

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi_1\left(\frac{|u(x)f(x)|}{C\|f\|_{L^u_{\Phi_2}(\mathbb{R}^n)}}\right) dx &\leq \int_{\mathbb{R}^n} \Phi_2\left(\frac{C|u(x)f(x)|}{C\|f\|_{L^u_{\Phi_2}(\mathbb{R}^n)}}\right) dx \\ &= \int_{\mathbb{R}^n} \Phi_2\left(\frac{|u(x)f(x)|}{\|f\|_{L^u_{\Phi_2}(\mathbb{R}^n)}}\right) dx \leq 1.\end{aligned}$$

By definition of $\|\cdot\|_{L^u_\Phi(\mathbb{R}^n)}$, we have $\|f\|_{L^u_{\Phi_1}(\mathbb{R}^n)} \le C\|f\|_{L^u_{\Phi_2}(\mathbb{R}^n)}$. This proves that $L^u_{\Phi_2}(\mathbb{R}^n) \subseteq L^u_{\Phi_1}(\mathbb{R}^n)$.

Next, since $(L^u_{\Phi_1}(\mathbb{R}^n), L^u_{\Phi_2}(\mathbb{R}^n))$ is a Banach pair, it follows from [2, Lemma 3.3] that (2) and (3) are equivalent. It thus remains to show that (3) implies (1).

Assume now that (3) holds. By Lemma 2.1, we have

$$\frac{1}{\Phi_1^{-1}\Big(\frac{1}{|B(a,r)|}\Big)} = \left\|\frac{\chi_{B(a,r)}}{u}\right\|_{L^u_{\Phi_1}(\mathbb{R}^n)} \le C\left\|\frac{\chi_{B(a,r)}}{u}\right\|_{L^u_{\Phi_2}(\mathbb{R}^n)} = \frac{C}{\Phi_2^{-1}\Big(\frac{1}{|B(a,r)|}\Big)}.$$

Since $\frac{1}{\Phi_1^{-1}\Big(\frac{1}{|B(a,r)|}\Big)} \le \frac{C}{\Phi_2^{-1}\Big(\frac{1}{|B(a,r)|}\Big)}$ is equivalent to $C\Phi_1^{-1}(\frac{1}{|B(a,r)|}) \ge \Phi_2^{-1}(\frac{1}{|B(a,r)|})$ for arbitrary $a \in \mathbb{R}^n$ and $r > 0$, by Lemma 1.3, we have

$$\Phi_1\Big(\frac{t_0}{C}\Big) \le \Phi_2(t_0),$$

for $t_0 = \Phi_2^{-1}(\frac{1}{|B(a,r)|})$. Since $a \in \mathbb{R}^n$ and $r > 0$ are arbitrary, we conclude that $\Phi_1(t) \le \Phi_2(Ct)$ for every $t > 0$. $\square$

Remark 2.3. *For $u(x) = 1$, Theorem 2.2 reduces to Theorem 2.5 in* [7].

Next, we also give the sufficient and necessary conditions for inclusion relation between weighted Orlicz spaces $L^{u_1}_{\Phi_1}(\mathbb{R}^n)$ and $L^{u_2}_{\Phi_2}(\mathbb{R}^n)$ with respect to Young functions Φ_1, Φ_2 and weights u_1, u_2. To get the result, we need the following lemma.

Lemma 2.4. *Let $u : \mathbb{R}^n \to (0,\infty)$ be a measurable function such that $u(x+y) \le u(x)\cdot u(y)$ for every $x, y \in \mathbb{R}^n$. If Φ is a Young function, then:*
(1) For all $f \in L^u_\Phi(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$, we have $\|T_x f\|_{L^u_\Phi(\mathbb{R}^n)} \le u(x)\|f\|_{L^u_\Phi(\mathbb{R}^n)}$, where $T_x f(y) = f(y-x)$.
(2) If $f \in L^u_\Phi(\mathbb{R}^n)$ and $f \ne 0$, then there exists a constant $C > 0$ (depending on f) such that

$$\frac{u(x)}{C} \le \|T_x f\|_{L^u_\Phi(\mathbb{R}^n)} \le Cu(x).$$

Proof.
(1) Let $f \in L^u_\Phi(\mathbb{R}^n)$ and $T_x f(y) = f(y-x)$. Then

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|u(v)f(v)|}{\|f\|_{L^u_\Phi(\mathbb{R}^n)}}\Big)dv \le 1.$$

Observe that (by setting $v := y - x$), we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi\Big(\frac{|u(y)T_x f(y)|}{u(x)\|f\|_{L^u_\Phi(\mathbb{R}^n)}}\Big)dy &= \int_{\mathbb{R}^n} \Phi\Big(\frac{|u(y)f(y-x)|}{u(x)\|f\|_{L^u_\Phi(\mathbb{R}^n)}}\Big)dy \\
&= \int_{\mathbb{R}^n} \Phi\Big(\frac{|u(v+x)f(v)|}{u(x)\|f\|_{L^u_\Phi(\mathbb{R}^n)}}\Big)dv \\
&\leq \int_{\mathbb{R}^n} \Phi\Big(\frac{|u(v)u(x)f(v)|}{u(x)\|f\|_{L^u_\Phi(\mathbb{R}^n)}}\Big)dv \\
&= \int_{\mathbb{R}^n} \Phi\Big(\frac{|u(v)f(v)|}{\|f\|_{L^u_\Phi(\mathbb{R}^n)}}\Big)dv \\
&\leq 1.
\end{aligned}$$

This shows that $\|T_x f\|_{L^u_\Phi(\mathbb{R}^n)} \leq u(x)\|f\|_{L^u_\Phi(\mathbb{R}^n)}$.
(2) Let $f \in L^u_\Phi(\mathbb{R}^n)$ and $f \neq 0$. Then there exists a constant $C > 0$ (depending on f) such that $\|f\|_{L^u_\Phi(\mathbb{R}^n)} \leq C$. By Lemma 2.4 (1), we have

$$\|T_x f\|_{L^u_{\phi,\Phi}(\mathbb{R}^n)} \leq Cu(x),$$

for every $x \in \mathbb{R}^n$. Since $f(x) \neq 0$ for every $x \in \mathbb{R}^n$, we have $\|T_x f\|_{L^u_\Phi(\mathbb{R}^n)} > 0$. Observe that

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi\Big(\frac{|u(x)f(v)|}{\sup\limits_{v\in\mathbb{R}^n} u(-v)\|T_x f\|_{L^u_\Phi(\mathbb{R}^n)}}\Big)dv &\leq \int_{\mathbb{R}^n} \Phi\Big(\frac{|u(x)f(v)|}{u(-v)\|T_x f\|_{L^u_\Phi(\mathbb{R}^n)}}\Big)dv \\
&\leq \int_{\mathbb{R}^n} \Phi\Big(\frac{|u(v+x)f(v)|}{\|T_x f\|_{L^u_\Phi(\mathbb{R}^n)}}\Big)dv \\
&\leq \int_{\mathbb{R}^n} \Phi\Big(\frac{|u(y)f(y-x)|}{\|T_x f\|_{L^u_\Phi(\mathbb{R}^n)}}\Big)dy \\
&= \int_{\mathbb{R}^n} \Phi\Big(\frac{|u(y)L_x f(y)|}{\|T_x f\|_{L^u_\Phi(\mathbb{R}^n)}}\Big)dy \\
&\leq 1.
\end{aligned}$$

This shows that $\frac{u(x)\|f\|_{L_\Phi(\mathbb{R}^n)}}{\sup\limits_{v\in R^n} u(-v)} \leq \|T_x f\|_{L^u_\Phi(\mathbb{R}^n)}$.

Choose $C := \max\Big\{C_1, \frac{\sup_{v\in\mathbb{R}^n} u(-v)}{\|f\|_{L_\Phi(\mathbb{R}^n)}}\Big\}$. Hence, we conclude that

$$\frac{u(x)}{C} \leq \|T_x f\|_{L^u_\Phi(\mathbb{R}^n)} \leq Cu(x),$$

as desired. □

Now, we present the sufficient and necessary conditions for the inclusion relation between weighted Orlicz spaces $L^{u_1}_{\Phi_1}(\mathbb{R}^n)$ and $L^{u_2}_{\Phi_2}(\mathbb{R}^n)$ with respect to Young functions Φ_1, Φ_2 and weights u_1, u_2.

Theorem 2.5. *Let Φ_1, Φ_2 be Young functions such that $\Phi_1 \prec \Phi_2$ and u_1, u_2 are measurable functions such that $u_i(x+y) \leq u_i(x) \cdot u_i(y)$ for every $x, y \in \mathbb{R}^n$, where $i = 1, 2$. Then the following statements are equivalent:*
(1) $u_1 \preceq u_2$.
(2) $L^{u_2}_{\Phi_2}(\mathbb{R}^n) \subseteq L^{u_1}_{\Phi_1}(\mathbb{R}^n)$.
(3) *There exists a constant $C > 0$ such that $\|f\|_{L^{u_1}_{\Phi_1}(\mathbb{R}^n)} \leq C\|f\|_{L^{u_2}_{\Phi_2}(\mathbb{R}^n)}$, for every $f \in L^{u_2}_{\Phi_2}(\mathbb{R}^n)$.*

Proof.

Assume that (1) holds. Let f be an element of $L^{u_2}_{\Phi_2}(\mathbb{R}^n)$. Since $\Phi_1 \prec \Phi_2$ and $u_1 \preceq u_2$, there exists constants $C_1, C_2 > 0$ such that $\Phi_1(t) \leq \Phi_2(C_1 t)$ for all $t > 0$ and $u_1(x) \leq C_2 u_2(x)$ for every $x \in \mathbb{R}^n$. Using a similar argument as in the proof of Theorem 2.2 we have

$$\|f\|_{L^{u_1}_{\Phi_1}(\mathbb{R}^n)} \leq C_1\|f\|_{L^{u_1}_{\Phi_2}(\mathbb{R}^n)} \leq C_1 C_2\|f\|_{L^{u_2}_{\Phi_2}(\mathbb{R}^n)}.$$

As before, we have that (2) and (3) are equivalent. It thus remains to show that (3) implies (1). Assume that (3) holds. By Lemma 2.4, we have

$$\frac{u_1(x)}{C} \leq \|T_x f\|_{L^{u_1}_{\Phi_1}(\mathbb{R}^n)} \leq C\|T_x f\|_{L^{u_2}_{\Phi_2}(\mathbb{R}^n)} \leq C u_2(x),$$

for every $x \in \mathbb{R}^n$. So, we obtain $u_1 \preceq u_2$. $\square$

Note that, for $\Phi_1(x) = \Phi_2(x)$ for every $x \in \mathbb{R}^n$, Theorem 2.5 reduces to Theorem 1.1.

Remark 2.6. *It follows from Theorems 2.2 and 2.5 that there cannot be an inclusion relation between $L^{u_1}_{p_1}(\mathbb{R}^n)$ and $L^{u_2}_{p_2}(\mathbb{R}^n)$ for distinct values of p_1 and p_2. In spite of that, for finite measure set X we can obtain an inclusion relation between $L^{u_1}_{p_1}(X)$ and $L^{u_2}_{p_2}(X)$ as presented in the next section.*

3. AN ADDITIONAL CASE

In the following, we will give a sufficient condition for Hölder's inequality in weighted Orlicz spaces which will be used to obtain an inclusion relation between $L^{u_1}_{p_1}(X)$ and $L^{u_2}_{p_2}(X)$.

Theorem 3.1. *(Hölder's inequality) Let X be a measurable set, Φ_1, Φ_2, Φ_3 be Young functions and $u_1, u_2, u_3 : X \to \mathbb{R}$ be measurable functions such that*

$$\Phi_1^{-1}(t)\Phi_2^{-1}(t) \leq \Phi_3^{-1}(t)$$

for every $t > 0$ and $u_3(x) \leq u_1(x)u_2(x)$ for every $x \in X$. If $f_1 \in L^{u_1}_{\Phi_1}(X)$ and $f_2 \in L^{u_2}_{\Phi_2}(X)$, then $f_1 f_2 \in L^{u_3}_{\Phi_3}(X)$ with

$$\|f_1 f_2\|_{L^{u_3}_{\Phi_3}(X)} \leq 2\|f_1\|_{L^{u_1}_{\Phi_1}(X)}\|f_2\|_{L^{u_2}_{\Phi_2}(X)}.$$

Proof. Let $s, t \geq 0$. Without loss of generality, suppose that $\Phi_1(s) \leq \Phi_2(t)$. By Lemma 1.2(3), we obtain

$$st \leq \Phi_1^{-1}(\Phi_1(s))\Phi_2^{-1}(\Phi_2(t)) \leq \Phi_1^{-1}(\Phi_2(t))\Phi_2^{-1}(\Phi_2(t)) \leq \Phi_3^{-1}(\Phi_2(t)).$$

Hence $\Phi_3(st) \leq \Phi_3(\Phi_3^{-1}(\Phi_2(t))) \leq \Phi_2(t) \leq \Phi_2(t) + \Phi_1(s)$. Since Φ is a convex function, we have

$$\begin{aligned}\int_X \Phi_3\left(\frac{|u_3(x)f_1(x)f_2(x)|}{2\|f_1\|_{L^{u_1}_{\Phi_1}(X)}\|f_2\|_{L^{u_2}_{\Phi_2}(X)}}\right) dx &\leq \frac{1}{2}\int_X \Phi_3\left(\frac{|u_3(x)f_1(x)f_2(x)|}{\|f_1\|_{L^{u_1}_{\Phi_1}(X)}\|f_2\|_{L^{u_2}_{\Phi_2}(X)}}\right) dx \\ &\leq \frac{1}{2}\int_X \Phi_3\left(\frac{|u_1(x)u_2(x)f_1(x)f_2(x)|}{\|f_1\|_{L^{u_1}_{\Phi_1}(X)}\|f_2\|_{L^{u_2}_{\Phi_2}(X)}}\right) dx.\end{aligned}$$

On the other hand, by Lemma 2.1, we obtain

$$\begin{aligned}\int_X \Phi_3\left(\frac{|u_1(x)u_2(x)f_1(x)f_2(x)|}{\|f_1\|_{L^{u_1}_{\Phi_1}(X)}\|f_2\|_{L^{u_2}_{\Phi_2}(X)}}\right) dx &\leq \int_X \Phi_1\left(\frac{|u_1(x)f_1(x)|}{\|f_1\|_{L^{u_1}_{\Phi_1}(X)}}\right) dx \\ &\quad + \int_X \Phi_2\left(\frac{|f_2(x)|}{\|f_2\|_{L^{u_2}_{\Phi_2}(X)}}\right) dx \leq 2,\end{aligned}$$

whenever $f_1 \in L^{u_1}_{\Phi_1}(X)$ and $f_2 \in L^{u_2}_{\Phi_2}(X)$. By the definition of $\|\cdot\|_{L^{u_1}_{\Phi_3}(X)}$, we have $\|f_1 f_2\|_{L^{u_3}_{\Phi_3}(X)} \leq 2\|f_1\|_{L^{u_1}_{\Phi_1}(X)}\|f_2\|_{L^{u_2}_{\Phi_2}(X)}$, as desired. $\square$

Corollary 3.2. *Let $X := B(a, r_0) \subseteq \mathbb{R}^n$ for some $a \in \mathbb{R}^n$ and $r_0 > 0$. If Φ_1, Φ_2 are two Young functions, $u_1, u_2 : X \to \mathbb{R}$ are measurable functions and there exist a Young function Φ and a weight $0 < u(x) \leq 1$ for every $x \in X$ such that $\Phi_1^{-1}(t)\Phi^{-1}(t) \leq \Phi_2^{-1}(t)$ for every $t \geq 0$ and $u_1(x) \leq u(x)u_2(x)$ for every $x \in X$, then*

$$L^{u_1}_{\Phi_1}(X) \subseteq L^{u_2}_{\Phi_2}(X)$$

with $\|f\|_{L^{u_2}_{\Phi_2}(X)} \leq \frac{2}{\Phi^{-1}(\frac{1}{|B(a,r_0)|})}\|f\|_{L^{u_1}_{\Phi_1}(X)}$ for $f \in L^{u_1}_{\Phi_1}(X)$.

Proof. Since $0 < u(x) \leq 1$ for every $x \in X$, we have $\|f\|_{L^{u_2}_{\Phi_2}(X)} \leq \|\frac{f}{u}\|_{L^{u_2}_{\Phi_2}(X)}$. Let $f \in L^{u_1}_{\Phi_1}(X)$. By Theorem 3.1 and choosing $g := \chi_{B(a,r_0)}$, we obtain

$$\begin{aligned}\|f\|_{L^{u_2}_{\Phi_2}(X)} &= \|f\chi_{B(a,r_0)}\|_{L^{u_2}_{\Phi_2}(X)} \\ &= \|fg\|_{L^{u_2}_{\Phi_2}(X)} \\ &\leq \left\|\frac{fg}{u}\right\|_{L^{u_2}_{\Phi_2}(X)} \\ &\leq 2\left\|\frac{g}{u}\right\|_{L^{u}_{\Phi}(X)}\|f\|_{L^{u_1}_{\Phi_1}(X)} \\ &= \frac{2}{\Phi^{-1}(\frac{1}{|B(a,r_0)|})}\|f\|_{L^{u_1}_{\Phi_1}(X)}.\end{aligned}$$

This shows that $L^{u_1}_{\Phi_1}(X) \subseteq L^{u_2}_{\Phi_2}(X)$. $\square$

We shall now discuss the inclusion properties of weighted weak Lebesgue spaces $L^{u_1}_{p_1}(X)$ and $L^{u_2}_{p_2}(X)$ with respect to distinct values of p_1 and p_2 as well as u_1 and u_2.

Corollary 3.3. *Let $X := B(a, r_0)$ for some $a \in \mathbb{R}^n$ and $r_0 > 0$. If $1 \le p_2 < p_1 < \infty$ and $u_1, u_2 : X \to \mathbb{R}$ are measurable functions such that $u_1(x) \le u_2(x)$ for every $x \in X$, then*

$$L^{u_1}_{p_1}(X) \subseteq L^{u_2}_{p_2}(X).$$

Proof. Let $\Phi_1(t) := t^{p_1}, \Phi_2(t) := t^{p_2}$, and $\Phi(t) := t^{\frac{p_1p_2}{p_1-p_2}}$ for every $t \ge 0$. Since $1 \le p_2 < p_1 < \infty$, we have $\frac{p_1p_2}{p_1-p_2} > 1$. Thus, Φ_1, Φ_2, and Φ are Young functions. Now, define $u(x) = \frac{u_1(x)}{u_2(x)}$ for every $x \in X$. Observe that, using the definition of Φ^{-1} and Lemma 1.2, we have

$$\Phi_1^{-1}(t) = t^{\frac{1}{p_1}},\ \Phi_2^{-1}(t) = t^{\frac{1}{p_2}},\ \text{and } \Phi^{-1}(t) = t^{\frac{p_1-p_2}{p_1p_2}}.$$

Moreover, $\Phi_1^{-1}(t)\Phi^{-1}(t) = t^{\frac{1}{p_1}} t^{\frac{p_1-p_2}{p_1p_2}} = t^{\frac{1}{p_2}} = \Phi_2^{-1}(t)$ and $u_1(x) = \frac{u_1(x)}{u_2(x)} u_2(x) = u(x)u_2(x)$. So it follows from Corollary 3.2 that

$$\|f\|_{L^{u_2}_{p_2}(X)} \le 2|B(a, r_0)|^{\frac{p_1-p_2}{p_1p_2}} \|f\|_{L^{u_1}_{p_1}(X)},$$

and therefore we can conclude that $L^{u_1}_{p_1}(X) \subseteq L^{u_2}_{p_2}(X)$. $\square$

4. CONCLUDING REMARKS

We have shown the inclusion properties of weighted Orlicz spaces for distinct Young functions Φ_1, Φ_2 and weights u_1, u_2. The inclusion properties of weighted Orlicz spaces are a generalization of inclusion properties of Orlicz spaces in [7] and inclusion properties of weighted Lebesgue spaces. In the proof of our results, we used the norm of characteristic function on $\mathbb{R}^n$ and estimated the norm of the translation functions in $\mathbb{R}^n$.

Furthemore, from Theorem 2.5 and Lemma 1.1, Theorem 2.8 in [10], we also have the following inclusion relations

$$\begin{array}{ccc} L^{u_2}_{\Phi_2} & \to & L^{u_1}_{\Phi_1} \\ \downarrow & \searrow & \downarrow \\ wL^{u_2}_{\Phi_2} & \to & wL^{u_1}_{\Phi_1} \end{array}$$

for $\Phi_1 \prec \Phi_2$ and $u_1 \preceq u_2$, where the arrows mean 'contained in' or 'embedded into'.

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